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On epimorphisms of groups*

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Abstract

We study the monoid $Ep(G)$ of epimorphisms for certain classes of groups G and we prove that the automorphisms and finitely many classes of epimorphisms generate $Ep(G)$.

1. Introduction

The set $Ep(G)$ of epimorphisms of a group G with binary operation the composition of functions is a monoid. If $Ep(G) = Aut(G)$, then every epimorphism is an automorphism and the group is called hopfian. So a group G is hopfian if every epimorphism $G \to G$ is an isomorphism. Equivalently, G is hopfian if and only if it is not isomorphic to any of its proper quotients.

The problem of existence of non-hopfian groups arose in the topological work of Hopf [2]. The simplest example of a non-hopfian group is the Baumslag-Solitar group

$$\langle b, t \mid t^{-1}b^2t = b^3\rangle.$$

In fact, the hopficity of Baumslag-Solitar groups was investigated by various authors and was settled by Collins and Levin (see [1] and the references therein).

We recall the following theorems:

Theorem 1.1 *[3] Let $G = \langle t, a \mid ta^{\kappa}t^{-1} = a^{\lambda}\rangle$ be a Baumslag-Solitar group. G is a non-hopfian group if and only if $\pi(\kappa) \neq \pi(\lambda)$, where $\pi(\xi)$ is the set of prime divisors of the integer ξ.*

Theorem 1.2 *[5] Assume that G is the HNN-extension $G = \langle K, t \mid tAt^{-1} = B\rangle$ where K is a polycyclic-by-finite group with $A \neq K \neq B$ and $h(K) - 1 \neq h(A) = h(B)$. If $\theta : G \to G$ is an epimorphism then $\theta(K) \leq gKg^{-1}$. In case $\theta(K) \leq K$ then $\theta(t) = gw_1t^{\pm 1}w_2g^{-1}$ where $w_1, w_2 \in K$ for some $g \in G$. Consequently, if $\theta(K) = gKg^{-1}$ then $\theta \in \mathrm{Aut}(G)$.*

The symbol $h(K)$ denotes the Hirsch number of K.

*Key Words and Phrases. HNN-extensions, Hopfian groups, Baumslag-Solitar groups, epimorphism , monoid

2. Equivalent epimorphisms

We begin with a definition

Definition 2.1 Let θ_1, θ_2 be two epimorphisms of the group G. The epimorphisms θ_1 and θ_2 are called *equivalent* if there exist automorphisms $\alpha, \beta \in Aut(G)$ such that $\alpha\theta_1\beta = \theta_2$. We denote $[\theta]$ the equivalence class of the epimorphism θ.

Proposition 2.1 *Let $G =< t, a \mid ta^{\kappa}t^{-1} = a^{\lambda} >$ be a Baumslag-Solitar group. If $\theta : G \longrightarrow G$ is an epimorphism then $\theta(a) = ga^{\xi}g^{-1}$ for some $g \in G$ and $\xi \in \mathbb{Z}$, $\xi \neq 0$.*

Proof. Let G be a group and H a subgroup of G. The commensurator of H is defined as follows:

$Comm_G(H) = \{h \in G \mid hHh^{-1} \cap H <_f H \text{ and } hHh^{-1} \cap H <_f hHh^{-1}\}$

The commensurator of a subgroup H is always a subgroup of G and contains H. If H is a normal subgroup of G, then $Comm_G(H) = G$. There are examples such that H is not a normal subgroup of G and $Comm_G(H) = G$.

From the normal form theorem for HNN-extensions in Baumslag-Solitar groups we have that $Comm_G(< a >) = G$. Now θ is an epimorphism, so it is easy to see that $Comm_G(< \theta(a) >= G$.

We concider the action of G on the standard tree T.

Now $< \theta(a) >$ is an infinite cyclic group, so there are two possibilities:

1 The group $< \theta(a) >$ stabilizes some vertex.

2 The minimal subtree is a double infinite path μ. If so, from well known results the minimal subtree of $Comm_G(< \theta(a) >) = G$ is the same path μ. From Bass-Serre theory we have that G is a polycyclic group, a contradiction.

Consequently, we have proved that the group $< \theta(a) >$ stabilizes some vertex, so $\theta(a) = ga^{\xi}g^{-1}$ for some $\xi \in \mathbb{Z}, \xi \neq 0$ and $g \in G$.

3. The theorem

The result of Proposition 2.1 can also be derived from Theorem 1.2. Theorem 1.2 on Baumslag-Solitar groups implies that if θ is an epimorphism, then there exists an integer ξ and an inner automorphism $\tau_{g^{-1}}$ such that:

$$\tau_{g^{-1}} \circ \theta(a) = a^{\xi} \;\; and \;\; \theta(t) = hw_1t^{\pm 1}w_2h^{-1} \;\; \text{where} \;\; w_1, w_2 \in < a > \;\; \text{for some} \;\; h \in G$$

Now θ and $\tau_{g^{-1}} \circ \theta$ belong to the same equivalence class. If $d = gcd(\xi, \kappa, \lambda) \neq 1$, there exists a prime p such that $p \mid d$ and the least power of a in $Im(\theta)$ is $a^d \neq a$, a contradiction because θ is an epimorphism. See also [1], Lemma 1.2 on page 387. If $\pi(n)$ is the set of prime divisors of the integer n and E is the set

$$E = (\pi(\kappa) - \pi(\lambda)) \cup ((\pi(\lambda) - \pi((\kappa))$$

then the existence of an epimorphism which is not an automorphism of G is equivalent to the existence of an epimorphism θ_{ξ} with the following properties:

1 $\theta_{\xi}(a) = a^{\xi}$ *and* $\theta_{\xi}(t) = hw_1t^{\pm 1}w_2h^{-1}$.

2 $w_1, w_2 \in K$ *for some* $h \in G$ and the prime divisors of ξ are in E.

It is easy to see that the set $\{\theta_\xi \mid \xi \in \mathbb{Z}$ and the prime divisors $are in E\}$ of epimorphisms of G is closed under the composition of maps and it is generated by the set of epimorphisms of G

$$E^* = \{\theta_p,\ \ p\ \text{ prime number and }\ p \in E\}$$

We have proved the following:

Theorem 3.1 *Let $G =< t, a \mid ta^\kappa t^{-1} = a^\lambda >$ be a Baumslag-Solitar group. Every epimorphism in the monoid $Ep(G)$ is a product of an inner automorphism and finitely many epimorpisms of type*

$$\theta_p : a \mapsto a^p, t \mapsto hw_1 t^{\pm 1} w_2 h^{-1} \ \text{ where }\ w_1, w_2 \in< a > \ \text{ for some }\ h \in G, p \in E$$

There is a natural set of epimorphisms of a non-Hopfian Baumslag-Solitar group. This set is

$$E^* = \{\phi_p : G \longrightarrow G, a \mapsto a^p, t \mapsto t, p \in E\}$$

It would be interesting to know whether every epimorphism of a non-Hopfian Baumslag-Solitar group is equivalent to a product of elements of E^* and inner automorphisms.

4. The case of HNN-extensions with base group a finitely generated abelian group

Theorem 1.2 [5] helps us to investigate a theorem analogous to Theorem 3.1.

Theorem 4.1 *Assume that G is the HNN-extension $G = \langle K, t \mid tAt^{-1} = B\rangle$ where K is a finitely generated torsion free abelian group with $A \neq K \neq B$ and $h(K) = h(A) = h(B)$. If $\theta : G \longrightarrow G$ is an epimorphism then $\theta(K) \leq gKg^{-1}$. In case $\theta(K) \leq K$ then $\theta(t) = gw_1 t^{\pm 1} w_2 g^{-1}$ where $w_1, w_2 \in K$ for some $g \in G$ and $\theta(K)$ is a subgroup of finite index in K.*

Proof. The proof follows from Theorem 1.2, combined with [4] Proposition 2 on page 1514, where it is proved that $\theta(K)$ is a subgroup of finite index in K.

Let G be a non-Hopfian group as in theorem 4.1. If θ_1 and θ_2 are two epimorphisms with the property $\theta_1(K) \leq K$ and $\theta_2(K) \leq K$ we define $\theta_1 \prec \theta_2$ if $\theta_1(K) \leq \theta_2(K)$. This is a partial order and a maximal element exists because a finitely generated abelian group satisfies the maximal condition* on subgroups.

*In fact every polycyclic-by-finite group satisfies the maximal conditions on subgroups [6]

We recall the proposition 8 from [4]

Proposition 4.1 *Assume that* G *is the HNN-extension* $G = \langle K, t \mid tAt^{-1} = B\rangle$ *where* K *is a finitely generated torsion free abelian group with* $A \neq K \neq B$ *and* $h(K) = h(A) = h(B)$*. Let* $\theta : G \longrightarrow G$ *be an epimorphism such that* $\theta(K) \leq K$*. Then the mapping* $\theta^* : G \longrightarrow G, t \mapsto t, k \mapsto \theta^{\lambda}(k), k \in K,\ \lambda = 1$ or $\lambda = 2$*, defines an epimorphism of* G*. Furthemore if* θ *is not an automorphism, then* θ^* *is not an automorphism.*

Let P be the set of epimorphisms and not automorphisms of G of type $\theta^* : G \longrightarrow G, t \mapsto t, k \mapsto \theta^{\lambda}(k), k \in K, \lambda = 1$ *or* $\lambda = 2$ with θ an epimorphism and not an automorphism . In order to study P, it is important to study monomorphisms $K \longrightarrow K$ with image a maximal subgroup, see [4].

References

1. D.J. Collins and F. Levin, Automorphisms and Hopficity of certain Baumslag- Solitar groups *Arch. Math. (Basel)* **40** *(1983), no. 5, 385–400*
2. H. Hopf, Beiträge zur Klassifizierung der Flächenabbildungen. *J. Reine Angew. Math.* **165** (1931), 225–236.
3. G.Baumslag and D.Solitar, Some two-generator one relator non-Hopfian groups.*Bull. Amer.Math. Soc.* **68***, 199-201 (1962)*
4. S. Andreadakis, E. Raptis and D. Varsos, Hopficity of certain HNN-extensions. *Comm. Algebra* **20** (1992), no. 5, 1511–1533.
5. V.Metaftsis, E.Raptis, D.Varsos, On the hopficity of HNN-extensions of polycyclic groups , to be appeared in *Communications in Algebra*
6. D.Segal, Polycyclic groups. *Cambridge Univ. Press 1983*

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