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Homogeneous Linear Matrix Difference Equations Of Higher Order:Regular Case

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Abstract

Abstract. In this paper the solutions of the matrix difference equations (regular case) $A_n\bar{x}_{k+n}+A_{n-1}\bar{x}_{k+n-1}+...+A_1\bar{x}_{k+1}+A_0\bar{x}_k=\bar{0}$ are developed (for the case det$A_n\neq 0$ and detA_n=0) with $\bar{x}_{k_0}$,$\bar{x}_{k_0+1}$,...,$\bar{x}_{k_0+n-1}$ the initial conditions sequence.Moreover we provide a proof that solution sequence is unique for consistent initial conditions and infinite for non-consistent initial conditions.Numerical examples are given.

Keywords: Weierstrass cannonical form,Difference equations,Matrix pencil,Zeta transform

1. Introduction

Consider the following homogeneous constant-coefficient-matrix difference equation of order n given by:

$A_n\bar{x}_{k+n}+A_{n-1}\bar{x}_{k+n-1}+...+A_1\bar{x}_{k+1}+A_0\bar{x}_k=\bar{0}$ and $\bar{x}_{k_0+i}$, (i=0,1,2,...,n-1) are the initial conditions, such that $\bar{x}_{k_0+i}$=$\phi_i\in R^m$ at the k_0+ith sampling time point.

where $A_i\in R^{m\times m}$, $i=1,2,...,n$ and $\bar{x}_k=\begin{bmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_m^k \end{bmatrix}$, $k\geq k_0$, integer with $\bar{x}_k$=$\bar{x}(T_k)$, T is the sampling period.

In the sequel we adopt the following notations :

$\bar{y}_k^1=\bar{x}_k$, $\bar{y}_k^2=\bar{x}_{k+1}$, ... , $\bar{y}_k^{n-1}=\bar{x}_{k+n-2}$, $\bar{y}_k^n=\bar{x}_{k+n-1}$

then ,

$$\bar{y}_{k+1}^1=\bar{x}_{k+1}=\bar{y}_k^2$$
$$\bar{y}_{k+1}^2=\bar{x}_{k+2}=\bar{y}_k^3$$

$\vdots$
$\bar{y}_{k+1}^{n-1} = \bar{x}_{k+n-1} = \bar{y}_k^n$
$A_n \bar{y}_{k+1}^n = A_n \bar{x}_{k+n} = -A_{n-1}\bar{y}_k^n - ... - A_1\bar{y}_k^2 - A_0\bar{y}_k^1$

Or in matrix form ,

$$\begin{bmatrix} I_m & 0_m & ... & 0_m & 0_m \\ 0_m & I_m & ... & 0_m & 0_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_m & 0_m & ... & I_m & 0_m \\ 0_m & 0_m & ... & 0_m & A_n \end{bmatrix} \begin{bmatrix} \bar{y}_{k+1}^1 \\ \bar{y}_{k+1}^2 \\ ... \\ \bar{y}_{k+1}^{n-1} \\ \bar{y}_{k+1}^n \end{bmatrix} = \begin{bmatrix} 0_m & I_m & \dots & 0_m \\ 0_m & 0_m & \dots & 0_m \\ \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \dots & I_m \\ -A_0 & -A_1 & \dots & -A_{n-1} \end{bmatrix} \begin{bmatrix} \bar{y}_k^1 \\ \bar{y}_k^2 \\ ... \\ \bar{y}_k^{n-1} \\ \bar{y}_k^n \end{bmatrix}$$

Let

$$A= \begin{bmatrix} I_m & 0_m & ... & 0_m & 0_m \\ 0_m & I_m & ... & 0_m & 0_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_m & 0_m & ... & I_m & 0_m \\ 0_m & 0_m & ... & 0_m & A_n \end{bmatrix}, B= \begin{bmatrix} 0_m & I_m & \dots & 0_m \\ 0_m & 0_m & \dots & 0_m \\ \vdots & \vdots & \ddots & \vdots \\ 0_m & 0_m & \dots & I_m \\ -A_0 & -A_1 & \dots & -A_{n-1} \end{bmatrix}, \bar{y}_k= \begin{bmatrix} \bar{y}_k^1 \\ \bar{y}_k^2 \\ ... \\ \bar{y}_k^{n-1} \\ \bar{y}_k^n \end{bmatrix}$$

so the system takes the form $A\bar{y}_{k+1} = B\bar{y}_k$, $\bar{y}_{k_0}$ initial conditions.
Where A,B are $(m \cdot n) \times (m \cdot n)$ matrices and $\bar{y}_k^n$ is $(m \cdot n) \times 1$ column, matrix.

2. Case of regular pencil

Definition 2.1 *The pencil sA-B is called regular if A,B are sqare matrices and* $det(sA - B) \neq 0$
Proposition 2.1 [1] *Let the pencil sA-B be regular then there exist non singular matrices P,Q such that* $PAQ = A_w$ *and* $PAB = B_w$*.Where* $sA_w - B_w$ *is the Weierstrass canonical form of the pencil sA-B.*

If $(s - a_i)^{t_i}$ are the finite elementary divisors ($i = 1, 2, ..., v$, $t_1 + t_2 + ...t_v = p$) and $\bar{s}^{q_j}$ the infinite elementary divisors ($j = 1, 2, ..., l$, $q_1 + q_2 + ... + q_l = q$) of the pencil sA-B then:

$$A_w = \begin{bmatrix} I_p & O \\ 0 & H_q \end{bmatrix}, B_w = \begin{bmatrix} J_p & O \\ 0 & I_q \end{bmatrix}, \ p + q = m \cdot n$$

where H_q=blockdiag$\{H_{q_1}, H_{q_2}, ..., H_{q_l}\}$, J_p=blockdiag$\{J_{t_1}(a_1), J_{t_2}(a_2), ..., J_{t_v}(a_v)\}$

$$H_{q_j}=\begin{bmatrix} 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & 1 \\ 0 & 0 & .. & 0 & 0 \end{bmatrix}, \; J_{t_i}(a_i)=\begin{bmatrix} a_i & 1 & \dots & 0 & 0 \\ 0 & a_i & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_i & 1 \\ 0 & 0 & \dots & 0 & a_i \end{bmatrix}$$

Now assume $\bar{y}_k$=Q$\bar{z}_k$ then :

A$\bar{y}_{k+1}$=B$\bar{y}_k$ $\Leftrightarrow$ PAQ_{k+1}=PBQ$\bar{z}_k$ $\Leftrightarrow$A$_w\bar{z}_{k+1} = B_w\bar{z}_k$

Proposition 2.2 *Consider the system* $A_n\bar{x}_{k+n}+A_{n-1}\bar{x}_{k+n-1}+...+A_1\bar{x}_{k+1}+A_0\bar{x}_k = \bar{0}$ *where* $\bar{x}_j$ *,* $(j = k_0, k_0+1, k_0+2, ..., k_0+n-1)$ *are the initial conditions.If* $detA_n \neq 0$ *the system has unique solution.*

Proof If $detA_n \neq 0 \Leftrightarrow detA \neq 0$ (of system the $A\bar{y}_{k+1} = B\bar{y}_k$, $\bar{y}_{k_0}$ initial conditions) $\Leftrightarrow$ q=0 because there does not exist infinite elementary divisors.That means p=m$\cdot$ n and so we have ,
$A_w\bar{z}_{k+1} = B_w\bar{z}_k \Leftrightarrow$
$\bar{z}_{k+1} = J_p\bar{z}_k \Leftrightarrow \bar{z}_k = J_p^k\bar{z}_{k_0} \Leftrightarrow$ $\bar{y}_k$=Q$J_p^k\bar{z}_{k_0}k \geq k_0$ .The solution is unique.

Proposition 2.3 *Let the system* $A_n\bar{x}_{k+n} + A_{n-1}\bar{x}_{k+n-1} + ... + A_1\bar{x}_{k+1} + A_0\bar{x}_k = \bar{0}$ *and* $\bar{x}_j$ *,* $(j = k_0, k_0+1, k_0+2, ..., k_0+n-1)$ *are the initial conditions.If* $detA_n = 0$ *then the solution is unique if and only if the initial conditions are consistent.*

Proof If $detA_n = 0 \Leftrightarrow$ q$\neq$0.Then we define $\bar{z}_k$=$\begin{bmatrix} \bar{z}_k^p \\ \bar{z}_k^q \end{bmatrix}$ where $\bar{z}_k^p$ column px1 and $\bar{z}_k^q$ column qx1 and we get :

$A_w\bar{z}_{k+1} = B_w\bar{z}_k \Leftrightarrow$

$$\begin{bmatrix} I_p & O \\ 0 & H_q \end{bmatrix}\begin{bmatrix} \bar{z}_{k+1}^p \\ \bar{z}_{k+1}^q \end{bmatrix}=\begin{bmatrix} J_p & O \\ 0 & I_q \end{bmatrix}\begin{bmatrix} \bar{z}_k^p \\ \bar{z}_k^q \end{bmatrix}$$

from where we get two equations : $\bar{z}_{k+1}^p$=$J_p\bar{z}_k^p$ and $H_q\bar{z}_{k+1}^q$=$\bar{z}_k^q$

The first equation gives the solution $_k^p = J_{p\,k_0}^{kp}$, k$\geq$k$_0$. For the second equation we apply the Zeta transform.Assume that q_{max}=maxq_j and we have

$H_q\bar{z}_{k+1}^q$=$\bar{z}_k^q$ $\Leftrightarrow$ H_qZ$\{\bar{z}_{k+1}^q\}$=Z$\{\bar{z}_k^q\}$ $\Leftrightarrow$ $H_q\left[zw(z) - z^{1-k_0}\bar{z}_{k_0}\right] = w(z)$

where $w(z) = \sum_{k=k_0}^{\infty}\frac{\bar{z}_k^q}{z^k}$, is by definition the Zeta transform of $\bar{z}_k^q$

$(zH_q - I_q)w(z)$=$z^{1-k_0}H_q\bar{z}_{k_0}^q$.It is easy to show that $det(zH_q - I_q) \neq 0$ and that

$(zH_q - I_q)^{-1}$=$-\sum_{n=0}^{q_{max}-1}(zH_q)^n$, while $H_q^n = 0$ for n$\geq q_{max}$$\Leftrightarrow$

$w(z) = (zH_q - I_q)^{-1}z^{1-k_0}H_q\bar{z}_{k_0}^q \Leftrightarrow$
$w(z) = -\sum_{n=0}^{q_{max}-1}(zH_q)^n z^{1-k_0}H_q\bar{z}_{k_0}^q = -\sum_{n=0}^{q_{max}-1}(z)^{n+1-k_0}H_q^{n+1}\bar{z}_{k_0}^q \Leftrightarrow$
$w(z) = -\sum_{n=1}^{q_{max}-1}(z)^{n-k_0}H_q^n\bar{z}_{k_0}^q \Leftrightarrow Z^{-1}\{w(z)\} = -\sum_{n=1}^{q_{max}-1}Z^{-1}\left\{(z)^{n-k_0}\right\}H_q^n\bar{z}_{k_0}$
z$_k^q = -\sum_{n=1}^{q_{max}-1}\delta_{n+k-k_0}H_q^n\bar{z}_{k_0}^q = 0$, because $\delta_{n+k-k_0} = \begin{cases} 1 & for \quad k = k_0 - n \\ 0 & for \quad k \neq k_0 - n \end{cases} \Leftrightarrow$
z$_k^q = 0$ and $\bar{z}_k^p = J_p^k\bar{z}_{k_0}^p$, k$\geq$k$_0$
so if $Q = [Q_pQ_q]$then ,
$\bar{y}_k = Q\bar{z}_k = [Q_pQ_q]\begin{bmatrix} \bar{z}_k^p \\ \bar{z}_k^q \end{bmatrix} = Q_p\bar{z}_k^p + Q_q\bar{z}_k^q = Q_pJ_p^k\bar{z}_{k_0}^p$
solution that exists if and only if $\bar{y}_{k_0} = Q_p\bar{z}_{k_0}^p$.That means $\bar{y}_{k_0} \in colspanQ_p$. The columns of Q_p are the p eigenvectors of finite elementary divisors (eigenvalues) of the pencil sA-B.In that case the system has the unique solution:

$\bar{x}_k = Q_p^1J_p^k\bar{z}_{k_0}^p$
where Q_p^1 is defined as Q_p=$\begin{bmatrix} Q_p^1 \\ Q_p^2 \end{bmatrix}$,Q$_p(n\cdot m)\times p$ matrix,and Q_p^1 $m\times p$ matrix.

Proposition 2.4 *Let the system $A\bar{y}_{k+1} = B\bar{y}_k$, $\bar{y}_{k_0}$ initial conditions.If $\bar{y}_{k_0} \notin colspanQ_p$ the system has infinite solutions.*

Proof In general for the equation $A\bar{y}_{k+1} = B\bar{y}_k$ if $\bar{y}_{k_0} \notin$colspan$Q_p \Leftrightarrow \bar{z}_{k_0}^q \neq 0 \Leftrightarrow_{k_0} = Q_p\bar{z}_{k_0}^p + Q_q\bar{z}_{k_0}^q$.In that case for the equation $A\bar{y}_{k+1} = B\bar{y}_k$ we have k$\neq k_0$ because if k=k_0 then $A\bar{y}_1 = B\bar{y}_{k_0}$ and $\bar{z}_{k_0}^q$=0 which is a contradiction.The system then takes the folowing form,

$u_{k-k_0-1}A\bar{y}_{k+1} = B\bar{y}_k - \delta_{k-k_0}B\bar{y}_{k_0}$, $k \geq k_0$

where $u_{k-k_0-1} = \begin{cases} 1 & for \quad k \geq k_0 + 1 \\ 0 & for \quad k = k_0 \end{cases}$ and $\delta_{k-k_0} = \begin{cases} 1 & for \quad k = k_0 \\ 0 & for \quad k \neq k_0 \end{cases}$

The solution of the equation is $\bar{y}_k = \bar{y}_{k,1} + \bar{y}_{k,2}$ where $\bar{y}_{k,1}$ solution of $u_{k-k_0-1}A\bar{y}_{k+1} = B\bar{y}_k$ and $\bar{y}_{k,2}$ partial solution of $u_{k-k_0-1}A\bar{y}_{k+1} = B\bar{y}_k - \delta_{k-k_0}B\bar{y}_{k_0}$.Assume $\bar{y}_k$=Q$\bar{z}_k$ then :

$u_{k-k_0-1}A\bar{y}_{k+1}$=B$\bar{y}_k \Leftrightarrow u_{k-k_0-1}PAQ\bar{z}_{k+1}$=PBQ$\bar{z}_k$

detA=0$\Leftrightarrow$ $q$$\neq$0.Then we define $\bar{z}_k = \begin{bmatrix} \bar{z}_k^p \\ \bar{z}_k^q \end{bmatrix}$ where $\bar{z}_k^p$ column px1 and $\bar{z}_k^q$ column qx1 and we get :

u$_{k-k_0-1}A_w\bar{z}_{k+1} = B_w\bar{z}_k \Leftrightarrow$

$$u_{k-k_0-1}\begin{bmatrix} I_p & O \\ 0 & H_q \end{bmatrix}\begin{bmatrix} \bar{z}^p_{k+1} \\ \bar{z}^q_{k+1} \end{bmatrix} = \begin{bmatrix} J_p & O \\ 0 & I_q \end{bmatrix}\begin{bmatrix} \bar{z}^p_k \\ \bar{z}^q_k \end{bmatrix}$$

from where we get two equations : $u_{k-k_0-1}\bar{z}^p_{k+1}{=}J_p\bar{z}^p_k$ and $u_{k-k_0-1}H_q\bar{z}^q_{k+1}{=}\bar{z}^q_k$

The first equation gives the solution $\bar{z}^p_k = u_{k-k_0-1}Q_pJ^k_p\bar{c}$, $k \geq k_0$. For the second equation by applying the Zeta transform we get $\bar{z}^q_k = \bar{0}$.So if $Q = [Q_pQ_q]$then ,
$\bar{y}_k = Q\bar{z}_k = [Q_pQ_q]\begin{bmatrix} \bar{z}^p_k \\ \bar{z}^q_k \end{bmatrix} = Q_p\bar{z}^p_k + Q_q\bar{z}^q_k = u_{k-k_0-1}J^k_p, k \geq$k$_0$.

For the partial solution we take $\bar{y}_{k,2} = \delta_{k-k_0}\bar{y}_{k_0}$. Then ,

$\bar{y}_k = \bar{y}_{k,1} + \bar{y}_{k,2} \Leftrightarrow \bar{y}_k = u_{k-k_0-1}Q_pJ^k_p{+}\delta_{k-k_0}\bar{y}_{k_0}$, $k \geq k_0$ and the dimension of the solution vector space is p.

Proposition 2.5 *Let the system* $A_n\bar{x}_{k+n} + A_{n-1}\bar{x}_{k+n-1} + ... + A_1\bar{x}_{k+1} + A_0\bar{x}_k = \bar{0}$ $k > k_0 + n$ *then for non cosistent initial conditions* $\bar{x}_{k_0+i}$ *, (i=0,1,...,n-1) the system has infinite solutions.*

$A_n\bar{x}_{k+n} + A_{n-1}\bar{x}_{k+n-1} + ... + A_1\bar{x}_{k+1} + A_0\bar{x}_k = \bar{0}$, $k > k_0 - 1 \Leftrightarrow$

$\sum_{i=0}^{n} A_i\bar{x}_{k+i} = \bar{0}$, $k > k_0 + n \Leftrightarrow$

u$_{k-k_0-n}\sum_{i=1}^{n} A_i\bar{x}_{k+i} + A_0\bar{x}_k = A_0\sum_{i=0}^{n-1}\delta_{k-k_0-i}\bar{x}_{k_0+i}$.

u$_{k-k_0-n} = \begin{cases} 1 & for \quad k \geq k_0 + n \\ 0 & for \quad k < k_0 + n \end{cases}$, $\delta_{k-k_0-i} = \begin{cases} 1 & for \quad k = k_0 + i \\ 0 & for \quad k \neq k_0 + i \end{cases}$

The solution of the equation is: $\bar{x}_k = \bar{x}_{k,1} + \bar{x}_{k,2}$.
Where $\bar{x}_{k,1}$ is the solution of $u_{k-k_0-n}\sum_{i=1}^{n} A_i\bar{x}_{k+i} + A_0\bar{x}_k = \bar{0}$ and $\bar{x}_{k,2}$ is partial solution of $A_n\bar{x}_{k+n} + A_{n-1}\bar{x}_{k+n-1} + ... + A_1\bar{x}_{k+1} + A_0\bar{x}_k = \bar{0}$.Then:

$\bar{x}_{k,1} = u_{k-k_0-n}Q^1_pJ^k_p\bar{c}$ and $\bar{x}_{k,2} = \sum_{i=0}^{n-1}\delta_{k-k_0-i}\bar{x}_{k_0+i}$

$\bar{x}_k = u_{k-k_0-n}Q^1_pJ^k_p\bar{c} + \sum_{i=0}^{n-1}\delta_{k-k_0-i}\bar{x}_{k_0+i}$, $k \geq k_0$.

The dimension of the solution vector space seems to be p but if some of the given initial conditions are consistent the dimension of the space that describes the solutions can be reduced.

3. Example

Let $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}_{k+3} + \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \bar{x}_{k+2} + \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix} \bar{x}_{k+1} + \begin{bmatrix} 4 & -2 \\ -1 & -1 \end{bmatrix} \bar{x}_k = \bar{0}$

Where $\bar{x}_k = \begin{bmatrix} x_1^k \\ x_2^k \end{bmatrix}$. Then $\bar{y}_k^1 = \bar{x}_k$, $\bar{y}_k^2 = \bar{x}_{k+1}$, $\bar{y}_k^3 = \bar{x}_{k+2} \Leftrightarrow$

$\bar{y}_{k+1}^1 = \bar{x}_{k+1} = y_k^2$
$\bar{y}_{k+1}^2 = \bar{x}_{k+1} = y_k^3$
$A_3\bar{y}_{k+1}^3 = A_3\bar{x}_{k+3} = -A_2\bar{x}_{k+2} - A_1\bar{x}_{k+1} - A_0\bar{x}_k = -A_2\bar{y}_k^3 - A_1\bar{y}_k^2 - A_0\bar{y}_k^1$

Or in matrix form ,

$$\begin{bmatrix} I_2 & 0_2 & 0_2 \\ 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & A_3 \end{bmatrix} \begin{bmatrix} \bar{y}_{k+1}^1 \\ \bar{y}_{k+1}^2 \\ \bar{y}_{k+1}^3 \end{bmatrix} = \begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ -A_0 & -A_1 & -A_2 \end{bmatrix} \begin{bmatrix} \bar{y}_k^1 \\ \bar{y}_k^2 \\ \bar{y}_k^3 \end{bmatrix}.$$

Assume

$$\text{A}=\begin{bmatrix} I_2 & 0_2 & 0_2 \\ 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & A_3 \end{bmatrix}, \bar{y}_k = \begin{bmatrix} \bar{y}_k^1 \\ \bar{y}_k^2 \\ \bar{y}_k^3 \end{bmatrix}, \text{B}=\begin{bmatrix} 0_2 & I_2 & 0_2 \\ 0_2 & 0_2 & I_2 \\ -A_0 & -A_1 & -A_2 \end{bmatrix}.$$

Then

$$\text{sA-B=s}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4 & 2 & 2 & -3 & -2 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix}.$$

The pencil sA-B has the following invariants:
s-1,s-2,s-3 finite elementary divisors and $\bar{s}^3$ infinte elementary divisor of degree 3.
There exist matrices non singular P,Q such that

PAQ=A_w and PBQ=B_w.Where ,

$$A_w=\begin{bmatrix} I_3 & 0_3 \\ 0_3 & H_3 \end{bmatrix} \text{ and } B_w=\begin{bmatrix} J_3 & 0_3 \\ 0_3 & I_3 \end{bmatrix}.$$

Let $\bar{y}_k$=Q$\bar{z}_k$ then :
A$\bar{y}_{k+1}$=B$\bar{y}_k \Leftrightarrow PAQ_{k+1}$=PBQ$\bar{z}_k$, k=0,1,2,...
We define $\bar{z}_k = \begin{bmatrix} \bar{z}_k^p \\ \bar{z}_k^q \end{bmatrix}$ where $\bar{z}_k^p$ column px1 and $\bar{z}_k^q$ column qx1
and :
$A_w\bar{z}_{k+1}$=PBQ$\bar{z}_k \Leftrightarrow \begin{bmatrix} I_3 & O \\ 0 & H_3 \end{bmatrix} \begin{bmatrix} {}^{p}_{k+1} \\ {}^{q}_{k+1} \end{bmatrix} = \begin{bmatrix} J_3 & 0_3 \\ 0_3 & I_3 \end{bmatrix} \begin{bmatrix} \bar{z}_k^p \\ \bar{z}_k^q \end{bmatrix}$ from where we get:

$\bar{z}^p_{k+1} = J_3\bar{z}^p_k$.Where $J_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow J_3^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$.

Q=$[Q_pQ_q]$ The columns of Q_p are the eigenvectors of the eigenvalues 1,2,3 of sA-B

$\bar{q}_1 = \begin{bmatrix} 3 \\ -5 \\ 3 \\ -5 \\ 3 \\ -5 \end{bmatrix}$ is the eigenvector of 1,$\bar{q}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 4 \\ -4 \end{bmatrix}$ is the eigenvector of 2 and $\bar{q}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -3 \\ 9 \\ -9 \end{bmatrix}$ is the eigenvector of 3.That means $Q_p = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -1 & -1 \\ 3 & 2 & 3 \\ -5 & -2 & -3 \\ 3 & 4 & 9 \\ -5 & -4 & -9 \end{bmatrix}$.

Example3.1 Let the initial values of the system be

$\bar{x}_0=\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\bar{x}_1=\begin{bmatrix} -2 \\ 0 \end{bmatrix}$, $\bar{x}_2=\begin{bmatrix} -10 \\ 8 \end{bmatrix}$ and $\bar{y}_0 = \begin{bmatrix} 1 \\ -3 \\ -2 \\ 0 \\ -10 \\ 8 \end{bmatrix}$.

Then $\bar{y}_0 \in colspan\mathrm{Q}_p$ (consistent initial conditions) and the solution of the system is $\bar{y}_k = Q_pJ_3^k\bar{z}_0$
and by calculating $\bar{z}_0$ we get $\bar{y}_0 = Q_p\bar{z}_0$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \\ 0 \\ -10 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ -5 & -1 & -1 \\ 3 & 2 & 3 \\ -5 & -2 & -3 \\ 3 & 4 & 9 \\ -5 & -4 & -9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \Leftrightarrow \bar{z}_0 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

and the solution of the system is $\bar{y}_k = Q_pJ_3^k\bar{z}_0 \Leftrightarrow \bar{y}_k = \begin{bmatrix} 3-2^k-3^k \\ -5+2^k+3^k \\ 3-2^{k+1}-3^{k+1} \\ -5+2^{k+1}+3^{k+1} \\ 3-2^{k+2}-3^{k+2} \\ -5+2^{k+2}+3^{k+2} \end{bmatrix} \Leftrightarrow$

$$\bar{x}_k = \begin{bmatrix} 3-2^k-3^k \\ -5+2^k+3^k \end{bmatrix}$$

Example3.2 Let the initial values of the system be

$$\bar{x}_0=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \bar{x}_1=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \bar{x}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \bar{y}_0=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Then $\bar{y}_0 \notin colspan\mathrm{Q}_p \Rightarrow$ non consistent initial conditions.Then the solution is

$$\bar{x}_k = u_{k+3}Q_3^1 J_3^k \bar{c} + \sum_{i=0}^{2} \delta_{k+i}\bar{x}_{k+i} = u_{k+3}\begin{bmatrix} 3c_1+2^k c_2+3^k c_3 \\ -5c_1-2^k c_2-3^k c_3 \end{bmatrix} + \delta_{k-2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The dimension of the solution vector space is 3.

Example3.3 Let $\bar{x}_0 = \bar{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ be non-consistent initial conditions of the system and $\bar{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ consistent.Then the solution is

$$\bar{x}_k = \begin{bmatrix} 3c_1+2^k c_2+3^k c_3 \\ -5c_1-2^k c_2-3^k c_3 \end{bmatrix}, k \geq k_0.$$

By replacing the consistent initial value $\bar{x}_2$ we take $c_1 = -1$ and $c_2 = 2 - \frac{3}{4}c_3$ and the solution is

$$\bar{x}_k = \begin{bmatrix} -3+2^{k+1}+(3^k-\frac{3}{4}2^k)c_3 \\ 5-2^{k+1}+(\frac{3}{4}2^k-3^k)c_3 \end{bmatrix}, k \geq k_0.$$

The dimension of the solution vector space is 1.

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