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Transition of a competitive oligopoly from identical to differentiated goods

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Abstract

There are various studies of oligopoly models with identical goods and their evolution in (discrete) time when more firms enter the market (with capacity constraints) is well-investigated. In this work we aim to introduce the important element of product differentiation and show the changes in the dynamic behavior of these oligopolies.

Keywords: Oligopoly, Product differentiation, Maps

1. Introduction

This paper deals with oligopoly models in discrete time. Such models have attracted much attention in the last couple of decades due to the resulting dynamical phenomena. Even in a very simple duopoly setting one may observe instabilities and chaos [11]. In [1, 2], based on [10], we considered the continuous case as it provides mathematically a more natural predecessor to discrete time. However, discrete time captures better the essence of an oligopoly setting: Firms are usually not in a position to alter functionality factors and strategic behaviour continuously but rather at discrete-time intervals.

In [5] the authors introduced a nonlinear map firstly with a single firm in the market (a monopoly) and thereafter investigated the entry of one firm at a time. Under what circumstances a second, a third, and so forth, firm may enter successfully the market and what are the conditions that lead to a firm exit? What happens when the number of firms increases? Can one detect more than one equilibrium points and what is the asymptotic behavior of the oligopoly as $t \to \infty$? Such an analysis becomes tractable with the introduction of appropriate capacity constraints [12] which enables one to compare a restricted number of large firms with an infinite number of small firms [13, 14, 15]. In Section 2 we present these results for identical goods whereas in Section 3 we shift to differentiated goods in an artificial way. This is done simply because we wish to explore possible changes in the dynamics and ultimate behavior in time of the oligopolies under consideration. In the last Section we make our speculations and give the directions of our research to be submitted in the near future.

2. Nonlinear model for identical goods

A model with a linear demand function and constant marginal costs presents relatively simple global dynamics. Indeed, as was evident in the analysis in [5], even a (slightly) nonlinear map produces at most an endless two-period oscillation. The possible entry or exit of a firm depended mainly on its marginal cost. As was implied by the Theocharis Problem, an increasing number of firms destabilises the Cournot equilibrium point.

In this section we investigate more the phenomena behind the gradual increase of the number of firms within a market. In so doing, we work with an isoelastic price function and introduce the notion of a capacity constraint which proves to be essential in our analysis. Constant marginal costs imply potentially infinitely large firms being able to produce any desired quantity. This is unreasonable and can be rectified by the addition of certain capacity limits [8]. This enables one to compare the limited number of large firms to infinitely many small firms. The latter lie behind the idea of perfect competition [13, 14, 15]. The last element of the new model is the diversification between a short run and a long run analysis.

There are n firms in a market that sell an identical good. The consumers are assumed to maximise Cobb-Douglas utility functions and this results in the isoelastic price function

$$p(Q) = \frac{1}{Q}$$

where $Q = \sum_{i=1}^{n} q_i$ and $Q_i = Q - q_i$.

We introduce a parameter k_i for firm i which represents its capacity limit. The capacity limit can be equally viewed as the capital stock – the larger capital stock the better for any firm. Of course in the short run the capital stock cannot change. However, in the long run each firm should be in a position to optimise its capacity stock and produce accordingly. To begin we assume that the capital stock is given. The proposed short-run cost function is[1]

$$C_i = c\frac{k_i^2}{k_i - q_i}.$$

[1] This cost function stems from the assumption that firms operate using a technology represented by a constant elasticity of substitution (CES) function. This function is a generalisation of Cobb-Douglas functions. Capital rent and wage rates (labour) must be considered. These production and utility functions are fundamental in standard economic theory. For a proper derivation consult [13, 15].

This signifies that firms are only distinguished by their capacities.
Obviously

$$\lim_{q_i \to k_i} C_i = \infty \quad \text{and} \quad \lim_{q_i \to 0} C_i = ck_i.$$

Therefore, as intuitively expected, infinite cost is met when production approaches capacity – fixed costs increase with production capacity. The short term marginal cost is

$$\frac{\partial C_i}{\partial q_i} = c\frac{k_i^2}{(k_i - q_i)^2}$$

and

$$\left.\frac{\partial C_i}{\partial q_i}\right|_{q_i=0} = c,$$

that is, c is the marginal cost at zero production.

In order to find the optimal capacity limit that firm i can choose from, we calculate

$$\frac{\partial C_i}{\partial k_i} = c\frac{k_i(k_i - 2q_i)}{(k_i - q_i)^2} = 0$$

and find that

$$k_i = 2q_i.$$

Substituting this value into the short-run cost function,

$$C_i = c\frac{k_i^2}{k_i - q_i},$$

we obtain the long-run cost function,

$$\bar{C}_i = 4cq_i.$$

These assumptions can be summarised into the following sentences: In the long run a firm can determine its optimal capacity limit through $k_i = 2q_i$. However, in the short run, the firm does not have that luxury. Consequently, since there are different cost functions for short- and long-run firm operation, there are two reaction functions that determine the output level in either case.

Short run:

$$q_i(t+1) = R_i\left(Q_i(t)\right) := \begin{cases} k_i\frac{\sqrt{\frac{Q_i}{c}} - Q_i}{k_i + \sqrt{\frac{Q_i}{c}}}, & Q_i(t) \le \frac{1}{c} \\ \epsilon, & Q_i(t) > \frac{1}{c} \end{cases} \qquad i = 1, n. \tag{2.1}$$

Long run:

$$q_i(t+1) = \bar{R}_i\left(Q_i(t)\right) := \begin{cases} \frac{1}{2}\sqrt{\frac{Q_i}{c}} - Q_i, & Q_i(t) \le \frac{1}{4c} \\ \epsilon, & Q_i(t) > \frac{1}{4c} \end{cases} \qquad i = 1, n. \tag{2.2}$$

Evidently these two maps, (2.1) and (2.2), describe the adjustment process for each firm in the short run and in the long run, respectively. This means that in the short run the firms take the capital stock as given whereas in the long run they optimise the stock and actually set it at a size twice the expected output level. The main question at this point is how often the firms reinvest. This can be resolved by the introduction of an indicator function [15]. This rule indicates when a given firm (re)adjusts its capital stock. Yet we assign another function as well, which is relevant to the successful entry of more and more firms. In this way we naturally move towards perfect competition.

Suppose that capital has a fixed duration of T periods and that the firms enter or reinvest within regular intervals of m periods.
Indicator: A firm invests (redefines its capacity) in time period, t, if and only if

$$(t - mi)\text{mod}T = 0. \tag{2.3}$$

Therefore the iteration process is

$$\begin{aligned} q_i(t+1) &= \delta(i,t)\bar{R}_i\left(Q_i(t)\right) + \left(1-\delta(i,t)\right)R_i\left(Q_i(t)\right) \\ k_i(t+1) &= 2\delta(i,t)\bar{R}_i\left(Q_i(t)\right) + \left(1-\delta(i,t)\right)k_i. \end{aligned} \tag{2.4}$$

Note that $\delta(i,t)$ is one whenever (2.3) is met, otherwise it takes the value zero. In what follows we assign $c = 0.25$ and $\epsilon = 10^{-6}$. If we assumed $\epsilon = 0$, we would allow for the firms to choose zero production levels and consequently exit the market (zero is a stable equilibrium state). In order to guarantee that the given oligopoly shall remain an oligopoly of the same size, we do not allow for the firms to exit, but to produce quantities close to zero but not zero (hence, the assumption $\epsilon = 10^{-6}$).

At this point, parenthetically, it is interesting to note that the choice $\epsilon = 0$ does not affect, obviously, the dynamics of a stable system. However, a system that exhibits more chaotic dynamics or an erratic behavior might benefit from such a choice in the sense that it might reach a stable state by reducing the number of active firms, hence change the structure of the oligopoly[2]. We could say that by allowing firms to exit the oligopoly we introduce an automatic stabilisation factor which tries to bring the system to a stable state by reducing the active players in it. In a future paper we should investigate further the effect of the choice $\epsilon = 0$ to include moderate and highly unstable systems.

[2]In a real life scenario this could occur, for example, when an oligopoly selling an identical product would experience the exit of some firms simply because the market could become saturated. This reduction in the numbers of firms would urge the remaining ones to differentiate (further) their product and therefore obtain higher profits and remain active.

Our modus operandi (regarding the evolution of a market as new firms appear) is the following: We start with one firm which determines the appropriate production level, q_1, and the long-run capital stock, $k_1 = 2q_1$. Then, another firm enters the market. It takes q_1 as given, calculates its long-run best response, q_2, and sets $k_2 = 2q_2$. The duopoly runs the short-run process and eventually settles at an equilibrium. Whenever the capital is depleted, the duopolists redefine their stocks according to (2.3). A third firm enters. It uses $Q_3 = q_1 + q_2$ to calculate the long-run output level, q_3, and sets $k_3 = 2q_3$. The short-run iteration process takes place once again and the triopoly equilibrium is finally reached. The existing firms always run the long-run iteration when indicated. It must be obvious that this process proceeds in similar manner until the monopoly has become an oligopoly. As more and more firms enter the market we observe that they become very small and, depending on the values assigned for the parameters in (2.3) we obtain the respective dynamical phenomena. It should be pointed out that we do not take into account any tendencies to firm collusions, evolution of technology, growing demand, barrier policies for firm entry, advertisement etc [9]. Our only goal is to provide an economically solid framework that shows the evolution of an oligopoly to perfect competition without the immediate distabilisation of the Cournot equilibrium point.

In order to ease the calculations involved and obtain graphs of important information we have developed a symbolic package called *Microeconomica* within the *Mathematica* environment[3].

We begin our numerical simulations[4] by choosing the investment period, $T = 8$. In Figure 1 we plot fifteen firms which enter the monopoly one by one every $n = 5$ time periods. This means that the monopoly becomes a duopoly in $t = 5$, a triopoly in $t = 10$ and so on. These firms reinvest every time period T again one at a time. We observe a considerable reclassification of all firms (according to their productivity levels) until, at some point, they reach the desired equilibrium state. At the equilibrium point all firms are obviously of identical size. It should be noted that we could run the same process with no matter how many firms; still we would obtain the same result. The question is: which is a sufficiently long investment period for the firms to converge to the situation depicted in Figure 1?

In our next simulation we take $T = 6$ and $m = 5$. This leads to a dramatic change from the previous situation. As can be surmised from all figures, the durability of capital, T, is the stabilising factor. In this case (see Figure 2) we do not allow sufficiently long

[3] *Microeconomica* is available through Dr Dimas.

[4] Figures 1, 2 and 3 are in the $(t,\, q_i)$-plane, Figure 4 in the $(t,\, u_i)$-plane whereas Figures 5 and 6 present a combination of such planes.

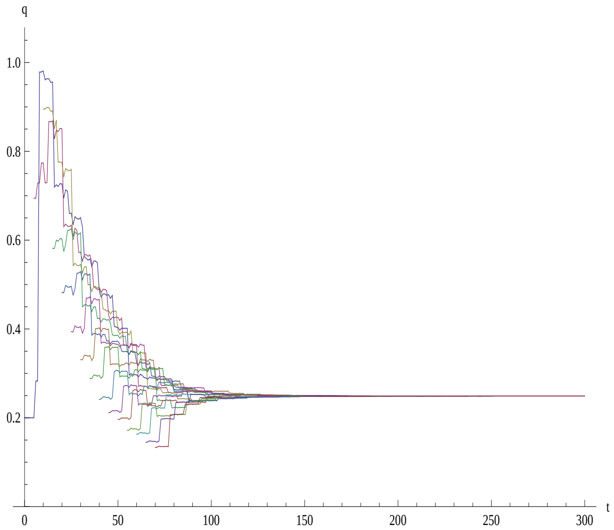

Figure 1: Fifteen firms which reinvest with $T = 8$ and $m = 5$ (in the $(t,\, q_i)$-plane).

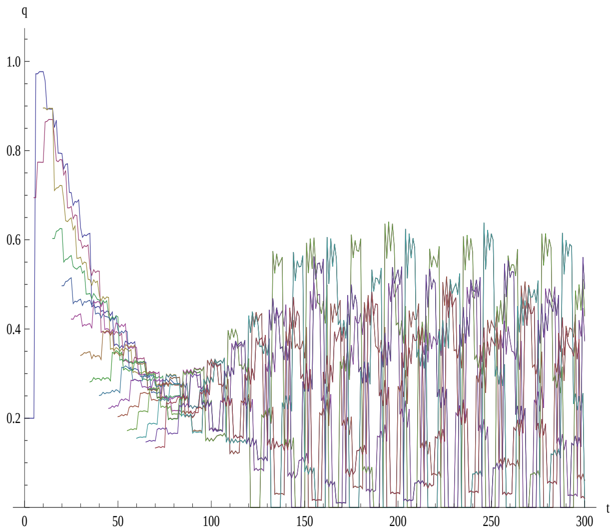

Figure 2: Fifteen firms with $T = 6$ and $m = 5$ (in the $(t,\, q_i)$-plane).

time, m, for the firms (fifteen in total) to reach some equilibrium state, since new firms enter quite frequently and reinvest very soon (some, in fact, exit the market). We note the groupwise synchronisation and the irregularity in the dynamics.

The sensitivity in T should be by now quite obvious. In Figure 3 we plot the situation with fifteen firms and $T = 7$ and $m = 5$. We observe a time series close to convergence but with endless oscillations. However, as pointed out above (and analytically in [15]), if we make $T = 8$, then the periodic solution reaches an equilibrium and the time series becomes convergent.

What is not revealed in [15] is the respective graphs for the profits, $u_i = pq_i - C_i$. These are depicted in Figure 4.

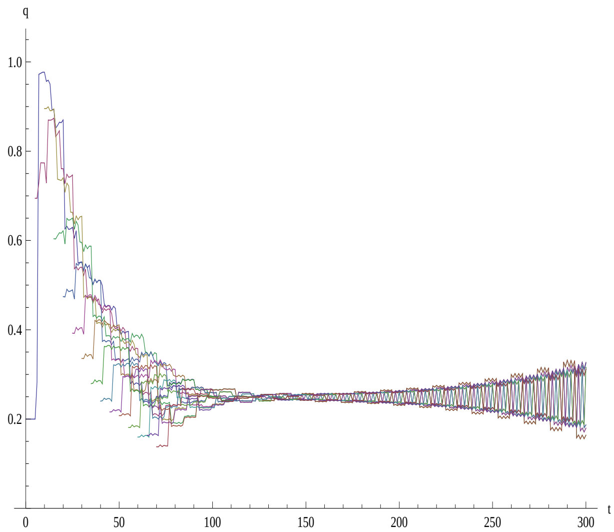

Figure 3: Fifteen firms with $T = 7$ and $m = 5$ (in the (t, q_i)-plane).

These three graphs show something which should have been anticipated but was not evident from an inspection of Figures 1, 2 and 3: The firms obtain profits close to zero. Especially in the case of identical firms and perfect competition, $T = 8$, the firms achieve an equilibrium state where each reaches the same production level for the same price. Therefore it is not unreasonable that the price attains a value close to the marginal cost with a direct consequence of almost zero profits.

3. Shifting to differentiated goods

Such phenomena, as outlined in the previous section, urge the firms in a market to differentiate their status, achieve higher profits and stand out from a situation

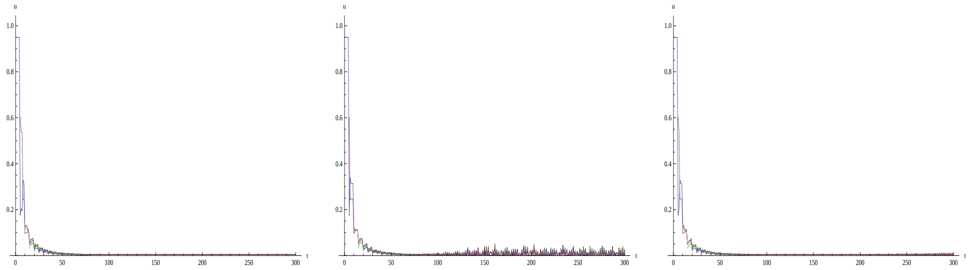

Figure 4: The profits ($(t,\, u_i)$-plane) of all fifteen firms for $T = 8$ (left), $T = 6$ (middle) and $T = 7$ (right) and $m = 5$.

of perfect competition. A straightforward way is to differentiate, even slightly, the product produced and sold. In reality a firm wishing to differentiate its product would firstly determine if such a move would indeed increase its profits. Otherwise it is apparent that such a differentiation would not take place because it would not be to the firm's benefit to alter its current status. Once this is established, the oligopoly would transform to an oligopoly of differentiated goods and it is quite probable that the other firms will react accordingly.

The degree of differentiation enters with the parameters, θ_i, as in [10, 3]. These parameters reflect how different the two products are (or, similarly, how much each firm takes into account the actions of the rival firms). It is therefore obvious that this differentiation enters, under the respective assumptions followed by each study, in the cost functions as well. However, these assumptions must necessarily fit the empirical data collected in experiments. With no such data we proceed in an artificial study: We consider price and cost functions that have the parameters θ_i with pre-assigned values. That means that the firms do not determine the optimal differentiation level by maximising the respective equations. These values are inserted by hand and we remark upon the resulting dynamics.

We proceed to a two-fold study: We examine a duopoly-triopoly setting both with identical and differentiated products. This is done in order to detect possible economic reasons for the firms to indeed wish to differentiate their products.

In Figure 5 we begin with a duopoly selling identical goods. After $m = 5$ time steps a third firm enters the market and produces the same good. Every $T = 9$ time steps each firm redefines its capacity limit according to (2.4) and (2.3). The graph does not add any more information than Figure 1 as it is essentially the same – the only difference is the number of firms and the value of T for the simulations.

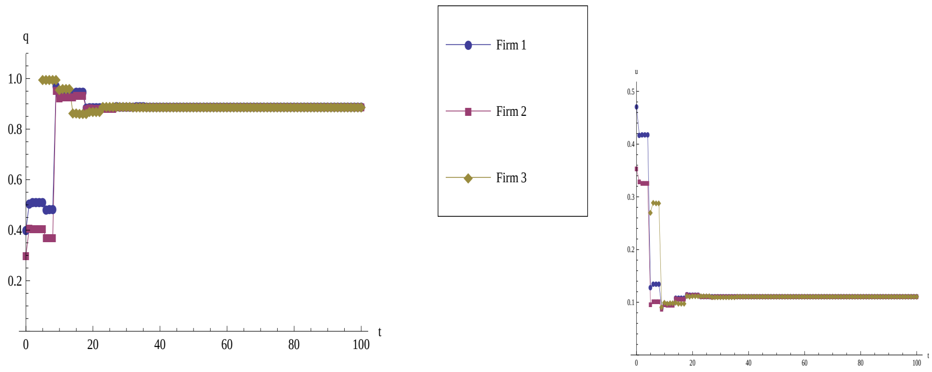

Figure 5: Entry of a third firm in a duopoly (of identical goods) with $T = 9$ and $m = 5$. Left: The (t, q_i)-plane. Right: The (t, u_i)-plane.

What is interesting is to see what happens when the third firm that enters the existing duopoly differentiates its product: The duopoly evolves as before and at $t = 5$ the price functions, from $P = 1/Q$, become

$$p_1 = \frac{1}{q_1 + q_2 + \theta_1 q_3}, \qquad p_2 = \frac{1}{q_1 + q_2 + \theta_2 q_3} \quad \text{and} \quad p_3 = \frac{1}{\theta_{31} q_1 + \theta_{32} q_2 + q_3}.$$

To keep things simple we assume that $\theta_1 = \theta_2$ (the initial duopoly produces the same identical good and behaves in an identical manner 'against' the new firm) and $\theta_{31} = \theta_{32} = \theta_3$ (the new firm does not view any of the existing firms differently). For our simulations, shown in Figure 6, we have chosen $\theta_1 = 0.96$ and $\theta_3 = 0.9$.

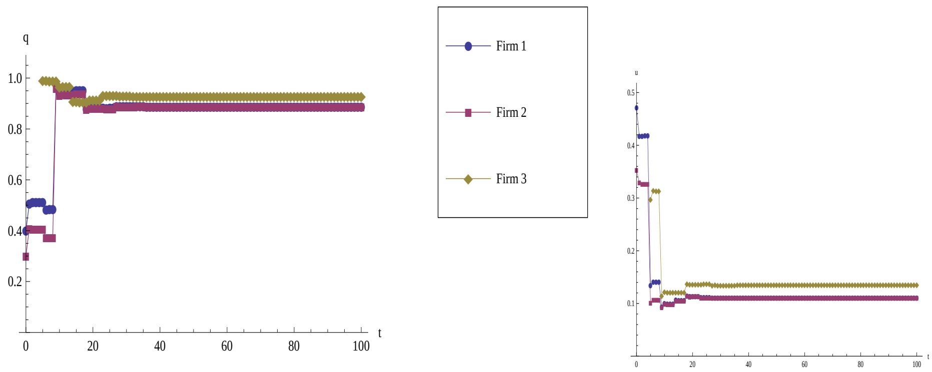

Figure 6: Entry of a third firm in a duopoly (with a differentiated good) with $T = 9$ and $m = 5$ (and $\theta_1 = 0.9$). Left: The (t, q_i)-plane. Right: The (t, u_i)-plane.

As becomes obvious from a comparison of Figures 5 and 6, when the third firm chooses to differentiate its product, it achieves higher profits than the two other firms. This is an indication of the importance towards product differentiation.

4. Concluding remarks

In this paper we examined the entry of a third firm in an existing duopoly that behaves in a Cournot-like [6, 7] manner. We proposed that the entrant will eventually differentiate its product in order to achieve higher profits. It may be argued that the existing duopoly can in fact guard itself from the entrant. This is a more complex situation and is discussed in [4].

Problematic as it may be, lacking the necessary empirical data to justify any choice of price and cost functions for oligopolies of differentiated goods, our simple and even fictitious study shows clearly that firms definitely wish, based on solid economic reasonance, to differentiate their products. In this way firms following similar strategies deviate from markets of perfect competition and form pure oligopolistic markets.
We stressed solely the static approach, i.e. the differentiation of the goods was predetermined from the start and remained constant throughout the study. The next step in our research is to implement the dynamic differentiation of goods via a suitable function that will determine on specific time intervals the differentiating factor for each firm. In order to achieve that we need to construct more realistic models for the determination of the differentiating function for each firm. During those time intervals the firms reevaluate their differentiating factor by maximising their profits as dictated by the respective differentiating function. The results, as pointed out above, are indicative of the further analysis that needs to be carried out in the future. In [10] the authors have provided analytic results to determine the stability or otherwise of the equilibrium points for certain duopoly models. We are currently working towards the resolution of these aspects for our multifirm setting and intend to submit this work in a separate article. In a multifirm model as described above the resulting dynamics is, clearly, richer and far more complex as it simulates a realistic scenario. The main obstacle for this endeavor is obviously the determination of the suitable differentiating functions and their dependence and compatibility with the dynamical system.

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