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A second order differential equation approach in the time domain for electromagnetic fields in a bounded chiral medium

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Abstract

In this work we consider the Drude-Born-Fedorov model for the evolution of electromagnetic fields in chiral media in the time domain. We reduce the problem to a second order Cauchy problem which we prove to be strongly well posed.

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1. Introduction

Since the mid 1980's, mainly due to vast progress in the related technology, the electromagnetic community focussed on the study of complex media. A very important class of linear complex media are the so-called chiral media. A rigorous mathematical analysis of wave propagation and scattering problems for chiral media in the frequency- domain (i.e. for time-harmonic fields) has been developed since the mid 1990's and has resulted to a rather well established theory. In the frequency domain, the "universally" considered constitutive laws, are the so called Drude-Born-Fedorov (DBF) relations. The mathematical analysis of problems related to chiral media in the time-domain is less developed, but is under strong current interest. Although the complete constitutive relations for chiral media in the time domain should be non-local in time (convolution terms appear), many authors have considered a local in time approximation to the constitutive relations, and have thus used the DBF relations in the time domain. To the best of our knowledge, the mathematical treatment in all related work is based on eventually considering a first-order abstract Cauchy problem.

In this work, by eliminating the electric field from Maxwell's equations under the DBF relations, we consider a second-order Cauchy problem and study its well-posedness.

The paper is organized as follows: in Section 2 we list the functional spaces and operators needed in this paper. In Section 3 we recall some technical results needed to set a proper mathematical framework. In Sections 4, 5 we formulate the problem in order to get a second order Cauchy problem. In Section 6 we establish the well-posedness of this second order Cauchy problem.

2. Basic Function Spaces

In what follows we assume all linear spaces considered to be over the complex field. Let Ω denote a bounded and simply connected domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. By $L^2(\Omega)$ we denote the usual Lebesque space of all square-integrable complex valued functions in Ω, equipped with the usual inner product $\langle \mathbf{a}, \mathbf{b}\rangle = \int_\Omega \mathbf{a}\cdot$

We start this section with the following Proposition that has been proved by Yoshida and Giga in [29].

Proposition 3.1. *In the space H, we consider the dense subspace H_{curl} and we define the operator $curl: H_{curl} \to H$. The following hold:*
The operator curl is closed and self-adjoint. Furthermore, curl is invertible and its inverse $curl^{-1}: H \to H_{curl}$ is compact. The spectrum $\sigma(curl)$ consists only of point spectrum $\sigma_\rho(curl) \subset \mathbb{R}$, and the set of the corresponding eigenfunctions gives an orthogonal complete basis of the space H.

We remind that $\sigma_\rho(curl)$ is the set of all eigenvalues of the operator $curl$ (hence $\sigma(curl) = \sigma_\rho(curl)$). Since the operator $curl$ is closed, its resolvent operator,

$$(\lambda I + curl)^{-1}: H \to H_{curl}$$

for $\lambda \in \rho(-curl) = \mathbb{C} \setminus \sigma(-curl)$, is not only densely defined but it is also bounded ([20]).

Proposition 3.2. *If $\lambda \in \rho(-curl)$ then the operators $(\lambda I + curl)^{-1}$ and curl commute on H_{curl}.*

Proof. It is clear that $(\lambda I + curl)^{-1}\mathbf{u} \in H_{curl}$ for $\mathbf{u} \in H_{curl}$. So

$$\begin{aligned}(\lambda I + curl)^{-1} curl\mathbf{u} &= (\lambda I + curl)^{-1}(\lambda I + curl)\mathbf{u} - \lambda(\lambda I + curl)^{-1}\mathbf{u}\\ &= (\lambda I + curl)(\lambda I + curl)^{-1}\mathbf{u} - \lambda(\lambda I + curl)^{-1}\mathbf{u}\\ &= (\lambda I + curl - \lambda I)(\lambda I + curl)^{-1}\mathbf{u}\\ &= curl(\lambda I + curl)^{-1}\mathbf{u},\end{aligned}$$

for every $\mathbf{u} \in H_{curl}$.

Remarks 3.3.
(i) It is well known ([20]) that if an operator

$$L: D(L) \subset H \to H$$

has a symmetric inverse with dense range, then the operator L^{-1} is also symmetric. Therefore, the resolvent operator $(\lambda I + curl)^{-1}$ is self-adjoint, for $\lambda \in \rho(-curl)$ since the operator $\lambda I + curl$ is self adjoint and invertible with dense range.
(ii)The operator $\mathcal{C}_\lambda = (\lambda I + curl)^{-1} curl$ is well defined with $D(\mathcal{C}_\lambda) = R(\mathcal{C}_\lambda) = H_{curl}$. Indeed,

$$\begin{aligned}D(\mathcal{C}_\lambda) &= \{\mathbf{u} \in D(curl) : curl\mathbf{u} \in D((\lambda I + curl)^{-1})\}\\ &= \{\mathbf{u} \in H_{curl} : curl\mathbf{u} \in H\} = H_{curl}\end{aligned}$$

and

$$R(\mathcal{C}_\lambda) \;=\; \mathcal{C}_\lambda(H_{curl}) \;=\; (\lambda I + curl)^{-1} curl(H_{curl}) \;=\; (\lambda I + curl)^{-1}(H) = H_{curl}.$$

(iii) We know that the operator $curl$ is closed, so the space $D(curl) = H_{curl}$ equipped with the inner product

$$\langle \mathbf{u}, \mathbf{v}\rangle_{H_{curl}} = \langle \mathbf{u}, \mathbf{v}\rangle_H + \langle curl\mathbf{u},\; curl\mathbf{v}\rangle_H$$

is a Hilbert space which will be denoted hereafter as $\mathcal{H}_c = (H_{curl},\; \|\cdot\|_{H_{curl}})$ where $\|\cdot\|_{H_{curl}}$ is the graph-norm of the operator $curl$.

Proposition 3.4. *If $\lambda \notin \sigma(-curl)$ then the operator*

$$\mathcal{C}_\lambda = (\lambda I + curl)^{-1} curl :\; \mathcal{H}_c \longrightarrow \mathcal{H}_c$$

is self-adjoint.

Proof. Let $\mathbf{u},\ \mathbf{v} \in \mathcal{H}_c$. Then

$$\langle (\lambda I + curl)^{-1} curl\mathbf{u},\; \mathbf{v}\rangle_{H_{curl}} \;=\; \langle (\lambda I + curl)^{-1} curl\mathbf{u},\; \mathbf{v}\rangle_H + \langle curl(\lambda I + curl)^{-1} curl\mathbf{u},\; curl\mathbf{v}\rangle_H.$$

For the second terms of the right hand side we have

$$\langle (\lambda I + curl)^{-1} curl\mathbf{u},\; \mathbf{v}\rangle_H = \langle curl(\lambda I + curl)^{-1}\mathbf{u},\; \mathbf{v}\rangle_H$$

$$= \langle (\lambda I + curl)^{-1}\mathbf{u},\; curl\mathbf{v}\rangle_H$$

$$= \langle \mathbf{u},\; (\lambda I + curl)^{-1} curl\mathbf{v}\rangle_H$$

and

$$\langle curl(\lambda I + curl)^{-1} curl\mathbf{u},\; curl\mathbf{v}\rangle_H \;=\; \langle (\lambda I + curl)(\lambda I + curl)^{-1} curl\mathbf{u},\; curl\mathbf{v}\rangle_H$$

$$-\lambda\langle (\lambda I + curl)^{-1} curl\mathbf{u},\; curl\mathbf{v}\rangle_H \;=\; \langle curl\mathbf{u},\; curl\mathbf{v}\rangle_H - \lambda\langle curl\mathbf{u},\; (\lambda I + curl)^{-1} curl\mathbf{v}\rangle_H$$

$$= \langle curl\mathbf{u},\; (I - \lambda(\lambda I + curl)^{-1}) curl\mathbf{v}\rangle_H \;=\; \langle curl\mathbf{u},\; (\lambda I + curl - \lambda I)(\lambda I + curl)^{-1} curl\mathbf{v}\rangle_H$$

$$= \langle curl\mathbf{u},\; curl(\lambda I + curl)^{-1} curl\mathbf{v}\rangle_H.$$

Then

$$\langle (\lambda I + curl)^{-1} curl\mathbf{u},\; \mathbf{v}\rangle_H + \langle curl(\lambda I + curl)^{-1} curl\mathbf{u},\; curl\mathbf{v}\rangle_H$$

$$= \langle \mathbf{u},\; (\lambda I + curl)^{-1} curl\mathbf{v}\rangle_H + \langle curl\mathbf{u},\; curl(\lambda I + curl)^{-1}) curl\mathbf{v}\rangle_H$$

$$= \langle \mathbf{u},\; (\lambda I + curl)^{-1} curl\mathbf{v}\rangle_{H_{curl}}.$$

We now have $\langle (\lambda I + curl)^{-1} curl\mathbf{u},\; \mathbf{v}\rangle_{H_{curl}} \;=\; \langle \mathbf{u},\; (\lambda I + curl)^{-1} curl\mathbf{v}\rangle_{H_{curl}}$. Then we obtain that $(\lambda I + curl)^{-1} curl :\; \mathcal{H}_c \longrightarrow \mathcal{H}_c$ is symmetric. Since $(\lambda I + curl)^{-1} curl$ is defined on the whole space $\mathcal{H}_c$, it is selfadjoint.

3. Formulation of the Problem

In this section we formulate the problem in order to get a second order Cauchy Problem. Consider a chiral medium filling a bounded domain Ω of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Assume that a bounded domain Ω in $\mathbb{R}^3$ is filled with a homogenous chiral medium. Ω is embedded in some medium (e. g. in vacuum) and we assume that the perfect conductor boundary condition is valid on $\partial\Omega$. The mathematical modeling of such media, is done through the modification of the constitutive relations for the well known Maxwell's equations. We assume that Maxwell's equations hold in $\Omega \subset \mathbb{R}^3$, for $t \geq 0$:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\mathbf{D} &= curl\mathbf{H} \\ \frac{\partial}{\partial t}\mathbf{B} &= -curl\mathbf{E} \end{aligned}\right\} \tag{1}$$

where $\mathbf{E} = \mathbf{E}(t,\ x)$ is the electric field,$\mathbf{H} = \mathbf{H}(t,\ x)$ is the magnetic field,$\mathbf{D} = \mathbf{D}(t,\ x)$ is the electric displacement and $\mathbf{B} = \mathbf{B}(t,\ x)$ is the magnetic induction. Maxwell's equations must be endowed with a set of constitutive relations. In our work we choose the Drude-Born-Fedorov (DBF) constitutive relations:

$$\mathbf{D} = \varepsilon(I + \beta curl)\mathbf{E} \tag{2}$$

$$\mathbf{B} = \mu(I + \beta curl)\mathbf{H} \tag{3}$$

where $\varepsilon > 0$ is the electric permittivity, $\mu > 0$ is the magnetic permeability and $\beta > 0$ is the chirality measure. We consider the Maxwell's equations (1), with initial data

$$\mathbf{E}(0,\ x) = \mathbf{E}_0, \quad \mathbf{H}(0,\ x) = \mathbf{H}_0$$

and the boundary conditions of a perfect conductor

$$\mathbf{E} \times \hat{\mathbf{n}} = \mathbf{0}, \quad \mathbf{H} \times \hat{\mathbf{n}} = \mathbf{J}_H \ \ on \ \ \partial\Omega.$$

Assumption 4.1. We assume that:
(1) $\mathbf{J}_H \in H(div,\ \partial\Omega)$ satisfy $div_\tau \mathbf{J}_H = 0$, where by div_τ, we denote the boundary divergence operator

$$div_\tau :\ H(div,\ \partial\Omega) \longrightarrow L^2(\partial\Omega)$$

(2) $curl:\ D(curl) = H_{curl} \longrightarrow H$
(3) $div\mathbf{D} = div\mathbf{B} = 0$ (divergence-free property)
(4) $\mathbf{E}_0 \cdot \hat{\mathbf{n}} = \mathbf{H}_0 \cdot \hat{\mathbf{n}} = 0$ *on* $\partial\Omega$ (initial-boundary conditions).

Proposition 4.2. *Assumption 4.1 and the D.B.F constitutive relations yield:*
(i) $div\mathbf{E} = div\mathbf{H} = 0$
(ii) $curl\mathbf{E} \cdot \hat{\mathbf{n}} = curl\mathbf{H} \cdot \hat{\mathbf{n}} = 0$, $\forall\ t > 0$ on $\partial\Omega$.
(iii)$\mathbf{E} \cdot \hat{\mathbf{n}} = \mathbf{H} \cdot \hat{\mathbf{n}} = 0$, $\forall\ t > 0$ on $\partial\Omega$.
(iv) $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$, $\forall\ t > 0$ on $\partial\Omega$.

Proof. The proof of *(i)* is obvious.
*(ii)*Using the identity

$$div_\tau(\mathbf{u} \times \hat{\mathbf{n}}) = curl\mathbf{u} \cdot \hat{\mathbf{n}} \ \ on \ \ \partial\Omega$$

we have

$$curl\mathbf{E}\cdot\hat{\mathbf{n}} = div_\tau(\mathbf{E}\times\hat{\mathbf{n}}) = 0, \quad forall\ \ t>0\ on\ \ \partial\Omega,$$
$$curl\mathbf{H}\cdot\hat{\mathbf{n}} = div_\tau(\mathbf{H}\times\hat{\mathbf{n}}) = div_\tau\mathbf{J}_H = 0, \quad forall\ \ t>0\ \ on\ \ \partial\Omega.$$

*(iii)*From (2) we have that

$$\mathbf{D}\cdot\hat{\mathbf{n}} = \varepsilon\mathbf{E}\cdot\hat{\mathbf{n}} + \varepsilon\beta curl\mathbf{E}\cdot\hat{\mathbf{n}}$$

and therefore $\mathbf{D}\cdot\hat{\mathbf{n}} = \varepsilon\mathbf{E}\cdot\hat{\mathbf{n}}$. Hence

$$\varepsilon\frac{\partial}{\partial t}(\mathbf{E}\cdot\hat{\mathbf{n}}) = \frac{\partial}{\partial t}\mathbf{D}\cdot\hat{\mathbf{n}} = curl\mathbf{H}\cdot\hat{\mathbf{n}} = 0$$

and therefore

$$\left.\begin{array}{rl}\dfrac{\partial}{\partial t}(\mathbf{E}\cdot\hat{\mathbf{n}}) & = 0\\ \mathbf{E}_0\cdot\hat{\mathbf{n}} & = 0\end{array}\right\}.$$

Hence $\mathbf{E}\cdot\hat{\mathbf{n}} = 0,\ \ forall\ \ t>0\ \ on\ \ \partial\Omega$. In a similar way we can prove that

$$\frac{\partial}{\partial t}(\mathbf{H}\cdot\hat{\mathbf{n}}) = 0$$

and therefore $\mathbf{H}\cdot\hat{\mathbf{n}} = 0,\ \ forall\ \ t>0\ \ on\ \ \partial\Omega$.
iii)From (3) we obtain

$$\mathbf{B}\cdot\hat{\mathbf{n}} = \mu\mathbf{H}\cdot\hat{\mathbf{n}} + \mu\beta curl\mathbf{H}\cdot\hat{\mathbf{n}}$$

and therefore $\mathbf{B}\cdot\hat{\mathbf{n}} = 0,\ \ forall\ \ t>0\ \ on\ \ \partial\Omega$.

Proposition 4.2 points out the reason for choosing the space H_{curl} as the domain of the operator *curl*. Maxwell equations (1), under DBF constitutive relations lead to the following initial-boundary value problem for $\mathbf{E}$, $\mathbf{H}$:

$$\left.\begin{array}{rll}\dfrac{\partial}{\partial t}(\varepsilon\mathbf{E}+\varepsilon\beta curl\mathbf{E}) = curl\mathbf{H} & in & \Omega,\ \ t>0\\ \dfrac{\partial}{\partial t}(\mu\mathbf{H}+\mu\beta curl\mathbf{H}) = -curl\mathbf{E} & in & \Omega,\ \ t>0\\ div\mathbf{E} = div\mathbf{H} = 0 & in & \Omega,\ \ t>0\\ \mathbf{E}\cdot\hat{\mathbf{n}} = \mathbf{H}\cdot\hat{\mathbf{n}} = curl\mathbf{E}\cdot\hat{\mathbf{n}} = curl\mathbf{H}\cdot\hat{\mathbf{n}} = 0 & on & \partial\Omega,\ \ t>0\\ \mathbf{E}(0,\ \cdot) = \mathbf{E}_0, \quad \mathbf{H}(0,\ \cdot) = \mathbf{H}_0 & in & \Omega\end{array}\right\} \qquad (4)$$

Remark 4.3. If $\beta \neq -\dfrac{1}{\lambda}$, where $\lambda\in\sigma_\rho(curl)$ then the operator $I+\beta curl:\ H_{curl}\longrightarrow H$ is invertible and the operator $(I+\beta curl)^{-1}:\ H\longrightarrow H_{curl}$ is bounded. It is obvious that the operator $\mathcal{C}_{\frac{1}{\beta}} = (I+\beta curl)^{-1}curl$ with $D(\mathcal{C}_{\frac{1}{\beta}}) = R(\mathcal{C}_{\frac{1}{\beta}}) = H_{curl}$ is well defined. Now we can see that the domain $D((\mathcal{C}_{\frac{1}{\beta}})^2)$ and the range $R((\mathcal{C}_{\frac{1}{\beta}})^2)$ of the operator $(\mathcal{C}_{\frac{1}{\beta}})^2 = [(I+\beta curl)^{-1}curl]^2$ coincide with H_{curl}.
Indeed

$$D((\mathcal{C}_{\frac{1}{\beta}})^2) = \{\mathbf{u}\in D(\mathcal{C}_{\frac{1}{\beta}}):\ \mathcal{C}_{\frac{1}{\beta}}\mathbf{u}\in D(\mathcal{C}_{\frac{1}{\beta}})\} = \{\mathbf{u}\in H_{curl}:\ (I+\beta curl)^{-1}curl\mathbf{u}\in H_{curl}\} =$$
$$\{\mathbf{u}\in H_{curl}:\ curl\mathbf{u}\in(I+\beta curl)H_{curl}\} = \{\mathbf{u}\in H_{curl}:\ curl\mathbf{u}\in H\} = H_{curl}$$
$$R((\mathcal{C}_{\frac{1}{\beta}})^2) = (\mathcal{C}_{\frac{1}{\beta}})^2(H_{curl}) = \mathcal{C}_{\frac{1}{\beta}}(\mathcal{C}_{\frac{1}{\beta}}(H_{curl})) = \mathcal{C}_{\frac{1}{\beta}}(H_{curl}) = H_{curl}.$$

4. The formulation of second order Cauchy Problem

In this section we eleminate **E** from Maxwell equations (4) to obtain a second order Cauchy Problem of the form:

$$u''(t) \ + \ Cu(t) = 0, \ u(0) = u_0 \ and \ u'(0) = u_1.$$

We need the following

Assumption 5.1. We assume that:

(1) $\beta \neq -\dfrac{1}{\lambda}$, where $\lambda \in \sigma_\rho(curl)$,
(2) $curl\mathbf{E}_0 \cdot \hat{\mathbf{n}} = curl\mathbf{H}_0 \cdot \hat{\mathbf{n}} = 0 \ \ on \ \ \partial\Omega$.

Then, from the first equation of (4) we obtain

$$\varepsilon(I + \beta curl)\frac{\partial}{\partial t}\mathbf{E} = curl\mathbf{H},$$

$$\frac{\partial}{\partial t}\mathbf{E} = \frac{1}{\varepsilon}(I + \beta curl)^{-1}curl\mathbf{H}.$$

Applying the *curl* operator we obtain

$$\frac{\partial}{\partial t}curl\mathbf{E} = \frac{1}{\varepsilon}curl(I + \beta curl)^{-1}curl\mathbf{H},$$

and therefore

$$-\frac{\partial^2}{\partial t^2}\mathbf{B} = \frac{1}{\varepsilon}curl(I + \beta curl)^{-1}curl\mathbf{H}.$$

From (2) we have

$$\mu(I + \beta curl)\frac{\partial^2}{\partial t^2}\mathbf{H} = -\frac{1}{\varepsilon}curl(I + \beta curl)^{-1}curl\mathbf{H},$$

$$\frac{\partial^2}{\partial t^2}\mathbf{H} = -\frac{1}{\varepsilon\mu}(I + \beta curl)^{-1}curl(I + \beta curl)^{-1}curl\mathbf{H}.$$

Hence

$$\frac{\partial^2}{\partial t^2}\mathbf{H} + [\frac{1}{\sqrt{\varepsilon\mu}}(I + \beta curl)^{-1}curl]^2\mathbf{H} = 0.$$

Since

$$\mathbf{E}_0 \cdot \hat{\mathbf{n}} = \mathbf{H}_0 \cdot \hat{\mathbf{n}} = curl\mathbf{E}_0 \cdot \hat{\mathbf{n}} = curl\mathbf{H}_0 \cdot \hat{\mathbf{n}} = 0 \ \ on \ \ \partial\Omega$$

and

$$\mathbf{B} = \mu(I + \beta curl)\mathbf{H}, \ for \ t \geq 0$$

we have

$$\frac{\partial}{\partial t}\mathbf{H} \mid_{t=0} = -\frac{1}{\mu}(I + \beta curl)^{-1}curl\mathbf{E}_0.$$

Hence we obtain the initial value problem

$$\left.\begin{aligned} \frac{\partial^2}{\partial t^2}\mathbf{H}(t) + [\frac{1}{\sqrt{\varepsilon\mu}}(I+\beta curl)^{-1}curl]^2\mathbf{H}(t) = 0 \\ \frac{\partial}{\partial t}\mathbf{H}\mid_{t=0} = -\frac{1}{\mu}(I+\beta curl)^{-1}curl\mathbf{E}_0 \\ \mathbf{H}(0) = \mathbf{H}_0 \end{aligned}\right\} \quad (5)$$

where $\mathbf{H}(t)$ stands for $\mathbf{H}(t,\ x)$,for arbitrary $x \in \Omega$.

5. Well-Posedness of the Problem

We study now the well-posedness of the following second order Cauchy Problem which is the same as (5).

$$\left.\begin{aligned} \frac{\partial^2}{\partial t^2}\mathbf{H}(t) = [\frac{i}{\sqrt{\varepsilon\mu}}(I+\beta curl)^{-1}curl]^2\mathbf{H}(t)\ (t \geq 0) \\ \frac{\partial}{\partial t}\mathbf{H}\mid_{t=0} = -\frac{1}{\mu}(I+\beta curl)^{-1}curl\mathbf{E}_0 \\ \mathbf{H}(0) = \mathbf{H}_0 \end{aligned}\right\} \quad (6)$$

We denote $\mathcal{A}_\beta := \dfrac{i}{\sqrt{\varepsilon\mu}}\mathcal{C}_{\frac{1}{\beta}}$, then the system (6) can be written in the form of a second order Cauchy Problem in the Hilbert space $\mathcal{H}_c$, as

$$(P_0) \quad \begin{cases} \dfrac{\partial^2}{\partial t^2}\mathbf{u}(t) = \mathcal{A}_\beta^2\mathbf{u}(t) \ \ (t \geq 0) \\ \mathbf{u}(0) = \mathbf{u}_0 \\ \dfrac{\partial}{\partial t}\mathbf{u}\mid_{t=0} = \mathbf{u}_1\ ,\ where\ \ \mathbf{u}_1 = i\sqrt{\dfrac{\varepsilon}{\mu}}\,\mathcal{A}_\beta\,\mathbf{E}_0. \end{cases}$$

with $D(\mathcal{A}_\beta) = H_{curl}$.
We know that the operator $\mathcal{A}_\beta:\ \mathcal{H}_c \longrightarrow \mathcal{H}_c$ is closed, with domain whole the space H_{curl}, then from the closed graph theorem is bounded, hence by [25] it is the infinitesimal generator of a uniformly continuous semigroup of operators $(U(t))_{t\geq 0}$ on $\mathcal{H}_c$.

Theorem 6.1. The Cauchy Problem (P_0) is well-posed.

Proof. Let

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{u}(t) \\ \dfrac{\partial}{\partial t}\mathbf{u}(t) \end{pmatrix} \in \mathcal{H}_c \times \mathcal{H}_c.$$

Then the problem (P_0) becomes

$$(P_1) \quad \mathbf{U}'(t) = M\ \mathbf{U}(t)\ (t \geq 0),\ \mathbf{U}(0) = f$$

where

$$M = \begin{pmatrix} 0 & I \\ \mathcal{A}_\beta^2 & 0 \end{pmatrix}, \; f = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \end{pmatrix},$$

with domain $D(M) = D(\mathcal{A}_\beta^2) \times D(\mathcal{A}_\beta) = H_{curl} \times H_{curl}$.
Since the operator $\mathcal{A}_\beta : \mathcal{H}_c \longrightarrow \mathcal{H}_c$ is bounded, the operator $M : D(M) = H_{curl} \times H_{curl} \longrightarrow H \times H$ is also bounded, hence it is the infinitesimal generator of a uniformly continuous semigroup of operators $(U(t))_{t\geq 0}$ on $\mathcal{H}_c \times \mathcal{H}_c$. It is clear that $\mathbf{u}_0 = \mathbf{H}(0) \in D(\mathcal{A}_\beta^2) = H_{curl}$ and $\mathbf{u}_1 = i\sqrt{\frac{\varepsilon}{\mu}}\, \mathcal{A}_\beta\, \mathbf{E}_0 \in D(\mathcal{A}_\beta) = H_{curl}$. Then (P_1) is well-posed, which implies the well posedness of (P_0).

Cosine functions constitute an approach to study abstract second order Cauchy Problems, analogous to the semigroup approach to first order problems. We describe the connection here in terms of mild solutions of second order problems. We will give now the definitions of mild and classical solutions of (P_0).

Definition 6.2. *A function $\boldsymbol{u} \in C(\mathbb{R}_+, \mathcal{H}_c)$ is called a mild solution of (P_0) if for all $t \geq 0$ we have*

$$\int_0^t \int_0^s \boldsymbol{u}(r) dr ds = \int_0^t (t-s)\boldsymbol{u}(s) ds \in D(\mathcal{A}_\beta^2)$$

and

$$\boldsymbol{u}(t) = \boldsymbol{u}_0 + t\boldsymbol{u}_1 + \mathcal{A}_\beta^2 \int_0^t (t-s)\boldsymbol{u}(s) ds.$$

Definition 6.3. *A function $\mathbf{u} \in C^2(\mathbb{R}_+, \mathcal{H}_c)$ is called a classical solution of (P_0) if:*
(1) $\mathbf{u}(t) \in D(\mathcal{A}_\beta^2)$ for all $t \geq 0$
(2) $\mathbf{u}(t)$ satisfies the equation of (P_0) for all $t \geq 0$.

Remark 6.4. If $\mathbf{u}$ is a classical solution, then integrating (P_0) twice shows that $\mathbf{u}$ is a mild solution. Conversely, if $\mathbf{u}$ is a mild solution and $\mathbf{u} \in C^2(\mathbb{R}_+, \mathcal{H}_c)$, then $\mathbf{u}$ is a classical solution. Remind that, a strongly continuous function $Cos : \mathbb{R}_+ \longrightarrow L(\mathcal{H}_c)$ is called a cosine function if $Cos(0) = I$ and $2Cos(t)Cos(s) = Cos(t+s) + Cos(t-s)$, $(t \geq s \geq 0)$.

In the following lemma we characterize cosine functions with bounded generators as follows.

Lemma 6.5. ([3] page 213.) *The following assertions are equivalent:*
(i) The operator $\mathcal{A}_\beta$ is bounded.
(ii) The operator $\mathcal{A}_\beta$ generates a cosine function Cos, such that

$$\lim_{t\downarrow 0} \|Cos(t) - I\| = 0.$$

Lemma 6.6. *If the operator $\mathcal{A}_\beta$ is the generator of a C_0-group of operators $(U(t))_{t\in\mathbb{R}}$ on $\mathcal{H}_c$, then $\mathcal{A}_\beta^2$ generates a cosine function Cos on $\mathcal{H}_c$ given by*

$$Cos(t) := \frac{1}{2}(U(t) + U(-t)), \; t \in \mathbb{R}.$$

The phase space is given by $\mathcal{H}_c \times \mathcal{H}_c$.
Indeed, we proved that the operator $\mathcal{C}_{\frac{1}{\beta}} : \mathcal{H}_c \longrightarrow \mathcal{H}_c$ is self adjoint. Then the operator $\mathcal{A}_\beta$ is skewadjoint, hence by Stone's Theorem is the generator of a C_0 unitary group of operators $(U(t))_{t\in\mathbb{R}}$ on the Hilbert space $\mathcal{H}_c$. From the example 3.14.15 in [3] we can verify that $\mathcal{A}_\beta^2$ generates a cosine function on $\mathcal{H}_c$ given by $Cos(t) := \frac{1}{2}(U(t) + U(-t)),\ t \in \mathbb{R}$.

Theorem 6.7. *The Cauchy problem* (P_0) *is classically well-posed in* $\mathcal{H}_c$.

Proof. From Lemma 6.6 we know that the operator $\mathcal{A}_\beta^2$ generates a cosine function Cos on $\mathcal{H}_c$ given by $Cos(t) := \frac{1}{2}(U(t) + U(-t)), \quad t \in \mathbb{R}$. Then from Corollary 3.14.8 in [3] we have that

$$\mathbf{u}(t) := Cos(t)\mathbf{u}_0 + \int_0^t Cos(s)\mathbf{u}_1 ds,\ t \geq 0$$

defines a unique mild solution of (P_0). Since $\mathbf{u}_1 \in D(\mathcal{A}) = \mathbf{H}_c$, then we have that $Cos(\cdot)\mathbf{u}_1 \in C^1(\mathbb{R}_+,\ \mathcal{H}_c)$, and hence $\int_0^t Cos(\cdot)\mathbf{u}_1 ds \in C^2(\mathbb{R}_+,\ \mathcal{H}_c)$. Since $\mathbf{u}_0 \in D(\mathcal{A}_\beta^2) = \mathbf{H}_c$, we have $Cos(\cdot)\mathbf{u}_0 \in C^2(\mathbb{R}_+,\ \mathcal{H}_c)$. It follows that $\mathbf{u} \in C^2(\mathbb{R}_+,\ \mathcal{H}_c)$. Therefore, since $\mathbf{u}$ is a unique mild solution and $\mathcal{A}_\beta^2$ is closed then $\mathbf{u}$ is a unique classical solution.

Remark 6.8. We note that the operator $\mathcal{A}_\beta^2$ is selfadjoint in $\mathcal{H}_c$ since $\mathcal{A}_\beta$ is skewadjoint, and therefore $\sigma(\mathcal{A}_\beta^2) \subset \mathbb{R}$. From lemma 6.6 we have that $\mathcal{A}_\beta^2$ generates a cosine function. Hence there exists $\omega > 0$ such that the spectrum $\sigma(\mathcal{A}_\beta^2)$ is contained in the parabola $\{x + iy : y^2 \leq -4\omega^2(x - \omega^2)\}$, and therefore $\sigma(\mathcal{A}_\beta^2) \subset \{x \in \mathbb{R} : x \leq \omega^2\}$. We observe that if λ is an eigenvalue of the operator *curl* then it is obvious that $\kappa = -[\frac{\lambda}{\sqrt{\varepsilon\mu}(1+\lambda\beta)}]^2,\ \lambda \neq -\frac{1}{\beta}$, is an eigenvalue of the operator $\mathcal{A}_\beta^2$ and $\kappa < \omega^2$ for every $\omega > 0$.Furthermore, since $\mathcal{A}_\beta^2$ is bounded then $|\frac{\lambda}{1+\lambda\beta}| \leq \|\mathcal{C}_{\frac{1}{\beta}}\|$ hence:

*(*1) If $\|\mathcal{C}_{\frac{1}{\beta}}\| < \frac{1}{\beta}$ then $\lambda \in (-\frac{1}{\beta},\ \frac{\|\mathcal{C}_{\frac{1}{\beta}}\|}{1-\beta\|\mathcal{C}_{\frac{1}{\beta}}\|}]$.

*(*2) If $\|\mathcal{C}_{\frac{1}{\beta}}\| > \frac{1}{\beta}$ then $\lambda \in (-\infty,\ \frac{\|\mathcal{C}_{\frac{1}{\beta}}\|}{1-\beta\|\mathcal{C}_{\frac{1}{\beta}}\|}] \cup [-\frac{\|\mathcal{C}_{\frac{1}{\beta}}\|}{1+\beta\|\mathcal{C}_{\frac{1}{\beta}}\|},\ +\infty)$.

Also the point spectrum $\sigma_\rho(curl) = \{\lambda_i\}_{i\in\mathbb{N}}$, and this can accumulate to both $-\infty$ and $+\infty$, if $\|\mathcal{C}_{\frac{1}{\beta}}\| > \frac{1}{\beta}$.

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References

1. W. Arendt, C. J. K. Batty *"Rank-1 perturbations of cosine functions and semigroups"*, J. Functional Anal. 238 (2006) 340-352.
2. W. Arendt, C. J. K. Batty *"Forms, Functional Calculus, Cosine Functions and Perturbation"*, in "Perspectives in Operator Theory" Banach Center Publ, 75, Polish Acad. Sci., Warsaw, 2007, 17-38.
3. W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander *"Vector-Valued Laplace Transforms and Cauchy Problems"*, Monographs in Mathematics Vol.96,Birkhauser Verlag,Basel,Boston,Berlin (2001).
4. C.Athanasiadis, G.F.Roach, I.G.Stratis *"A Time Domain analysis of wave motions in chiral materials"*, Math. Nachr., 250 (2003),3-16.
5. C. Athanasiadis, G. F. Roach *"Time-depent potential scattering in chiral media"*, J. Math. Anal. Appl., 310 (2005), 1-15.
6. H. Brezis *"Analyse Fonctionnelle - Théorie et Applications"*, Masson, Paris, (1983).
7. M. Cessenat *"Mathematical Methods in Electromagnetism"*, World Scientific, Singapore, (1996).
8. P.Ciarlet, G.Legendre *"Well posedness of the Drude-Born-Fedorov model for chiral media"*, Math. Models Meth.Appl.Sci., 17 (2007),461-484.
9. P. Courilleau, T. Horsin Molinaro, I. G. Stratis *"On the controllability of Maxwell's equationsin a class of complex media"*, Bull. Greek Math. Soc., 54, (2007), 97-113.
10. R. Dautray - J. L. Lions *"Mathematical Analysis and Numerical Methods for Science and Technology"*, Volume 2: "*Functional and Variational Methods*", Springer - Verlag, Berlin, Heidelberg, (1988).
11. R. Dautray - J. L. Lions *"Mathematical Analysis and Numerical Methods for Science and Technology"*, Volume 3: "*Spectral Theory and Applications*", Springer - Verlag, Berlin Heidelberg, (1990).
12. K.J. Engel - R. Nagel *"One-Parameter Semigroups for Linear Evolution Equation"*, Springer - Verlag, New York, (2000).
13. H. O. Fattorini *"Second Order Linear Differential Equations in Banach Spaces"*, Elsevier and Science Publishers B. V., Amsterdam (1985).
14. D. J. Frantzeskakis, A. Ioannidis, G. F. Roach, I. G. Stratis, A. N. Yannacopoulos *"On the error of the optical response approximation in chiral media"*, Appl. Anal. 82, (2003), 839-856.
15. J. A. Goldstein *"Semigroups of Linear Operators and Applications"*, Oxford University Press, (1985).
16. A. Karlsson, G. Krinstensson *"Constitutive relations,dissipation and reciprocity for the Maxwell equations in the time domain"*, LUDETX/(TEAT-7005),1-38,2000.
17. T. Kato *"Perturbation Theory for Linear Operators"*, Springer - Verlag, Berlin Heidelberg, New York, Tokyo (1984).
18. A. Lakhtakia *"A mini-review on isotropic chiral mediums",in "Electromagnetic Fields in Unconventional Materials and Structures" (eds. O. N. Singh, A. Lakhtakia)* Wiley, New York, (2000),125-149.
19. R. Leis *"Initial Boundary Value Problems in Mathematical Physics"*, B. G. Teubner,Stuttgart:John Willey and Sons, Chichester, (1986).
20. K. B. Liaskos, I. G. Stratis, A. N. Yannacopoulos *"Stochastic integrodifferent equations in Hilbert spaces with applications in electromagnetics"*, J. Integral Equations Appl., in press.
21. P. Monk *"Finite Element Methods for Maxwell's Equations"*, Oxford Science Publica-

tions, (2003).
22. D. Mugnolo *"Matrix methods for wave equations"*, Math. Z 253, (2006), 667-680.
23. F. Neubrander *"Well-Posedness of Higher order Abstract Cauchy Problems"*, Trans 295, 1, (1986), 257-290.
24. F. Neubrander *"Integrated semigroups and their application to complete second order Cauchy Problems"*, Semigroup Forum, 38 (1989), 233-251.
25. A. Pazy *"Semigroups of Linear Operators and Applications to Partial Differential Equations"*, Springer - Verlag, New York, (1983).
26. R.Picard *"On the spectrum of curl"*, Preprint Technisce Universitat Dresden,Math-Anal-04-95 (1995).
27. G. F. Roach *"An introduction to Echo Analysis:Scattering Theory and Wave Propagation"*, Springer Verlag,London,2008.
28. C. C. Travis, G. F. Webb *"Cosine Families and Abstract Nonlinear Second Order Differential Equations"*, Acta Mathematica Hungaricae 32, (1978), 75-96.
29. Z. Yoshida, Y. Giga *"Remarks on spectra of operator Rot"*, Math. Z.,204 (1990), 235-245.

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