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Topologically Q-algebras

Abdelhak Najmi

Abstract

It is shown that topologically Q-algebras inherit most properties of Q-algebras. In particular, we give characterizations of topologically Q-algebras in terms of equicontinuity of the set of nonzero continuous multiplicative linear functionals of the algebra and / or quasi-inverse closedness of the algebra (Theorem 2.29 and Corollary 2.33). Furthermore, we give a characterization of almost commutative topological algebras (Proposition 2.35, Corollary 2.36).

1. Introduction and preliminaries

In all what follows by a locally convex algebra, *l.c.a.* in brief, we mean a complex algebra with a *locally convex topology* for which the product is separately continuous. A *l.c.a.* is said to be *m-convex*, *l.m.c.a.* in brief , if the origin 0 admits a fundamental system of idempotent neighborhoods see [7], [8]. By a *topological algebra A* we mean a topological $\mathbb{C}$-vector space being also an algebra, with separately continuous ring multiplication, having non-empty *spectrum* $\mathcal{X}(A)$ (: set of nonzero continuous multiplicative linear functionals of A) endowed with the *Gel'fand topology.* The respective *Gel'fand map* of A is given by

$$\begin{aligned} g \quad &: \quad A \longrightarrow C(\mathcal{X}(A)) : x \longmapsto g(x) := \widehat{x} : \mathcal{X}(A) \longrightarrow \mathbb{C} \\ &: \quad f \longmapsto \widehat{x}(f) := f(x) \end{aligned}$$

The range of g, denoted $\hat{A}$, is called the *Gel'fand transform algebra* of A, topologized as a *locally m-convex algebra* by the inclusion

$$\hat{A} \subset C_c(\mathcal{X}(A))$$

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where the algebra $C(\mathcal{X}(A))$ (: of continuous functions on $\mathcal{X}(A)$) carries the topology " c " of compact convergence in $\mathcal{X}(A)$. A topological algebra A is called *semi-simple* if the respective Gel'fand map g is one-to-one, while A is called *bounded*, if the Gel'fand transform algebra $\hat{A}$ consists of complex-valued continuous bounded functions on $\mathcal{X}(A)$, equivalently, if the spectrum $\mathcal{X}(A)$ is a bounded subset of A'_s (: weak dual of A). Whenever every bounded subset of $\mathcal{X}(A)$ is *equicontinuous* then, A is said to be *spectrally barrelled* [7]. In such an algebra *equicontinuous*, *weakly relatively compact* and *weakly bounded subsets* of $\mathcal{X}(A)$ *are identical.* A topological algebra is said to be *simplicial* [2] *or normal* [8] , if any proper closed right or left and regular ideal is contained in a *maximal closed* ideal of the same type. A topological algebra is said to be *topologically simplicial* , if any proper closed right or left and regular ideal is contained in a *closed maximal* ideal of the same type. We quote here that *every simplicial algebra is also a topologically simplicial one* and the *converse is not in general true.* A topological algebra A is said to be *t-acceptable* if any regular maximal and closed unilateral ideal of A is two-sided. Like it is done by Y. Tsertos [10], let W_0 be the set of neighborhoods of 0 in A. We will note by g_V the function from A into $\mathbb{R}^*_+$, set of positive nonzero real numbers, such that for every $x \in A : g_V(x) = \inf A_x(V)$, where $A_x(V) = \{r > 0 : x \in rV\}$. Let $i(A)$ be the set of all one-sided regular and closed ideals of A. Let B be a subset of A, then $\overline{B}^A$ will denote the closure of B in A, yet, simply $\overline{B}$. An element $x \in A$, is *right (resp. left) quasi-invertible* if there exists an $y \in A$ such that $x \circ y := xy - x - y = 0$ (resp. $y \circ x = 0$); it is *quasi-invertible* if it is both right and left quasi-invertible. The set of all quasi-invertible (resp. left quasi-invertible, right quasi-invertible) elements of A will be denoted by $QinvA$ (resp. $LqinvA$, $RqinvA$). An algebra A is said to be a *Q-algebra* if $QinvA$ is open. An element $x \in A$, is *topologically right (resp. left) quasi-invertible* if there exists a net $(y_\alpha)_\alpha \subset A$ such that $\lim_\alpha (x \circ y_\alpha) = 0$ (resp. $\lim_\alpha (y_\alpha \circ x = 0)$; it is *topologically quasi-invertible* if it is both topologically right and left quasi-invertible. The set of all topologically quasi-invertible (resp. topologically left quasi-invertible, topologically right quasi-invertible) elements of A will be denoted by $TqinvA$ (resp. $TlqinvA$, $TrqinvA$). In particular, when A has a unit element e then $x \in A$ is *topologically right* (resp. *left*) *invertible* if there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ such that $(x_\lambda x)_{\lambda \in \Lambda}$ (resp. $(x x_\lambda)_{\lambda \in \Lambda}$) converges to e. It is *topologically invertible* if it is both topologically right and left invertible. In this context the terminology "*advertibly null net*" (A. Mallios), appropriately specialized, each time, is also of use. So, given a topological algebra A, a net $(x_\delta)_\delta$ in A is called *advertibly null* (resp. *invertibly unital* when A is unital), if there exists an element $x \in A$ such that both of the nets $(x \circ x_\delta)_\delta$ and $(x_\delta \circ x)_\delta$ are null (resp. $(x x_\delta)_\delta$ and $(x_\delta x)_\delta$ are unital) i.e., they converge to the zero (resp. unit) element of A. Now, a topological algebra A is said to be *advertibly complete* (resp. *invertibly complete* when A is unital), if every Cauchy net in A, which is also advertibly null (resp. invertibly unital), converges. When A is unital we note the set of all invertible (resp. left invertible, right invertible, topologically invertible, topologically left invertible, topologically right invertible) elements of A by $InvA$ (resp. $LinvA$, $RinvA$, $TinvA$, $TlinvA$, $TrinvA$). We will say that a closed two-sided (resp. left-sided, right-sided) ideal I is *topologically quasi-regular* (resp. *topologically left quasi-regular, topologically right quasi-regular*) if $I \subset TqinvA$ (resp. $I \subset TlqinvA$, $I \subset TrqinvA$). Similarly when A is unital will say that a closed two-sided (resp. left-sided, right-sided) ideal I is *topologically regular* (resp. *topologically left regular, topologically right regular*) if $I \subset TinvA$ (resp. $I \subset TlinvA, I \subset TrinvA$). Given an algebra A, we mean by its unitization the algebra $A_1 := A \oplus \mathbb{C}e$, obtained by adjunction of a unit element e to A. The element e which appears in the next formulas (1.2) and (1.4) is that of A_1. We can remark that for every $x \in A$ we have: $x \in QinvA$ (resp. $LqinvA$, $RqinvA$, $TlqinvA$, $TrqinvA$, $TqinvA$) if, and only if

$e - x \in InvA_1$ (resp. $LinvA_1$, $RinvA_1$, $TlinvA_1$, $TrinvA_1$, $TinvA_1$); so we have the next equalities:

$$\left.\begin{array}{r} x \in LinvA \Leftrightarrow Ax = A \\ x \in RinvA \Leftrightarrow xA = A \\ x \in InvA \Leftrightarrow Ax = xA = A \end{array}\right\} \text{ if } A \text{ has a unit.} \tag{1}$$

$$\left.\begin{array}{r} x \in LqinvA \Leftrightarrow A(e-x) = A \\ x \in RqinvA \Leftrightarrow (e-x)A = A \\ x \in QinvA \Leftrightarrow A(e-x) = (e-x)A = A \end{array}\right\} \text{ if } A \text{ hasn't a unit.} \tag{2}$$

$$\left.\begin{array}{r} x \in TlinvA \Leftrightarrow \overline{Ax} = A \\ x \in TrinvA \Leftrightarrow \overline{xA} = A \\ x \in TinvA \Leftrightarrow \overline{Ax} = \overline{xA} = A \end{array}\right\} \text{ if } A \text{ has a unit.} \tag{3}$$

$$\left.\begin{array}{r} x \in TlqinvA \Leftrightarrow \overline{A\,(e-x)} = A \\ x \in TrqinvA \Leftrightarrow \overline{(e-x)\,A} = A \\ x \in TqinvA \Leftrightarrow \overline{A\,(e-x)} = \overline{(e-x)\,A} = A \end{array}\right\} \text{ if } A \text{ hasn't a unit.} \tag{4}$$

We will say that A is an *advertive algebra* [2](the notion is essentially due to A. Beddaa ([5, Definition 1.2, p. 57]) if, $TqinvA = QinvA$ and more precisely if A is unital: A is an *invertive algebra* if $TinvA = InvA$. A topological algebra A is said to be *quasi-inverse* (resp. *topologically quasi-inverse*) *closed* [6], if for every $x \in A$: $1 \notin \widehat{x}(\mathcal{X}(A))$ implies that $x \in QinvA$ (resp. $x \in TqinvA$). If A is unital, we have the following precision: A is said to be *inverse* (resp. *topologically inverse*) *closed* if for every $x \in A$: $0 \notin \widehat{x}(\mathcal{X}(A))$ implies that $x \in InvA$ (resp. $x \in TinvA$). We note that *a quasi-inverse (resp. inverse) closed algebra is necessarily topologically quasi-inverse (resp. inverse) closed.* We can remark here that *the converse is not true* (cf. Example 2.32). The *spectrum* (resp. *topological spectrum*) of an element $x \in A$ will be denoted by $Sp_A(x)$ (resp. $Sp^t_A(x)$). If we put

$$\Phi_x = \begin{cases} \emptyset & \text{if } A \text{ is unital and } x \in InvA \\ \{0\} & \text{in the other cases} \end{cases}$$

$$\Phi^t_x = \begin{cases} \emptyset & \text{if } A \text{ is unital and } x \in TinvA \\ \{0\} & \text{in the other cases} \end{cases}$$

then we have

$$Sp_A(x) = \left\{\lambda \in \mathbb{C} \setminus \{0\} : \frac{x}{\lambda} \notin QinvA\right\} \cup \Phi_x \tag{5}$$

The *spectral radius* is defined by

$$\rho_A(x) = \sup\{|\lambda| : \lambda \in Sp_A(x)\} \tag{6}$$

Similarly we define

$$Sp^t_A(x) = \left\{\lambda \in \mathbb{C} \setminus \{0\} : \frac{x}{\lambda} \notin TqinvA\right\} \cup \Phi^t_x \tag{7}$$

The *topological spectral radius* is defined by

$$\rho^t_A(x) = \sup\left\{|\lambda| : \lambda \in Sp^t_A(x)\right\} \tag{8}$$

The *topological radical* $radA$ of A is by definition the intersection of the kernels of all continuous irreducible representations of A on separated vector space [1]. It is a closed subset of A. If A has no continuous irreducible representations at all then, we say that A *is topologically radical* (: $A = radA$). We note by $RadA$ the *Jacobson's radical* of A. Recall that if A is a topological algebra and $A^{'}$ its topological dual, *the polar* B° of a subset B of A is, by definition,

$$B^\circ = \{x' \in A' : \ sup\,\{|< x, x' >| : x \in B\} \leq 1\} \tag{9}$$

Several authors, among others, R. Hadjigeorgiou [6], A. Mallios [7], E. A. Michael [8], Tsertos, Y. [10], S. Warner [11] and W. Zelazko [12], worked on Q-algebras. They considered characterizations of this sort of algebras, in certain particular classes of topological algebras. One follows here a similar advance for studying *tQ-algebras* (cf. Definition 2.1). Here it is shown that *the class of tQ-algebras contains strictly that of Q-algebras* (cf. Example 2.3). For *advertive* algebras *these two classes coincide*. So, for advertive algebras, one thus deduces the most part of the results found by the aforementioned authors. Besides, we have $QinvA \subset TqinvA$, if, and only if, for every $x \in A$, $Sp^t_A(x) \subset Sp_A(x)$. We can remark that a normed algebra is Q if, and only if, for every $x \in A$ we have $Sp^t_A(x) = Sp_A(x)$. The main results on *tQ-algebra*s are summarized in Lemmas 2.8, 2.9, 2.10, 2.39; Corollaries 2.11, 2.15, Proposition 2.16 and Proposition 2.26. Furthermore, we give a characterization of (topologically) almost commutative algebras (cf. Proposition 2.35 and Corollary 2.36).

2. Topologically Q-algebras

Definition 2.1 Let A be a topological algebra. The algebra A is said to be a *topologically Q-algebra*, *tQ-algebra*, in brief, if $TqinvA$ is open.

Firstly we give some examples of topologically Q-algebras.

Example 2.1 Every Q-algebra is a *tQ-algebra.*

Below, we give an example of a *tQ-algebra*, which is not a Q-algebra.

Example 2.2 Let $A = \mathbb{C}[t]$ be the unital algebra of polynomial functions of one indeterminate definite on $[0, 1]$ with complex coefficients and equipped with the following algebra norm: $P \longrightarrow \|P\| = \sum_{i=0}^n |a_i|$; with $P(t) = \sum_{i=0}^n a_i t^i$ for every $t \in [0, 1]$. All non-zero continuous multiplicative linear functionals of A are of the form δ_z with $|z| \leq 1$ and $\delta_z(P) = P(z)$. Then we have $\mathcal{X}(A) = \{\delta_z : |z| \leq 1\}$. The set of all invertible elements of A is not other than $InvA = \mathbb{C}\backslash\{0\}$. Hence $InvA$ is not open , of course, in the normed topology of A, as before. Let P be a polynomial of degree ≥ 1 and not having any root in $[0, 1]$. By Stone-Weierstrass theorem, there is a sequence $(P_n)_n$ in A which converges towards the inverse of P in the algebra of continuous functions on $[0, 1]$. The sequence $(PP_n)_n$ converges towards 1 in A. So $TinvA \supset InvA$ and $TinvA \neq InvA$. As we have $TinvA = Inv(\widetilde{A}) \cap A$, then $TinvA$ is open. So A is a topologically Q-algebra.

Example 2.3 Let A be a *tQ-algebra* which is not a Q-algebra and B any Q-algebra. Then $A \times B$ is a *tQ-algebra* which is not a Q-algebra.

Proposition 2.1 *Every normed algebra A is a tQ-algebra.*

Proof. Since in a unital normed algebra A, with $\widetilde{A}$ its completion, we have $TinvA \supseteq Inv\left(\widetilde{A}\right) \cap A$, then $TinvA$ is always open. Consequently in a non unital normed algebra we have also $TqinvA \supseteq Qinv\left(\widetilde{A}\right) \cap A$, so $TqinvA$ is also always open. The equality holds true in the case we deal with Cauchy nets (cf. [6, p. 52, Lemma 2.4]). □

Because $Qinv\left(\widetilde{A}\right) \cap A \subset TqinvA$, the preceding result can be generalized as follows.

Remark 2.1 (A. Mallios) *A topological algebra, with completion (whenever exists) a Q-algebra, is a tQ-algebra.*

Lemma 2.1 *If A is a complex algebra and P a polynomial, then $Sp_A^t\left(P\left(x\right)\right) = P\left(Sp_A^t\left(x\right)\right).$*

Proof.

1 *Unital case.* Let e be the unit of A and $\lambda \in Sp_A^t\left(x\right)$. Then $P\left(x\right) - P\left(\lambda\right)e = \left(x - \lambda e\right)Q\left(x\right)$ with $Q\left(x\right)$ a polynomial in x. As $x - \lambda e \notin TinvA$, it is also so for $P\left(x\right) - P\left(\lambda\right)e = \left(x - \lambda e\right)Q\left(x\right).$ Hence $P\left(Sp_A^t\left(x\right)\right) \subset Sp_A^t\left(P\left(x\right)\right).$ Conversely, let $\mu \in Sp_A^t\left(P\left(x\right)\right).$ If P is a constant, the result is obvious. Suppose that P is not constant and let n be the degree of P. Since the field of complex numbers is algebraically closed, they exist $\alpha_0, \left(\lambda_i\right)_{i=1,\ldots,n} \in \mathbb{C}$ such that $P\left(x\right) - \mu e = \alpha_0 \prod_{i=1}^{n}\left(x - \lambda_i e\right)$. As $P\left(x\right) - \mu e \notin TinvA$, there exists $i_0 \in \{1, ...n\}$ such that $\left(x - \lambda_{i_0} e\right) \notin TinvA$. Hence $\lambda_{i_0} \in Sp_A^t\left(x\right)$ and we have $P\left(\lambda_{i_0}\right) = \mu$. So $\mu \in P\left(Sp_A^t\left(x\right)\right).$

2 *Non unital case.* In this case, the result is still true for P without constant term. The proof remains the same one.

□

The following result has an intersection with a recent result of H. Arizmendi et al. [4].

Lemma 2.2 *Let A be a topological algebra and $TqinvA$ the set of its topologically quasi-invertible elements. Then the following assertions are equivalent:*

1 *$TqinvA$ is an open set of A; i.e. A is tQ-algebra.*

2 *$TqinvA$ has a non-empty interior.*

3 *$TqinvA$ is a neighborhood of zero in A.*

4 *There exists $V \in W_0$ such that $\rho_A^t \leq g_V$ (Y. Tsertos).*
In particular,

5 *Let $(A, (p)_{p\in\Gamma})$ be a locally convex algebra. Then A is a tQ-algebra if, and only if, there exist $p \in \Gamma$ and $M > 0$ such that $\rho_A^t \leq M.p$*

6 *Let $(A, (p)_{p\in\Gamma})$ be a locally m-convex algebra. Then A is a tQ-algebra if, and only if, there exists $p \in \Gamma$ such that $\rho_A^t \leq p$.*

Proof. (1) $\Longrightarrow$ (2) It is obvious.

(1) $\Longrightarrow$ (3). Since $0 \in A$ is a (topologically) quasi-invertible element, we have the implication as well.

(3) $\Longrightarrow$ (2). It is still obvious.

(2) $\Longrightarrow$ (1). Thus, for every $x \in A$, consider the continuous functions $\phi_x : A \longrightarrow A$ and $\psi_y : A \longrightarrow A$ such that $\phi_x(y) := x \circ y$ and $\psi_x(y) := y \circ x$. If $x \in A$ then, $\phi_x^{-1}(TqinvA) \subset TrqinvA$ (resp. $\psi_x^{-1}(TqinvA) \subset TlqinvA$); hence, for any elements x and y in A, we have $\phi_x^{-1}(TqinvA) \cap \psi_y^{-1}(TqinvA) \subset TqinvA$. Suppose now that x is an interior point of $TqinvA$ and let z be any point of $TqinvA$ thereby, there exist two nets $(t_\lambda)_\lambda$ and $(s_\lambda)_\lambda$ such that $\lim_\lambda \phi_z(t_\lambda) = \lim_\lambda \psi_z(s_\lambda) = 0$. Then, $\lim_\lambda \phi_{x\circ t_\lambda}(z) = \lim_\lambda (x \circ t_\lambda) \circ z = \lim_\lambda x \circ (t_\lambda \circ z) = x \circ 0 = x$ and $\lim_\lambda \psi_{s_\lambda \circ x}(z) = \lim_\lambda z \circ (s_\lambda \circ x) = \lim_\lambda (z \circ s_\lambda) \circ x = x$. So, for every $V \in O(x)$ (: set of open neighborhoods of x), there exists λ such that for every $\beta \geq \lambda$, $\phi_{x\circ t_\beta}(z) \in V$ and $\psi_{s_\beta\circ x}(z) \in V$. Hence z is an interior point of $\phi_{x\circ t_\beta}^{-1}(TqinvA) \cap \psi_{s_\beta \circ x}^{-1}(TqinvA) \subset TqinvA$. Whence, the conclusion.

By the preceding lemma, in the case of a polynomial, the "Spectral mapping theorem" remains true for topologically quasi-invertible elements. Hence for any $x \in A$ if we replace $Sp_A(x), \rho_A(x)$ and $Qinv(A)$ respectively by $Sp_A^t(x), \rho_A^t(x)$ and $Tqinv(A)$, then the proofs of (1) $\Longleftrightarrow$ (4), (5) and (6) are practically and respectively the same ones as those ((I), (II) and (III)) given by Tsertos ([10]). □

Lemma 2.3 *Let A be a topological algebra and consider the following subset of A:*

$$T(A) = \left\{x \in A : \rho_A^t(x) \leq 1\right\} \tag{10}$$

where $\rho_A^t(x)$ is given by (1.8). Then, the following two assertions are equivalent:

1 *A is a tQ-algebra*

2 *$T(A)$ is a neighborhood of zero in A.*

Proof. (1) $\Longrightarrow$ (2). If A is a *tQ-algebra*, by the preceding lemma, $TqinvA$ is a neighborhood of zero in A, so that there exists a balanced absorbing neighborhood U of $0 \in A$ contained in $TqinvA$. We shall show that $U \subset T(A)$: Indeed, if $x \in U$ and $\rho_A^t(x) > 1$, then by equation (1.8) there would exist $\lambda \in Sp_A^t(x)$ with $|\lambda| > 1$. But, $\frac{x}{\lambda} \notin TqinvA$; that is a contradiction, since $\frac{x}{\lambda} \in U \subset TqinvA$, because $\left|\frac{1}{\lambda}\right| < 1$ and U is balanced. Thus, $U \subset T(A)$. Hence, the implication.

(2) $\Longrightarrow$ (1). On the other hand, if (2) is valid, then $0 \in A$ is an interior point of $T(A)$; hence, the set $\frac{1}{2}T(A)$ has a non-empty interior too; and however is contained in $TqinvA$. Since otherwise, if an element $\frac{x}{2} \in \frac{1}{2}T(A) \cap (A \setminus TqinvA)$ then, $2 \in Sp^t_A(x)$; so that $\rho^t_A(x) \geq 2$, that is, a contradiction to equation (2.1). Hence, the conclusion. □

Lemma 2.4 *Every element x of a tQ-algebra has a compact topological spectrum $Sp^t_A(x)$.*

Proof. By Lemma 2.8, the set $TqinvA$ is a neighborhood of $0 \in A$. So, there exists a balanced, absorbing neighborhood U of $0 \in A$ with $U \subset TqinvA$. Moreover, if $x \in A$, there exists an $\alpha > 0$ with $\alpha x \in U$. Thus, we show that

$$\rho^t_A(x) \leq \frac{1}{\alpha} \tag{11}$$

Indeed, if $\rho^t_A(x) > \frac{1}{\alpha}$, there would exist $\lambda \in Sp^t_A(x)$ with $|\lambda| > \frac{1}{\alpha}$. So that $\left|\frac{1}{\lambda\alpha}\right| < 1$ and U is balanced. One concludes that $\frac{1}{\lambda\alpha}.\alpha x = \frac{x}{\lambda} \in U \subset TqinvA$, that is , a contradiction to $\lambda \in Sp^t_A(x)$; and this proves (2.2). □

Corollary 2.1 *Let A be a tQ-algebra. Moreover consider the following subset of A*

$$TB(A) = \{x \in A : \rho^t_A(x) < +\infty\} \tag{12}$$

Then, $A = TB(A)$. In other words, a tQ-algebra is topologically spectrally bounded.

Proposition 2.2 *Let A be a topological algebra and $x \in A$. Moreover, consider the following assertions:*

1 $x \in TqinvA$.

2 x is not a unit element of A modulo any regular proper and closed one-sided ideal of A.

3 x is not a unit element of A modulo any regular maximal and closed one-sided ideal of A.

Then, (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). *Furthermore, if A is a simplicial algebra, then all the above assertions are equivalent.*

Proof. (1) $\Rightarrow$ (2). If x is a unit element of A modulo, for example, a closed proper right ideal I of A, one has $\{y - xy : y \in A\} := (e - x)A \subset I$. Since x is, for example, right topologically quasi-invertible, then by equations (1.4) we have $\overline{(e - x)A} = A$. Hence $I = A$; contradiction.

(2) $\Rightarrow$ (1). If $x \notin TqinvA$, suppose, for example, that x is not left topologically quasi-invertible. Then $\overline{A(e - x)} \neq A$. Hence $\overline{A(e - x)}$ is a closed regular left ideal of A with right unit element x of A modulo $\overline{A(e - x)}$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2). Suppose that x is a unit element of A modulo a regular proper closed one-sided ideal of A. Suppose that the algebra A is simplicial, then, the element x is a unit element of A modulo a regular maximal and closed one-sided ideal of A. □

If A is unital, the last proposition gives rise to the following corollary.

Corollary 2.2 *Let A be a unital topological algebra and $x \in A$. Moreover, consider the following assertions:*

1 $x \in TinvA$.

2 x does not belong to any proper and closed one-sided ideal of A.

3 x does not belong to any maximal and closed one-sided ideal of A.

Then, (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). *Furthermore, if A is a simplicial algebra, then all the above assertions are equivalent.*

Remark 2.2 Contrary to a topological Q-algebra where every regular maximal ideal is closed, in a *tQ-algebra*, we may have regular maximal ideals which are not closed. Indeed, let A be the algebra of Example 2.3. All nonzero multiplicative linear functionals of A are of the form δ_z, $z \in \mathbb{C}$ with $\delta_z(P) = P(z)$. All nonzero continuous multiplicative linear functionals of A are of the form δ_z with $|z| \leq 1$. Finally all regular maximal ideals of A are of the form $Ker\delta_z$ with $z \in \mathbb{C}$. But only the sets $Ker\delta_z$ with $|z| \leq 1$, are closed.

Corollary 2.3 *Let A be a topological algebra. Then, A is a tQ-algebra if, and only if, the set $\bigcup_{I \in i(A)} I$ is closed.*

Proof. The equivalence (1) $\Longleftrightarrow$ (2) of Proposition 2.12 means that $TqinvA = A \backslash \bigcup_{I \in i(A)} I$, hence, the conclusion. □

We know that any topological Q-algebra is simplicial. We have a parallel result for *tQ-algebras*.

Proposition 2.3 *Every topological tQ-algebra A is topologically simplicial.*

Proof. If A is topologically radical, then $TqinvA = radA$. So A is trivially topologically simplicial. Now if A is not topologically radical, it admits proper regular closed one sided ideals. By definition, put $i(A) = i_l(A) \cup i_r(A)$ where $i_l(A)$ (resp. $i_r(A)$) is the set of all regular proper and closed left (resp. right) ideal of A. Let, for example, $J \in i_l(A)$ and u a right unit element of A modulo J. Let $\mathcal{F} = \{I \in i_l(A) : J \subset I\}$. So $\mathcal{F} \neq \emptyset$. Let $(I_k)_{k \in K}$ be a totally ordered subfamily of $\mathcal{F}$. Consider $I_0 = \bigcup_{k \in K} I_k$. If $\overline{I_0}^A = A$, then $\overline{\bigcup_{I \in i(A)} I} = A$; which contradicts the fact that $\bigcup_{I \in i(A)} I$ is closed in A. So $\overline{I_0}^A$ is a regular closed, and maximal left ideal of A which contains J. Consequently, A is topologically simplicial. □

Using Proposition 2.5 one has the following corollary.

Corollary 2.4 *Every normed algebra is topologically simplicial.*

Remark 2.3 Every topological algebra whose completion is a Q-algebra is topologically simplicial (Proposition 2.16 and Remark 2.6). It also has an equicontinuous spectrum (Proposition 2.26) and a continuous Gel'fand map.

The following statement is fulfilled by a *tQ-algebra* (cf. Proposition 2.16).

Corollary 2.5 *Let A be a topologically simplicial algebra and $x \in A$. Then the following assertions are equivalent:*

1 $x \in TqinvA$.

2 x is not a unit element of A modulo any regular proper and closed one-sided ideal of A.

3 x is not a unit element of A modulo any regular closed and maximal one-sided ideal of A.

Lemma 2.5 *Let A be a topological algebra and $x \in A$. Then, $x \in TqinvA$ (resp. $x \in TinvA$ if A is unital) implies that $1 \notin \widehat{x}(\mathcal{X}(A))$ (resp. $0 \notin \widehat{x}(\mathcal{X}(A))$).*

Proof. If x is topologically quasi-invertible, then there exists a net $(x_\lambda)_\lambda$ such that $x_\lambda \circ x \longrightarrow 0 \longleftarrow x \circ x_\lambda$. Let $f \in \mathcal{X}(A)$, then for example, $f(x \circ x_\lambda) = f(x)f(x_\lambda) - f(x) - f(x_\lambda) \longrightarrow 0$. So, obviously $1 \notin \widehat{x}(\mathcal{X}(A))$. The other case is treated in the same way. □

Proposition 2.4 *Let A be a topological algebra. Then, for every $x \in A$ one has $\widehat{x}(\mathcal{X}(A) \cup \Phi_x^t) \subset Sp_A^t(x)$. Furthermore, if A is a Gel'fand-Mazur simplicial t-acceptable algebra, then for every $x \in A$ one has the relation*

$$\widehat{x}\left(\mathcal{X}(A) \cup \Phi_x^t\right) = Sp_A^t(x), \tag{13}$$

Hence, for every $x \in A$ one has the relation

$$\rho_A^t(x) = \sup_{f \in \mathcal{X}(A)} |f(x)|\,. \tag{14}$$

Proof. If, for some $f \in \mathcal{X}(A)$, $f(x) = 0$, then x can not be topologically invertible. So that $0 \in Sp_A^t(x)$. On the other hand, if $\widehat{x}(f) = f(x) = \lambda \neq 0$ in $\mathbb{C}$, where $f \in \mathcal{X}(A)$, then $f(\frac{x}{\lambda}) = 1$. Hence, by the preceding lemma, $\frac{x}{\lambda} \notin TqinvA$; so, $\lambda \in Sp_A^t(x)$. Furthermore, if $\lambda \in Sp_A^t(x)$ with $\lambda \neq 0$, then $\frac{x}{\lambda} \notin TqinvA$. So, by Proposition 2.12 and the fact that A is a Gel'fand-Mazur t-acceptable algebra, there exists an $f \in \mathcal{X}(A)$ such that $f(\frac{x}{\lambda}) = 1$. Hence, $\lambda \in \widehat{x}(\mathcal{X}(A))$. Consequently, we have the relation (2.4). Now, (2.5) is straightforward from (2.4). □

Remark 2.4 By equation (2.5), a simplicial Gel'fand-Mazur t-acceptable topological algebra is bounded if, and only if, $A = TB(A)$ (cf. Corollary 2.11). Furthermore: *Any topological algebra satisfying (2.5) is bounded if, and only if, it is topologically spectrally bounded.*

Corollary 2.6 *Let A be a topological Gel'fand-Mazur simplicial t-acceptable topological algebra with continuous Gel'fand map (hence also if it is, respectively, Frechet, barrelled, m-barrelled, spectrally barrelled). Then, the following assertions are equivalent:*

1 A is a tQ-algebra

2 $A = TB(A)$

Proof. By Corollary 2.11, it is enough to prove that(2) $\Longrightarrow$ (1). Indeed, by the preceding proposition, for every $x \in A$ we have $\rho_A^t(x) = \sup\{|\widehat{x}(f)| : f \in \mathcal{X}(A)\}$. Since $A = TB(A)$, ρ_A^t is a semi-norm on A which is continuous at 0, due to the continuity of the Gel'fand map. Hence, $T(A)$ is already a neighborhood of 0. □

Proposition 2.5 *In a topological algebra A the two following assertions are equivalent:*

1 A is topologically quasi-inverse closed.

2 For every $x \in A$, *we have* $\widehat{x}\,(\mathcal{X}(A) \cup \Phi_x^t) = Sp_A^t(x)$.

Proof. By Proposition 2.21, for every $x \in A$ we have $\widehat{x}\,(\mathcal{X}(A) \cup \Phi_x^t) \subset Sp_A^t(x)$. Suppose that the assertion (1) is satisfied. If $\lambda \in Sp_A^t(x)$ with $\lambda \neq 0$, then $\frac{x}{\lambda} \notin TqinvA$. By hypothesis, $1 \in \widehat{\frac{x}{\lambda}}\,(\mathcal{X}(A))$. Hence, there is an $f \in \mathcal{X}(A)$ such that $f(x) = \lambda$. So, $\lambda \in \widehat{x}\,(\mathcal{X}(A) \cup \Phi_x^t)$. Conversely, if $1 \notin \widehat{x}\,(\mathcal{X}(A))$, then $1 \notin Sp_A^t(x)$. Hence, $x \in TqinvA$. So, A is topologically quasi-inverse closed. □

Corollary 2.7 *In a topological algebra A the following assertions are equivalent:*

1 A is quasi-inverse closed.

2 For every $x \in A$, *we have* $\widehat{x}\,(\mathcal{X}(A) \cup \Phi_x) = Sp_A(x)$.

Proposition 2.6 *Let A be a topological tQ-algebra. Then* $\mathcal{X}(A)$ *is an equicontinuous subset of* A' *and hence, relatively compact in* A'_s *(: the weak topological dual of A).*

Proof. If U is balanced neighborhood of $0 \in A$ consisting of topologically quasi-invertible elements of A, then one has the relation

$$U \subset (\mathcal{X}(A))^\circ = \{x \in A : \sup\{|f(x)| : f \in \mathcal{X}(A)\} \leq 1\} \tag{15}$$

where $(\mathcal{X}(A))^\circ$ denotes the polar set of $\mathcal{X}(A)\,(\subset A')$ in A. Indeed, if $x \in U$ and $|f(x)| > 1$ for some $f \in \mathcal{X}(A)$, then $f(x) := \lambda \neq 0$ and $\left|\frac{1}{\lambda}\right| < 1$. Thus, $\frac{x}{\lambda} \in U$ and $f(\frac{x}{\lambda}) = 1$, a contradiction since $\frac{x}{\lambda} \in TqinvA$ (Lemma 2.20). Therefore, one has $|f(x)| \leq 1$ for every $f \in \mathcal{X}(A)$ and $x \in U$, which proves (2.6). Hence, $(\mathcal{X}(A))^\circ$ is a neighborhood of zero in A. Therefore, $\mathcal{X}(A) \subset (\mathcal{X}(A))^{\circ\circ}$ is an equicontinuous subset of A'. □

Proposition 2.7 *Let A be a simplicial t-acceptable Gel'fand-Mazur topological algebra. Then, the following assertions are equivalent:*

1 $x \in TqinvA$.

2 $1 \notin \widehat{x}\,(\mathcal{X}(A))$.

Proof. By Proposition 2.12 A is not topologically radical. If $A = TqinvA$, then A must be an ideal of topologically quasi-invertible elements. So, A must be topologically radical; which is not the case. Then, there exists $x \in A$ such that $x \notin TqinvA$. As A is simplicial, by Proposition 2.12, for example, x is a right unit element of A modulo a maximal and closed ideal M. But A is a Gel'fand-Mazur t-acceptable topological algebra, then M give rise to a nonzero continuous character of A. Whence, $\mathcal{X}(A)$ is not empty. So, the assertion (3) of Proposition 2.12 is equivalent to $1 \notin \widehat{x}\,(\mathcal{X}(A))$. Indeed, if $x \in TqinvA$, then there exists a net $(y_\alpha)_\alpha \subset A$ such that $\lim_\alpha (xy_\alpha - x - y_\alpha) = 0$. If $1 \in \widehat{x}\,(\mathcal{X}(A))$, by applying an $f \in \mathcal{X}(A)$ to the preceding limit, one leads to a contradiction. Conversely, if $x \notin TqinvA$, by Proposition 2.12, there exists a regular maximal and closed one sided J of A such that x is a unit of A modulo J. But, by assumption, J is two-sided. So, there exists a nonzero continuous character $f \in \mathcal{X}(A)$ such that $f(x) = 1$. Whence, $1 \in \widehat{x}\,(\mathcal{X}(A))$. □

Corollary 2.8 *Let A be a simplicial and unital t-acceptable Gel'fand-Mazur topological algebra. Then the following assertions are equivalent:*

1 $x \in TinvA$.

2 $0 \notin \widehat{x}\,(\mathcal{X}(A))$.

Theorem 2.1 *Let A be a topological algebra. Moreover, consider the following statements:*

1 $\mathcal{X}(A)$ is an equicontinuous subset of A' and A is topologically quasi-inverse closed.

2 A is a tQ-algebra.

Then $(1) \Longrightarrow (2)$. *Furthermore, if A is a simplicial Gel'fand-Mazur t-acceptable topological algebra, then* $(2) \Longrightarrow (1)$ *as well. So the two assertions are equivalent.*

Proof. $(1) \Longrightarrow (2)$. As $\mathcal{X}(A)$ is equicontinuous, then $U = \frac{1}{2}\,(\mathcal{X}(A))^\circ$ is by hypothesis a neighborhood of zero in A such that: if $x \in U$, then $|f(x)| \leq \frac{1}{2}$; that is $f(x) \neq 1$, for every $f \in \mathcal{X}(A)$. So, by the fact that A is topologically quasi-inverse closed, $x \in TqinvA$. Whence, by Lemma 2.8, A is a *tQ-algebra*.

$(2) \Longrightarrow (1)$. By Proposition 2.26, $\mathcal{X}(A)$ is equicontinuous, hence, by Propositions 2.21 and 2.24, A is topologically quasi-inverse closed. □

Example 2.4 Let $(A, \tau) = (H(\mathbb{C}), (p_k)_k)$, with $p_k(f) = \sup_{|z| \leq k} |f(z)|$, be the algebra of holomorphic functions on $\mathbb{C}$. Hence A is a *Frechet l.m.c.a.*. As A is complete, then it is invertibly complete. Besides, A is not a Q-algebra, because it contains elements with unbounded spectrum. Even if A is topologically inverse closed, which it can be deduced by Proposition 2.24, appropriately specialized to this case, and the fact that A is a commutative (hence, t-acceptable) and simplicial Gel'fand-Mazur algebra, A is not a *tQ-algebra*. Because $InvA = TinvA$. Consequently, $\mathcal{X}(A)$ is not equicontinuous.

Corollary 2.9 *One has the following results.*

1 *In every simplicial Gel'fand-Mazur t-acceptable topologically non radical algebra the set $\mathcal{X}(A)$ is not empty.*

2 *Every simplicial Gel'fand-Mazur t-acceptable topological algebra is topologically quasi-inverse closed.*

3 *Let A be a simplicial Gel'fand-Mazur t-acceptable tQ-algebra. Then A is Q-algebra if, and only if, for every $x \in A : Sp_A(x) = \widehat{x}\,(\mathcal{X}(A) \cup \Phi_x)$.*

Proof. The assertion (1) is deduced from Proposition 2.12. Now, the assertion (2) is deduced from (2) $\Longrightarrow$ (1) of Proposition 2.24 and Proposition 2.21. Now we prove the assertion (3). Necessary condition. If A is a Q-$algebra$, by Proposition 2 ([2]), one has $QinvA = TqinvA$ and hence for every $x \in A$, $Sp_A(x) = Sp^t_A(x)$. So, the assertion derives from Proposition 2.21, being a Q-algebra simplicial. Sufficient condition. Suppose that $Sp_A(x) = \widehat{x}(\mathcal{X}(A) \cup \Phi_x)$ for every $x \in A$. By Proposition 2.21, we have $Sp^t_A(x) = \widehat{x}(\mathcal{X}(A) \cup \{0\})$ for every $x \in A$. Consequently, $Sp^t_A(x) = Sp_A(x)$ for every $x \in A$. So A is Q-$algebra$. □

Example 2.5 Let A be the commutative algebra of Example 2.3. By Proposition 2.21, for every $P \in A$ we have $\widehat{P}\,(\mathcal{X}(A) \cup \Phi^t_P) = Sp^t_A(P)$. Hence, by an analogue of Proposition 2.24 specialized to unital case , it is topologically inverse closed. But A is not inverse closed. Indeed, let $P : t \longrightarrow t - 2$. Then, we have $\widehat{P}\,(\mathcal{X}(A)) = \mathcal{D}\left((-2, 0)\,, 1\right)$ (: closed disk of center $(-2, 0)$ and radius 1). Hence, $0 \notin \widehat{P}\,(\mathcal{X}(A))$ and $P \notin InvA = \mathbb{C}\backslash\{0\}$. We can remark here that $Sp^t_A(P) \subset Sp_A(P)$ and $Sp^t_A(P) \neq Sp_A(P)$.

Based on Theorem 2.29, the following corollary is obvious:

Corollary 2.10 *Let A be a Gel'fand-Mazur simplicial t-acceptable topological algebra. Then, the following assertions are equivalent:*

1 *A is a tQ-algebra.*

2 *$\mathcal{X}(A)$ is equicontinuous.*

Proposition 2.8 *If the completion $\widetilde{A}$ of a topological algebra A is t-acceptable and simplicial Gel'fand-Mazur topological algebra then,*

1 *A is simplicial.*

2 *If in addition $\mathcal{X}(A) = \mathcal{X}(\widetilde{A})$ within a continuous bijection and A is a tQ-algebra, then it is also so for $\widetilde{A}$.*

Proof. Let A be a topological algebra and suppose that $\widetilde{A}$ is t-acceptable simplicial Gel'fand-Mazur topological algebra.

1 For example, let I be a regular, proper and closed right ideal of A. Then $\overline{I}^{\widetilde{A}}$ is of the same type in $\widetilde{A}$. Otherwise, $I = \overline{I}^{A} := \overline{I}^{\widetilde{A}} \cap A = \widetilde{A} \cap A = A$. Which is absurd. Besides, since $\widetilde{A}$ is simplicial there exists a regular maximal right ideal J which is closed and such that it contains $\overline{I}^{\widetilde{A}}$. Now, since $\widetilde{A}$ is t-acceptable J is two-sided. By the fact that $\widetilde{A}$ is a Gel'fand-Mazur algebra, there exists a character g of $\widetilde{A}$ such that $J = \ker g$. Let f be the restriction of g to A. Then f is a character of A such that $\ker f$ contains I. Hence A is simplicial.

2 Assuming that A is a tQ-algebra, by Proposition 2.26, $\mathcal{X}(A)$ is equicontinuous. Hence, by hypothesis and A. Mallios [7, p. 146, Lemma 2.2], $\mathcal{X}(\widetilde{A})$ is equicontinuous too, providing that $\widetilde{A}$ is a tQ-algebra in view of Corollary 2.33.

3 Let $f \in \mathcal{X}(A)$ and $M = Ker(f)$. Then the closed ideal $\overline{M}^{\widetilde{A}}$ of $\widetilde{A}$ is maximal (as right and as left ideal), otherwise $M := \overline{M}^{\widetilde{A}} \cap A = A$; which is impossible. By assumption $\widetilde{A}$ is a Gel'fand-Mazur algebra. So there exists a continuous character $\widetilde{f}$of $\widetilde{A}$ such that $\overline{M}^{\widetilde{A}} = Ker(\widetilde{f})$. Conversely, every continuous character $\widetilde{f}$ of $\widetilde{A}$ define a continuous character f of A such that $f = \widetilde{f}_{/A}$. Now if in addition A is a *tQ-algebra*. To show that $\widetilde{A}$ is a *tQ-algebra*, it is enough, by the previous corollary, to show that the set of continuous characters of $\widetilde{A}$ is equicontinuous. But by assumption we have $\mathcal{X}(A) = \mathcal{X}(\widetilde{A})$ within a continuous bijection. So the equicontinuity of $\mathcal{X}(A)$ implies that one of $\mathcal{X}(\widetilde{A})$ (cf. [7, p. 146, Lemma 2.2]).

□

Now we turn our attention to advertibly complete Gel'fand-Mazur topological algebras which are closely related with (topologically) almost commutativity.

Proposition 2.9 *Let A be a Gel'fand Mazur topological algebra. Then, the following statements are equivalent:*

1 A is t-acceptable algebra.

2 A is topologically almost commutative (i.e. $B := A/radA$ is commutative).

Proof. By Corollary 1 of [1], the topological radical of A is closed. Hence B is a topological algebra. Suppose that A is t-acceptable algebra. Then, $radA = \bigcap_{f\in\mathcal{X}(A)} Ker(f)$,which is an improvement of a result of Abel [3, Theorem 5, p. 19] obtained in the *commutative* case. So one has $xy-yx \in radA$ for every $x, y \in A$, hence, the commutativity of B. Conversely, suppose that B is commutative. Hence B is t-acceptable algebra. Let M be a closed regular one sided ideal of A which is maximal as a unilateral ideal. Then, $\pi(M)$ is a regular and maximal ideal in B, where π is the canonical map from A onto B. Indeed, if there is an ideal M' of B such that $B \neq M' \supset \pi(M)$, then $\pi^{-1}\left(M'\right)$ is a maximal ideal of A such that $\pi^{-1}\left(M'\right) \supset \pi^{-1}\left(\pi(M)\right) \supset M$. By the fact that M is maximal, we have $\pi^{-1}\left(M^{'}\right)=\pi^{-1}\left(\pi(M)\right)=M$. Since $\pi(M)$ is two-sided, it is also so for M. One deduces that the algebra A is also t-acceptable algebra. □

Furthermore, in the above proposition, if A is advertive and simplicial (hence, it is advertibly complete), i.e. $QinvA = TqinvA$ (so we have $radA = RadA$ [9, Proposition 14, p. 324]), then one has the following corollary:

Corollary 2.11 *Let A be an advertive and simplicial Gel'fand Mazur topological algebra. Then, the following statements are equivalent:*

1 A is t-acceptable.

2 A is almost commutative.

Proposition 2.10 *Let A be a topological algebra. Moreover consider the following statements:*

1 $A/RadA$ is advertibly complete and $RadA$ is closed.

2 A is advertibly complete.

Then $(1)\Longrightarrow(2)$. *Furthermore, if A is simplicial and advertive algebra, then* $(2)\Longrightarrow(1)$ *as well. So, the two assertions are equivalent.*

Proof. $(1)\Longrightarrow(2)$. Let $(x_\lambda)\subset A$ be a Cauchy net and $x\in A$ such that $x\circ x_\lambda\longrightarrow 0$ and $x_\lambda\circ x\longrightarrow 0$. Then, $\widetilde{x}\circ\widetilde{x_\lambda}\longrightarrow\widetilde{0}$ and $\widetilde{x_\lambda}\circ\widetilde{x}\longrightarrow\widetilde{0}$. Furthermore, $(\widetilde{x_\lambda})$ is a Cauchy net. Hence, there exists $\widetilde{y}\in A/RadA$ such that $\widetilde{x_\lambda}\longrightarrow\widetilde{y}$. Then $x_\lambda\longrightarrow y$, that is A is advertibly complete with y the quasi-inverse of x.

$(2)\Longrightarrow(1)$. As A is advertive and simplicial, then $RadA = radA$ [9, Proposition 14 , p. 324]. Hence, $RadA$ is closed. So $A/RadA$ is a topological algebra. Let $\widetilde{x_\lambda}\subseteq A/RadA$ a Cauchy net with $\widetilde{x_\lambda}\circ\widetilde{x}\longrightarrow 0\longleftarrow\widetilde{x}\circ\widetilde{x_\lambda}$ for some $x\in A$. Then, $x_\lambda\circ x\longrightarrow 0\longleftarrow x\circ x_\lambda$, with (x_λ) a Cauchy net of the advertibly complete algebra A, hence $x_\lambda\longrightarrow y\in A$, such that $x\circ y=y\circ x=0$. Thus, $\widetilde{x_\lambda}\longrightarrow\widetilde{y}$ with $\widetilde{x}\circ\widetilde{y}=\widetilde{y}\circ\widetilde{x}=0$, which proves the assertion. □

Corollary 2.12 *Let A be an advertive topological algebra. Moreover consider the following statements:*

1 A is advertibly complete.

2 $A/RadA$ is advertibly complete.

Then, $(2)\Longrightarrow(1)$. *Furthermore, if A is simplicial, then* $(1)\Longrightarrow(2)$ *as well. So, the two assertions are equivalent.*

Lemma 2.6 *Let A be a non unital topological algebra and A_1 its unitization. Then A is tQ-algebra if, and only if, A_1 is tQ-algebra.*

Proof. $TinvA_1=\bigcup_{\lambda\in C^*}\lambda\left(e-TqinvA\right)$. Hence if $TqinvA$ is open, it is also so for $TinvA_1$. Conversely , one has $TqinvA = A\cap(e-TinvA_1)$. Hence if $TinvA_1$ is open, it is also so for $TqinvA$. □

Proposition 2.11 *Let A be a topological algebra and consider the following assertions:*

1 *A is quasi-inverse closed.*

2 *A is advertibly complete.*

Then, $(1) \Longrightarrow (2)$. *Furthermore, if A is an advertive Gel'fand-Mazur and simplicial t-acceptable algebra, then* $(2) \Longrightarrow (1)$ *as well. So the two preceding assertions are equivalent.*

Proof. We know that A is advertibly complete if, and only if, its unitization $A_1 = A \oplus \mathbb{C}e$ is invertibly complete. Besides, let $x_1 = x + \beta e \in A_1$ with $x \in A$ and $\beta \neq 0$. By Lemma 2.39, we can remark that x_1 is topologically invertible if, and only if, $-\frac{x}{\beta}$ is topologically quasi-invertible. Furthermore, we remark that A is quasi-inverse closed if, and only if, A_1 is inverse closed. We will use these preliminaries in the direct implication.

$(1) \Longrightarrow (2)$. Let $(x_\lambda) \subset A_1$ be a Cauchy net and $x \in A_1$ such that $xx_\lambda \longrightarrow e$ and $x_\lambda x \longrightarrow e$. Then, $0 \notin \widehat{x}\left(\mathcal{X}(A_1)\right)$. So, by hypothesis, $x \in InvA_1$. Let x^{-1} be its inverse. Then, for example, $\left(x^{-1}x\right) x_\lambda = x_\lambda \longrightarrow x^{-1}e = x^{-1}$. Whence, A_1 is invertibly complete.

$(2) \Longrightarrow (1)$ Let $x \in A$ and suppose that $1 \notin \widehat{x}\left(\mathcal{X}(A)\right)$. If $x \notin TqinvA$, then $x \notin QinvA$. Hence, for example, $I = \{y - yx : y \in A\}$ is a regular left ideal with right unit element x of A modulo I and $J \neq A$, where J is the closure of I in A. Consequently, J is a regular and closed left ideal with right unit element x of A modulo J. Since A is simplicial, then there exists a regular, closed and maximal left ideal M, with right unit element x of A modulo M which contains J. But A is t-acceptable, so M is two-sided. Hence, by the fact that A is a Gel'fand-Mazur algebra, M defines an $f \in \mathcal{X}(A)$ such that $M = Kerf$. Consequently, $f(x) = 1$. Contradiction. Hence, $x \in TlqinvA$. We proceed in a similar way to prove that $x \in TrqinvA$. Hence $x \in TqinvA$. Since by hypothesis A is advertive, then $TqinvA = QinvA$, which proves the assertion. □

Theorem 2.2 *Let A be a topological algebra. Moreover, consider the following assertions:*

1 *The completion $\widetilde{A}$ of A is tQ-algebra.*

2 *$\mathcal{X}(A)$ is equicontinuous.*

3 *$\mathcal{X}(A)$ is weakly relatively compact subset of A'_s.*

4 *$\mathcal{X}(A) \cup \{0\}$ is weakly compact subset of A'_s.*

5 *A is a bounded algebra.*

Then, one has the following implications:

a) $(1) \Longrightarrow (2)$ *if $\mathcal{X}(A) = \mathcal{X}(\widetilde{A})$ up to a continuous bijection.*

b) $(2) \Longrightarrow (3) \Longleftrightarrow (4)$.

c) $(3) \Longrightarrow (2)$ *if the Gel'fand map is continuous. So under this condition we have:* $(2) \Longleftrightarrow (3) \Longleftrightarrow (4)$.

d) $(5) \Longrightarrow (2)$ *if A is spectrally barrelled. So, in this case, one has*

$$(2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5) \Longrightarrow (2)$$

e) (2) $\Longrightarrow$ (1) *if* $\widetilde{A}$ *is topologically quasi-inverse closed and* $\mathcal{X}(A) = \mathcal{X}(\widetilde{A})$ *up to a continuous bijection.*

f)

$$(1) \Longleftrightarrow (2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5)$$

if A *is spectrally barrelled such that* $\widetilde{A}$ *is topologically quasi-inverse closed and* $\mathcal{X}(A) = \mathcal{X}(\widetilde{A})$ *up to a continuous bijection.*

Proof. By Proposition 2.26, the set $\mathcal{X}(\widetilde{A})$ is equicontinuous. By assumption $\mathcal{X}(\widetilde{A}) = \mathcal{X}(A)$ up to a continuous bijection [7: p. 146, Lemma 2.2]. So $\mathcal{X}(A)$ is also equicontinuous. One always has (2) $\Longrightarrow$ (3) $\Longleftrightarrow$ (4), while (3) $\Longrightarrow$ (2) under the continuity of the Gel'fand map [7: p. 182, Theorem 1.1], so in this case (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4). If A is spectrally barrelled, then by the very definitions (5) $\Longrightarrow$ (2), hence in a spectrally barrelled algebra one gets (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) $\Longrightarrow$ (2). Now, if $\widetilde{A}$ is topologically quasi-inverse closed and $\mathcal{X}(\widetilde{A}) = \mathcal{X}(A)$ up to a continuous bijection, then (2) $\Longrightarrow$ (1) by applying (1) $\Longrightarrow$ (2) of Theorem 2.29 for $\widetilde{A}$. In conclusion, all the above assertions are equivalent in a spectrally barrelled algebra A whose completion $\widetilde{A}$ is topologically quasi-inverse closed and $\mathcal{X}(\widetilde{A}) = \mathcal{X}(A)$ up to a continuous bijection. □

Based on Corollary 2.11, Proposition 2.26 and Remark 2.22, one can immediately state the following result.

Corollary 2.13 *Let* A *be a topological algebra and consider the following assertions:*

1 A *is tQ-algebra.*

2 $\mathcal{X}(A)$ *is equicontinuous.*

3 $\mathcal{X}(A)$ *is weakly relatively compact.*

4 $\mathcal{X}(A)$ *is weakly bounded.*

5 $A = TB(A)$.

Then, one has

(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) *and* (1) $\Longrightarrow$ (5).

If, A *satisfies* (2.5)*, then* (4) $\Longleftrightarrow$ (5)*, while* (5) $\Longrightarrow$ (1) *if, in addition, the Gel'fand map of* A *is continuous. So, all the preceding are equivalent in a topological algebra* A *having continuous Gel'fand map and satisfying* (2.5).

Since *relation* (2.5) *is fulfilled in a simplicial t-acceptable Gel'fand-Mazur algebra* (cf. Proposition 2.21), *one obtains as a byproduct the next.*

Corollary 2.14 *In a simplicial t-acceptable Gel'fand-Mazur algebra* A *with continuous Gel'fand map the following assertions are equivalent:*

1 A *is tQ-algebra.*

2 $\mathcal{X}(A)$ *is equicontinuous.*

3 $\mathcal{X}(A)$ *is weakly relatively compact.*

4 $\mathcal{X}(A)$ is weakly bounded.

5 $A = TB(A)$.

As *a commutative locally m-convex algebra is always Gel'fand-Mazur* (cf. [7, p. 308, Definition 9.5 along with p. 71, Theorem 7.1]), one immediately gets at the following.

Corollary 2.15 *In any simplicial t-acceptable locally m-convex algebra A with continuous Gel'fand map the five following assertions are equivalent:*

1 A is tQ-algebra.

2 $\mathcal{X}(A)$ is equicontinuous.

3 $\mathcal{X}(A)$ is weakly relatively compact.

4 $\mathcal{X}(A)$ is weakly bounded.

5 $A = TB(A)$.

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⋄ Abdelhak Najmi
Ecole Normale Supérieure. Takaddoum B. P. 5118, 10105 Rabat, Morocco
`najmiabdelhak@yahoo.fr`