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Remarks on locally A-convex algebras

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Abstract

Properties of A-convex algebras (m-convexity, Q-property, barrelledness, etc), in particular with respect to weak topologies, are reexamined. The proofs are direct and short. Some improvements are obtained. The Grothendieck completion is examined. Necessary and sufficient conditions are given to ensure that the completion of a locally A-convex algebra is of the same type.

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1. Introduction. A. C. Cochran, R. Keown and C. R. Williams introduced [3] the class of locally A-convex algebras. In [4], the first author recognized among them the uniformly A-convex ones. Then he came with a paper on weak A-convex algebras [5], continuing the work of S. Warner ([16], [17]) on weak m-convex ones. Our approach allows direct proofs. It also enlighten the known examples. Of course the principal ingredient in the proofs remains Warner's one, that is the ideals of finite codimension. But here the use of appropriate families of seminorms shortens the proofs; and we do not appeal to polarity hence, in particular, to the theorem of bipolars.

Due to technical reasons, one asks for additional conditions, on the seminorms, than just those defining the topology considered. This is the case with renormalizations (cf. e.g. [7]). We obtain that if an algebra is m-convex and uniformly A-convex, then its topology can be given by a family of seminorms satisfying the inequalities associated with both structures (Proposition 3.5).

There is a well known theorem of Gleason-Kahane-Zelazko on linear functionals which are necessarily multiplivative (i.e., characters). We give a remark in the spirit of C. Le page approach [8], that is to get an equality from an inequality (Proposition 4.1).

It is known that a barrelled locally A-convex algebra is necessarily m-convex [10]. Actually it is sufficient to put the hypothesis on the factors associated with its Arens-Michael decomposition (Proposition 5.1).

Given a Hausdorff locally convex algebra $(E, (|.|_\lambda)_\lambda)$, we show that if every $N_\lambda = \{x \in E : |x|_\lambda = 0\}$ contains a closed two-sided ideal of finite codimension, then it is necessarily a weak m-convex algebra (Proposition 6.1). We also come with remarks enlightening the known examples of weak A-convex algebras. The latter are in an a priori m-convex context. In particular, for the uniformly A-convex case, the setting is that one of normed algebras.

It is known that the Grothendieck completion (G-completion) of a locally convex algebra with (jointly) continuous product, in particular of an m-convex algebra, is of the same type. This is not the case for A-convex algebras (see the examples in

Section 7); the G-completion could not be even an algebra. The aim here is to clarify the situation in that context. A necessary and sufficient condition is given to ensure that the sequential completion (S-completion; denoted $\widehat{E_s}$) or the Mackey completion (M-completion; denoted $\widehat{E_M}$) of a locally A-convex algebra is an algebra of the same type. This is done by means of appropriate bounded subsets, of an associated m-convex topology (Proposition 7.6). The conclusions have to be compared with those in [2] and [6].

2. Definitions. Let (E, τ) be a locally convex algebra ($l.c.a.$), with a separately continuous product, whose topology τ is given by a family $(p_\lambda)_\lambda$ of seminorms. The algebra (E, τ) is said to be locally A-convex ($l - A - c.a.$; [3], [4]) if, for every x and every λ, there is $M(x, \lambda) > 0$ such that

$$\max\left[p_\lambda(xy), p_\lambda(yx)\right] \leq M(x, \lambda)p_\lambda(y); \forall y \in E.$$

In the case of a single vector norm, $(E, \|.\|)$ is called an A-normed algebra. If $M(x, \lambda) = M(x)$ depends only on x, we say that (E, τ) is a locally uniformly A-convex algebra ($l.u - A - c.a.$; [4]). If it happens that, for every λ,

$$p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y); \forall x, y \in E,$$

then (E, τ) is named a locally m-convex algebra ($l.m.c.a.$; cf. [10], [9]). Recall also that a $l.c.a.$ has a continuous product if, for evry λ, there is λ' such that

$$p_\lambda(xy) \leq p_{\lambda'}(x)p_{\lambda'}(y); \forall x, y \in E.$$

3. m-convexity and uniform A-convexity. It is a common problem to ask whether the topology of a $l.c.a.$ can be given by a family $(p_\lambda)_\lambda$ of seminorms satisfying specific conditions. Here is the question we deal with. There are $l.m.c.a$'s which are also $l.u - A - c.a$'s the topologies of which are defined by families $(p_\lambda)_\lambda$ of seminorms such that

$$\forall \lambda : p_\lambda(xy) \leq p_\lambda(x)p_\lambda(y); \forall x, y \in E. \ (1)$$

$$\forall \lambda, \exists M(x) > 0 : p_\lambda(xy) \leq M(x)p_\lambda(y); \forall x, y \in E. \ (2)$$

Example 3.1. Let $C[0, 1]$ be the complex algebra of continuous functions on the intervall $[0, 1]$. Endow it with the seminorms $(|.|_{K_d})_{K_d}$, where $|f|_{K_d} = \sup\{|f(x)| : x \in |.|_{K_d}\}$ with K_d a denumerable compact subset of $[0, 1]$. Then (1) and (2) are satisfied.

Example 3.2. Let $(E, \|.\|)$ is a unital commutative and semisimple Banach algebra. Endow it with the topology defined by its carrier space $\mathcal{M}(E)$ i.e., by the family $(|.|_\chi)_\chi$ of seminorms, where $|x|_\chi = |\chi(x)|$ with $\chi \in \mathcal{M}(E)$. Here $|\chi(xy)| = |\chi(x)| . |\chi(y)|$ and $|\chi(xy)| \leq \rho(x) |\chi(y)|$, where ρ is the spectral radius.

Example 3.3. Consider $\mathcal{K}(R)$ the measure space of complex continuous functions, on R, with compact support. Endow it with the topology of uniform convergence on compacta. This algebra is not unital.

Example 3.4. The previous examples concern commutative algebras. To have a non commutative situation, consider a closed subalgebra of the C^*-algebra $\mathcal{L}(H)$ of bounded operators on a Hilbert space H i.e., a von Neumann algebra. Endow it with the representation topology defined by the family $(|.|_\pi)_\pi$ of seminorms, where $|T|_\pi = \|\pi(T)\|$ with π running over the collection of representations of $\mathcal{L}(H)$.

One then wonders if this is always possible. The question is answered in positive in the unital case.

Proposition 3.5. Let (E, τ) be a unital *l.m.c.a.* which is also a $l.u - A - c.a.$. Then there is a family $(q_\lambda)_\lambda$ of seminorms defining τ and satisfying both (1) and (2).

Proof. First, consider that E is unital. Let $(p_\lambda)_\lambda$ be a family of seminorms defining τ and such that

$$\forall x, \exists M(x) > 0, \forall \lambda : p_\lambda(xy) \leq M(x)p_\lambda(y); \forall y \in E.$$

Without loss of generality, we may suppose $p_\lambda(e) = 1$, for every λ. Put

$$q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \leq 1\}.$$

Then $(E, (q_\lambda)_\lambda)$ is a *l.m.c.a.* and $(q_\lambda)_\lambda$ defines the coarsest m-convex topology $M(\tau)$ which is stronger than τ. But τ is itself m-convex. Hence it is given by $(q_\lambda)_\lambda$. Now $q_\lambda(x) \leq M(x)$, for every x and every λ. So

$$q_\lambda(xy) \leq q_\lambda(x).q_\lambda(y) \leq M(x)q_\lambda(y); \forall y \in E.$$

If E is not unital, take its unitization $E_1 = E \times C$ and $(p^1_\lambda)_\lambda$ the family of seminorms giving its topology, where $p^1_\lambda(x, \alpha) = p_\lambda(x) + |\alpha|$. The algebra $(E, (p^1_\lambda)_\lambda)$ is a unital $l - A - c.a.$ Then defining q^1_λ, as usual, one obtains the corresponding m-convex topology $M^1(\tau)$. Next one checks that

$$q^1_\lambda(xy) \leq M(x)q^1_\lambda(y) \leq M(x)q_\lambda(y); \forall y \in E.$$

Remark 3.6. Notice that we also have $q_\lambda(e) = 1$, for every λ, and $q_\lambda(x) \leq \|x\|_o$, for every x, where $\|x\|_o = \sup\{q_\lambda(x) : \lambda\}$. Also $x \longrightarrow \|x\|_o$ is an algebra norm. Now it is clear that if the topology of a *l.m.c.a.E* is given by submultiplicative seminorms p_λ such that $\sup\{q_\lambda(x) : \lambda\} < +\infty$, for every x, then it is uniformly A-convex. So a *l.m.c.a.* is a $l.u - A - c.a$ if, and only if, its topology can be given by a family $(q_\lambda)_\lambda$ of seminorms satisfying

1) $q_\lambda(xy) \leq q_\lambda(x).q_\lambda(y); \forall x, y \in E.$
2) $q_\lambda(e) = 1; \forall \lambda.$
3) $q_\lambda(xy) \leq M(x)q_\lambda(y); \forall x, y \in E.$
4) $x \longrightarrow \|x\|_o = \sup\{q_\lambda(x) : \lambda\}$ is an algebra norm.

This is exactly what happens in the examples above. It is also worthwhile to notice that the situation here is very comfortable. A more difficult problem, stated by W. Zelazko, and solved by A. Fernandez and V. Müller in [7], concerns only 1) and 2) in the setting of *l.c.a.*'s with jointly countinuous product.

4. Weak A-seminorms. Actually, a linear form defining an A-seminorm is necessairily a character up to a multiplication by a constant.

Proposition 4.1. Let f be a non zero linear functional on a unital Banach algebra E. If, for every x, there is $M(f, x) > 0$ such that

$$|f(xy)| \leq M(f, x) |f(y)| ; \forall y \in E,$$

then f is a multiple of a character.

Proof. Putting $y = e$, one has $|f(x)| \leq M(f, x) |f(e)|$, for every x. So $f(e) \neq 0$ for otherwise f should be identically zero. Now we also have $|f(e)| \leq M(f, x^{-1}) |f(x)|$, for every invertible x in E. Hence $f(x) \neq 0$ for every such x. Applying the theorem of Gleason-Kahane-Zelazko, one obtains that $g = [f(e)]^{-1} f$ is a character.

5. m-barrelledness and m-convexity. According to a Michael's result [10], any m-barrelled A-convex algebra is m-convex . Actually the denomination 'm-barrelled' is due to A. Mallios ([9], Theorem 5.2, p. 36). Also (ibid., Definition 5.1, p. 31 and Theorem 5.1, p. 34), there is made an interesting connection with a seminal I. M. Gelfand idea displaying a Banach algebra as an algebra of operators. Now, here is a slight improvement of Michael's result mentioned above. Let $(E, (p_\lambda)_\lambda)$ be a $l-A-c.a.$ and consider the quotient algebras $E_\lambda = E/N_\lambda$ where $N_\lambda = \{x \in E : p_\lambda(x) = 0\}$.

Proposition 5.1. If every N_λ is m-barrelled, then $(E, (p_\lambda)_\lambda)$ is m-convex.

Proof. Consider, for every p_λ, the associated algebra seminorm q_λ. One has $N_\lambda^{'} = \{x \in E : q_\lambda(x) = 0\} = N_\lambda$. So $E_\lambda = E/N_\lambda = E/N_\lambda^{'}$. We do have on E_λ the A-norm $\overline{p_\lambda}$ and the algebra norm $\overline{q_\lambda}$ which is exactly the associated one to $\overline{p_\lambda}$. The unit ball of $\overline{q_\lambda}$ is an m-barrell for $\overline{p_\lambda}$. Whence the result.

Remark 5.2. Here is an axample where the algebra is not m-barrelled while the E_λ's are. Consider Exampe 3.2 above. The E_χ's are isomorphic to the complex field C. But $(E, (|.|_\chi)_\chi)$ can not be m-barrelled for otherwise it should be normable and hence of finite dimension.

Remark 5.3. Professor A. Mallios pointed out that Proposition 5.1 is an illustration of a result of him ([9], Theorem 6.1, p. 280) involving the notion of Γ-completeness [ibid. p 280, Definition 6.1] he has introduced; that is, in our context, $E_\lambda = E/N_\lambda$ is a normed algebra for every λ ([9], p. 280). The P-completeness, that is $E_\lambda = E/N_\lambda$ is complete for every λ, is another notion (of completeness [3]) to ensure m-convexity. So we are led to a comparison. Consider the following conditions every one of which implies the multiplicative convexity of an A-convex algebra.
(i) E is m-barrelled.
(ii) Every $E_\lambda = E/N_\lambda$ is m-barrelled.
(iii) E is P-complete.
(iv) E is Γ-complete.

Each one of the conditions (i), (ii) and (iii) implies (iv). Also (i) impplies (ii). But none of the inverse implications is true.

6. Weak A-convex algebras. We begin with an essential result the seminal idea of which is due to Warner ([16], [17]).

Proposition 6.1. Let $(E, (p_\lambda)_\lambda)$ be a $l.c.a.$. If every N_λ contains a closed two-sided ideal J_λ of finite codimension, then $(E, (p_\lambda)_\lambda)$ is a weak m-convex algebra.

Proof. For every λ, consider the finite dimensional algebra $F_\lambda = E/J_\lambda$. The quotient topology is Hausdorff, for J_λ is closed. Now any product on F_λ is continuous. So we actually have a Banach algebra. To finish the proof it remains to show that, for every λ and every ε, the neighborhood $U = U(\lambda, \varepsilon) = \{x : p_\lambda(x) < \varepsilon\}$ contains an idempotent neighborhood. Let $\|.\|$ be an algebra norm defining the topology of $F_\lambda = s(E_\lambda)$ where $s : E \longrightarrow F_\lambda$ is the canonical surjection. Since s is open, $s(U)$ is a neighborhood of zero in F_λ. Put $B(\alpha) = \{s(y) : \|s(y)\| < \alpha\} \subset s(U)$, with $0 < \alpha < 1$. Then

$$s^{-1}(B(\alpha)) \subset s(s(U)) = U + J_\lambda.$$

Of course $s^{-1}(B(\alpha))$ is an idempotent neighborhood of zero and $U + J_\lambda \subset U$, for $J_\lambda \subset N_\lambda$. Now since F_λ is of finite dimension its topology is the weak one. So arguing as above one shows the weakness of the topology of $(E, (p_\lambda)_\lambda)$.

Remark 6.2. The condition of finite codimensionality is necessary. Indeed if $(E, (|.|_n)_n)$ is a B_0-algebra which is not m-convex, then every N_n contains a closed two-sided ideal but not all the N_n's are of finite codimension.

Remark 6.3. Proposition 6.1 implies that $(E, \sigma(E, E'))$ is m-convex if, and only if, it is isomorphic (as a topological algebra) to a subalgebra of a product of finite dimensional Banach algebras. By the way, the classical example handled by S. Warner is the algebra $C[X]$ of complex polynomials which is isomorphic to the algebra $C^{(N)}$ of complex sequences with only a finite number of non zero elements.

We now apply the previous proposition to weak topologies. Given a complex algebra E, let E^* be its algebraic dual and E' a total subspace of E^* i.e., E' separates the points of E. The weak topology compatible with the duality $\left\langle E, E' \right\rangle$ is denoted by $\sigma = \sigma(E, E')$.

Proposition 6.4. The following assertions are equivalent.
(i) (E, σ) is A-convex.
(ii) (E, σ) is m-convex.
(iii) (E, σ) has a continuous product.

Proof. (i)⇒(ii) Let $(p_\lambda)_\lambda$ be a family of seminorms defining σ. Then any N_λ is a closed two-sided ideal. Moreover, for every λ, there are f_1,..., f_n in E' and $k > 0$ such that

$$p_\lambda(x) \leq k \sup\{|f_i(x)|\,; \forall x \in E\}.$$

Whence $\cap \ker f_i \subset N_\lambda$. Hence N_λ is of finite codimension. We conclude by Proposition 6.1.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let $(p_\lambda)_\lambda$ be a family of seminorms defining σ. One shows, as in (i)⇒(ii), that N_λ is of finite codimension. Now the product being continuous there is, for every λ, a λ' such that

$$p_\lambda(xy) \leq p_{\lambda'}(x) p_{\lambda'}(y); \forall x, y \in E.$$

But then $EN'_\lambda \subset N_\lambda$ and $N'_\lambda E \subset N_\lambda$. So the closed two-sided ideal J_λ generated by N'_λ is contained in N_λ; yet, it is of finite codimension for $N'_\lambda \subset J_\lambda$. We conclude once again by Proposition 6.1.

Remark 6.5. The equivalence of (i) and (ii) can also be obtained as follows. One has $N'_\lambda = \{x \in E : q_\lambda(x) = 0\} = N_\lambda$. Hence $E_\lambda = E/N_\lambda = E/N'_\lambda$. So the quotient norms $\overline{p_\lambda}$ and $\overline{q_\lambda}$ are equivalent, for E_λ is finite dimensional. Whence the conclusion.

We now examine the strength of the barrelledness and the Q-property in the context of this section. We begin with the uniformly A-convex case. Let E be a complex unital algebra, E^* its algebraic dual and E' a total subspace of E^*. If (E, σ) is a $l.u-A-c.a.$, then it can be endowed with an algebra norm $\|.\|_0$ which is stronger than σ ([12], [13]). Hence $E' \subset E'_{\|\cdot\|_0}$. So we are in the setting of an a priori normed algebra $(E, \|.\|)$ with E' a total vector subspace of $E'_{\|\cdot\|}$.

Proposition 6.6. Let $(E, \|.\|)$ be a unital normed algebra and E' a total vector subspace of E^*. If $(E, \sigma(E, E'))$ is m-convex and barrelled then E is finite dimensional.

Proof. Let $(p_\lambda)_\lambda$ be a family of submultiplicative seminorms defining $\sigma(E, E')$. So, for every λ,

$$p_\lambda(xy) \leq p_\lambda(x) p_\lambda(y); \forall x, y \in E.$$

Now

$$\forall\lambda, \exists k_\lambda > 0 : p_\lambda(xy) \le k_\lambda \|x\| \, p_\lambda(y); \forall y.$$

Whence, taking $y = e$,

$$\forall\lambda, \frac{1}{k_\lambda} p_\lambda(x) \le \|x\|\,; \forall x.$$

So $x \longrightarrow \|x\|^{'} = \sup\{\frac{1}{k_\lambda} p_\lambda(x), \lambda\}$ is a vector space norm stronger than $\sigma(E, E^{'})$ and having σ-barrells as a 0-neighborhood basis. Hence $(E, \sigma(E, E^{'}))$ is normable ([12], [13]). Whence the conclusion.

We now consider the general A-convex case. If $(E, \sigma(E, E^{'}))$ is a $l - A - c.a.$, then E can be endowed with a locally m-convex topology $M(\tau)$ which is stronger than $\sigma(E, E^{'})$; see [11]. Hence $E^{'} \subset E^{'}_{M(\tau)}$. So we have an a priori *l.m.c.a.* (E, τ) with $E^{'}$ a total vector subspace of $E^{'}_\tau$.

Proposition 6.7. If $(E, \sigma(E, E^{'}))$ is an m-convex and semisimple Q-algebra then E is finite dimensional.

Proof. Let $(p_\lambda)_\lambda$ be a directed family of submultiplicative seminorms defining $\sigma(E, E^{'})$. By the Q-property, there is λ and $\varepsilon > 0$ such that $e + \{x : p_\lambda(x) < \varepsilon\} \subset G(E)$ (: the group of invertible elements of E). But $Rad(E) = \{x : e + xy \in G(E), \forall y \in E\}$. So one has $N_\lambda \subset Rad(E)$. And we have seen that every N_λ is of finite codimension. Now $Rad(E) = \{0\}$ by hypothesis.

Here are examples as a matter of illustration.

Example 6.8. Let $(E, \|.\|)$ be a unital commutative and semisimple normed Q-algebra. Its carrier space $\mathcal{M}(E)$ separates the points of E. Endow E with the topology $\sigma = \sigma(E, E, \langle\mathcal{M}(E)\rangle)$ where $\langle\mathcal{M}(E)\rangle$ is the vector space generated by $\mathcal{M}(E)$ in the algebraic dual E^* of E. The latter becomes a *l.c.a.* with a continuous product. By a result of S. Warner ([16]; see also Proposition 5.4) it is a *l.m.c.a.*. Let $(p_\lambda)_\lambda$ be a directed family of submultiplicative seminorms defining its topology. By hypothesis, there is, for every λ, a $k_\lambda > 0$ such that

$$p_\lambda(xy) \le k_\lambda \|x\| \, p_\lambda(y); \forall x, y \in E.$$

Now, as in Proposition 6.6, one obtains a vector space norm and a basis of 0-neighborhoods consisting of barrells for σ. So if E is infinite dimensional, then (E, σ) can not be barrelled. It can not also be a Q-algebra; for otherwise, by Tsertos inequality ([15]), there should exist f_1, ..., f_n in $\langle\mathcal{M}(E)\rangle$ and $k > 0$ such that

$$\rho(x) \le k \sup\{|f_i(x)| : 1 \le i \le n\}; \forall x \in E.$$

But then E should be finite dimensional.

Now, observe that β_σ designating the boundednes radius for (E, σ), one has

$$\beta_\sigma(x) = \sup(\limsup |f(x^n)|^{\frac{1}{n}}) \geqslant \sup |\chi(x)| = \rho(x).$$

But we always have $\beta_\sigma \le \rho$. So $\beta_\sigma = \rho$. Hence (E, σ) is advertibly complete.

Example 6.9. The algebra $C[0, 1]$ of Example 3.1 is an m-convex algebra which is also uniformly A-convex. It can not be barrelled for otherwise it should be normable. But then there should exist a denumerable compact subset of $[0, 1]$ defining a norm. This can not happen by a separation lemma of Urysohn type.

7. Completion of A-convex algebras. First, examples corresponding to different situations.

Example 7.1. Let $E = C[0,1]$ be the complex algebra of continuous functions on the intervall $[0,1]$. Endow it with the vector space norm $\|.\|_1$ i.e., $\|f\|_1 = \int_0^1 |f(t)|\,dt$. Then $(E, \|.\|_1)$ is an A-normed algebra for the usual operations. Its G-completion is $L^1[0,1]$ which is not an algebra for the usual product.

Example 7.2. Consider $(L^\omega[0,1], (\|.\|_n)_n)$ the Arens algebra [1], where $\|f\|_n = \left(\int_0^1 |f(t)|^n\,dt\right)^{\frac{1}{n}}, n \in N^*$. It is known that $C[0,1]$ endowed with the induced topology is dense in $L^\omega[0,1]$. So here is a $l.u-A-c.a.$ the G-completion of which is an algebra. But, it is not of the same type, since as metrizable and complete (a B_0-algebra) it should be normable (cf [12] or [13]). Notice that here the product on $C[0,1]$ is continuous and so hypocontinuous.

Example 7.3. The algebra $C_b(R)$ of complex continuous and bounded functions on the real field R, endowed with the family $(|.|_n)_n$ of seminorms given by $|f|_n = \sup\{|f(x)| : |x| \leq n\}$, is a $l.u-A-c.a.$ the G-completion of which is m-convex but not uniformly A-convex.

Example 7.4. Given a non complete normed algebra E and a complete uniformly A-convex algebra F, the G-completion of $E \times F$ is $\widehat{E \times F} = \widehat{E} \times F$. It is a $l.u-A-c.a.$, hence of the same type as $E \times F$.

Now we provide conditions in order the M-completion or the S-completion to keep the same structure of a given $l-A-c.a.$. For convenience, we recall some facts from [11], [12] and [13], needed in the sequel. If $(E, (p_\lambda)_\lambda)$ is a unital $l-A-c.a.$, then it can be endowed with a stronger m-convex topology $M(\tau)$, where τ is the topology of E. It is determined by the family $(q_\lambda)_\lambda$ of seminorms given by

$$q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \leq 1\}.$$

If $(E, (p_\lambda)_\lambda)$ is uniformly A-convex, then there is also an algebra norm $\|.\|_0$ on E, stronger than $M(\tau)$, given by

$$\|x\|_0 = \sup\{q_\lambda(x) : \lambda\}.$$

Proposition 7.5. Let (E, τ) be a unital $l-A-c.a.$.

(i) If (E, τ) is S-complete, then τ and $M(\tau)$ have the same bounded sequences.

(ii) If (E, τ) is M-complete, then τ and $M(\tau)$ have the same bounded sets.

(iii) If (E, τ) is an M-complete $l.u-A-c.a.$, then τ and $\|.\|_0$ have the same bounded sets.

Proof. (i) Let $(x_n)_n$ be a τ-bounded sequence. Its absolutely convex hull D is a closed disc for τ. But a basis of 0-neighborhoods for $M(\tau)$ consists of τ-barrells. And since (E, τ) is S-complete, these barrells absorb D. So $(x_n)_n$ is $M(\tau)$-bounded.

(ii) We apply arguments as in (i), using also the fact that the bound structure of an M-complete $l.c.s.$ admits a basis of closed bounded discs.

(iii) Along the lines of (ii).

Now we can state the following, where $\mathcal{B}$ indicates the von Neumann bound structure.

Proposition 7.6. Let (E, τ) be a unital $l-A-c.a.$. Then,

(i) $\widehat{E_M}$ is a $l-A-c.a.$ if, and only if, $\mathcal{B}\tau = \mathcal{B}M(\tau)$.

(ii) $\widehat{E_M}$ is a $l.u-A-c.a.$ if, and only if, $\mathcal{B}\tau = \mathcal{B}\,\|.\|_0$.

(iii) $\widehat{E_S}$ is a $l-A-c.a.$ if, and only if, τ and $M(\tau)$ have the same bounded sequences.

Proof. **(i)** We only show the sufficiency. First the product is hypocontinuous. Indeed, if $(x_i)_i$ and $(y_i)_i$ are bounded nets tending to zero, then they are also bounded for $M(\tau)$. But one has

$$p_\lambda(x_i y_i) \leq q_\lambda(x_i) p_\lambda(y_i); \forall i.$$

Whence the conclusion. So $\widehat{E_M}$ is an algebra by Theorem 4.2, p. 30, in [9]. Now for $\widehat{x} \in \widehat{E_M}$, take a bounded net such that $\widehat{x} = \lim x_i$. By hypothesis, it is also bounded for $M(\tau)$. And one has

$$p_\lambda(x_i y) \leq q_\lambda(x_i) p_\lambda(y); \forall y.$$

Then

$$p_\lambda(x_i y) \leq M(\lambda, \widehat{x}) p_\lambda(y); \forall y.$$

Whence

$$\widehat{p_\lambda}(\widehat{x}\widehat{y}) \leq M(\lambda, \widehat{x}) \widehat{p_\lambda}(\widehat{y}); \forall \widehat{x}, \widehat{y} \in \widehat{E_M}.$$

(ii) The same lines as in (i).

(iii) First $\widehat{E_S}$ is an algebra. Indeed for $\widehat{x}, \widehat{y} \in \widehat{E_S}$, let $(x_n)_n$ and $(y_n)_n$ be sequences for which x and y are the respective limits. It follows from the inequality

$$p_\lambda(x_n y_n - x_m y_m) \leq p_\lambda(x_n - x_m) q_\lambda(y_m) + q_\lambda(x_m) p_\lambda(y_n - y_m),$$

that $(x_n y_n)_n$ is a Cauchy sequence. Put $xy = \lim x_n y_n$, noticing that it does not depend of the choice of $(x_n)_n$ nor $(y_n)_n$. The end of the proof is as in (i).

Remark 7.7. A. Mallios considered [9], p. 30, the completion of topological algebras (not necessairily locally convex). He showed that if the product is hypocontinuous then the S-completion is an algebra. However hypocontinuity is not sufficient to preserve the whole topological algebra structure. It does not, for instance, imply that $\mathcal{B}\tau = \mathcal{B}M(\tau)$. This is shown by Example 7.2. In this case the product is (jointly) continuous so, if we had $\mathcal{B}\tau = \mathcal{B}M(\tau)$, then $L^\omega[0,1]$ should be m-convex; which is not the case.

Remark 7.8. The algebra $(C[0,1], \|.\|_1)$, in Example 7.1, is a $l.u-A-c.a.$ with a separately continuous product, but not (jointly) continuous. So the same is not hypocontinuous, due to the metrizability.

Remark 7.9. In [14], there is an example of a unital and complete $l.u-A-c.a.$ in which the product is hypocontinuous but not (jointly) continuous. It is the multiplier algebra of a Hilbertian algebra.

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