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Lie algebra automorphisms as Lie point symmetries and the solution space for some vacuum Bianchi spacetime geometries

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Abstract

Lie group symmetry analysis for systems of coupled, nonlinear ordinary differential equations is performed in order to obtain the entire solution space to Einstein's field equations for certain vacuum Bianchi spacetime geometries. The symmetries used are the automorphisms of the Lie algebra of the corresponding three-dimensional isometry group acting on the hyper-surfaces of simultaneity for each Bianchi Type, as well as the scaling and the time reparametrization symmetry. The method is applied to Bianchi Types I and II.

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1. Introduction

Automorphisms have been identified as a tool of a unified treatment for Bianchi geometries since the early 1960Us [1]. In 1979, Harvey [2] found the automorphisms of all three-dimensional Lie algebras, while the corresponding results for the four-dimensional Lie algebras have been presented in [3]. In Jantzen's tangent space approach the automorphism matrices are considered as the means for achieving a convenient parametrization of a full scale factor matrix in terms of a desired, diagonal matrix [4, 5, 6]. Siklos used these time-dependent automorphisms as a tool for the proper choice of variables aiming at a simplification of the ensuing equations [7], while Samuel and Ashtekar were the first to look upon automorphisms from a space viewpoint [8]. The notion of *Time-Dependent Automorphism Inducing Diffeomorphisms*, i.e., coordinate transformations mixing space and time in the new spatial coordinates and inducing automorphic motions on the scale-factor matrix, the lapse, and the shift vector, has been developed in [9]. The use of these covariances enables one to set the shift vector to zero without destroying manifest spatial homogeneity. At this stage one can use the “rigid” automorphisms, i.e., the remaining “gauge” symmetry, as Lie point symmetries of Einstein’s field equations in order to reduce their order and ultimately completely integrate them [10]. For a detailed discussion of the relevant theory see, e.g., [11], [12].

The paper is organized as follows: in Section 2, we give our method. In Sections 3-4, the application of the method to Bianchi Types I, and II is presented. Conclusions are presented Section 5.

2. The Method

It is known, that the line element for spatially homogeneous spacetime geometries with a simply transitive action of the corresponding isometry group [13], [14], assumes the form

$$ds^2 = \left(N^\alpha N_\alpha - N^2\right) dt^2 + 2N_\alpha \sigma^\alpha_i dx^i dt + \gamma_{\alpha\beta}\sigma^\alpha_i \sigma^\beta_j dx^i dx^j \tag{1}$$

$$d\boldsymbol{\sigma}^\alpha = C^\alpha_{\beta\gamma}\boldsymbol{\sigma}^\beta \wedge \boldsymbol{\sigma}^\gamma \Leftrightarrow \sigma^\alpha_{i,j} - \sigma^\alpha_{j,i} = 2C^\alpha_{\beta\gamma}\sigma^\gamma_i \sigma^\beta_j . \tag{2}$$

(Small Latin letters denote world space indices while small Greek letters count the different basis one-forms).

Subsequently, Einstein’s field equations are given in terms of the extrinsic curvature $K_{\alpha\beta}$ and the Ricci tensor $\mathbf{R}_{\alpha\beta}$ of the hypersurface of simultaneity [9]:

$$E_o \doteq K^{\alpha\beta}K_{\alpha\beta} - K^2 - \mathbf{R} = 0 \tag{3}$$

$$E_\alpha \doteq K^\mu_\alpha C^\epsilon_{\mu\epsilon} - K^\mu_\epsilon C^\epsilon_{\alpha\mu} = 0 \tag{4}$$

$$\begin{aligned} E_{\alpha\beta} \doteq \dot{K}_{\alpha\beta} + N\left(2K^\tau_\alpha K_{\tau\beta} - KK_{\alpha\beta}\right) + \\ + 2N^\rho\left(K_{\alpha\nu}C^\nu_{\beta\rho} + K_{\beta\nu}C^\nu_{\alpha\rho}\right) - N\mathbf{R}_{\alpha\beta} = 0 \end{aligned} \tag{5}$$

$$K_{\alpha\beta} = -\frac{1}{2N}\left(\dot{\gamma}_{\alpha\beta} + 2\gamma_{\alpha\nu}C^\nu_{\beta\rho}N^\rho + 2\gamma_{\beta\nu}C^\nu_{\alpha\rho}N^\rho\right) \tag{6}$$

$$\begin{aligned} \mathbf{R}_{\alpha\beta} = \quad & C^\kappa_{\sigma\tau}C^\lambda_{\mu\nu}\gamma_{\kappa\alpha}\gamma_{\beta\lambda}\gamma^{\sigma\nu}\gamma^{\tau\mu} + 2C^\kappa_{\lambda\beta}C^\lambda_{\alpha\kappa} + 2C^\mu_{\kappa\alpha}C^\nu_{\beta\lambda}\gamma_{\mu\nu}\gamma^{\kappa\lambda} + \\ & 2C^\lambda_{\kappa\beta}C^\mu_{\mu\nu}\gamma_{\alpha\lambda}\gamma^{\kappa\nu} + 2C^\lambda_{\kappa\alpha}C^\mu_{\mu\nu}\gamma_{\beta\lambda}\gamma^{\kappa\nu}. \end{aligned} \tag{7}$$

In [9], particular spacetime coordinate transformations have been found, which reveal as symmetries of (3), (4), and (5) the following induced transformations of the dependent variables N, N_α, $\gamma_{\alpha\beta}$:

$$\tilde{N} = N, \quad \tilde{N}_\alpha = \Lambda^\rho_\alpha\left(N_\rho + \gamma_{\rho\sigma}P^\sigma\right), \quad \tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \gamma_{\alpha\beta} \tag{8}$$

where the matrix Λ and the triplet P^α must satisfy the integrability conditions:

$$\Lambda^\alpha_\rho C^\rho_{\beta\gamma} = C^\alpha_{\mu\nu}\Lambda^\mu_\alpha \Lambda^\nu_\beta \tag{9a}$$

$$2P^\mu C^\alpha_{\mu\nu}\Lambda^\nu_\beta = \dot{\Lambda}^\alpha_\beta . \tag{9b}$$

For all Bianchi Types, this system of equations admits solutions that contain three arbitrary functions of time plus several constants depending on the automorphism group of each Bianchi Type. The three functions of time are distributed among Λ and P^α (which also contains derivatives of these functions). So one can use this freedom either to simplify the form of the scale factor matrix or to set the shift vector N^α to zero. The second action can always be taken, since, for every Bianchi Type, all three functions appear in P^α. In this work we adopt the latter point of view. Having used the freedom stemming from the three arbitrary functions in order to set the shift vector to zero, there is still a remaining "gauge" freedom consisting of constant Λ^α_β (automorphisms of the Lie algebra defined by $C^\alpha_{\beta\gamma}$'s). Indeed, the system (9) accepts the solution $\Lambda^\alpha_\beta = \text{const.}$, $P^\alpha = \mathbf{0}$ ("Rigid" symmetries [15]).

The generators of the corresponding motions $\tilde{\gamma}_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu \gamma_{\alpha\beta}$ induced in the space of the dependent variables $\gamma_{\alpha\beta}$'s are:

$$X_I = \lambda^\rho_{I\alpha} \gamma_{\rho\beta} \frac{\partial}{\partial\gamma_{\alpha\beta}} \tag{10}$$

$$\lambda^\alpha_{I\rho} C^\rho_{\beta\gamma} = \lambda^\rho_{I\beta} C^\alpha_{\rho\gamma} + \lambda^\rho_{I\gamma} C^\alpha_{\beta\rho}. \tag{11}$$

These generators define a symmetry Lie group of motions (with an associated Lie algebra C^I_{JK}) on the configuration space spanned by the $\gamma_{\alpha\beta}$'s. If a generator is brought to its normal form (i.e., $\frac{\partial}{\partial z_i}$), then the Einstein field equations, written in terms of the new dependent variables, will not explicitly involve z_i. They thus become a first order system in the function $\dot{z}_i$ [17]. If the aforesaid Lie algebra is abelian, then all generators can be brought to their normal form simultaneously. If the Lie algebra is non-abelian, then we can diagonalize in one step those generators corresponding to any eventual abelian subgroup. The rest of the generators (not brought in their normal form) continue to define a symmetry of the reduced system of the field equations if the Lie algebra of the $X_{(I)}$'s is solvable [11]. One can thus repeat the previous step by choosing one of these remaining generators and bring it to its normal form. This choice will of course depend upon the simplifications brought to the system at the previous level. Finally, if the Lie algebra does not contain any abelian subgroup, one can always choose one of the generators, bring it to its normal form, reduce the system of equations, and search for its symmetries (if there are any). Lastly, two further symmetries of (3), (4), and (5) are also present and can be used in conjunction with the constant automorphisms: The time reparametrization $t \to t + \alpha$, owing to the non-appearance of time in these equations (the system being autonomous), and the scaling by a constant $\gamma_{\alpha\beta} \to \lambda\gamma_{\alpha\beta}$ (homothety) as can be straightforwardly verified. Hence, in every Bianchi Type there are, added to the $X_{(I)}$ generators, also the following generators:

$$Y_1 = \frac{\partial}{\partial t} \tag{12}$$

$$Y_2 = \gamma_{11}\frac{\partial}{\partial\gamma_{11}} + \gamma_{12}\frac{\partial}{\partial\gamma_{12}} + \gamma_{13}\frac{\partial}{\partial\gamma_{13}} + \gamma_{22}\frac{\partial}{\partial\gamma_{22}} + \gamma_{23}\frac{\partial}{\partial\gamma_{23}} + \gamma_{33}\frac{\partial}{\partial\gamma_{33}} \tag{13}$$

These generators commute with each other, as well as with the $X_{(I)}$'s:

$$[X_I, Y_\alpha] = 0 \qquad \{I = 1,2,3,4 \,|\, \alpha = 1,2\} \tag{14}$$

3. Application of the method

3.1. Bianchi Type I

The Bianchi Type I model is characterized by the following structure constants, basis 1-forms and Killing fields:

$$C^\alpha{}_{\beta\gamma} = 0 \qquad \text{for every value of} \quad \alpha,\, \beta,\, \gamma \tag{15}$$

$$\boldsymbol{\sigma}^1 = \mathbf{d}\,x, \quad \boldsymbol{\sigma}^2 = \mathbf{d}\,y, \quad \boldsymbol{\sigma}^3 = \mathbf{d}\,z \tag{16}$$

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z \tag{17}$$

From equations (9) we have that the triplet $P^\alpha(t)$ is arbitrary while the automorphisms $\Lambda^\alpha_\beta \in GL_3(\mathbb{R})$ are constants. Hence, the symmetry group contains the nine prameters of $GL_3(\mathbb{R})$ plus the shift parameter corresponding to the generator (12). Using the triplet $P^\alpha(t)$, we can set the shift vector N^α to zero by applying (8). Furthermore, in order to simplify the field equations, we choose the time gauge $N^2 = \gamma$, $\gamma = \det(\gamma_{\alpha\beta})$. In this case, eq. (4) is satisfied identically, while equations (3), (5) take the form

$$\gamma^{\mu\alpha}\,\gamma^{\nu\beta}\,\dot\gamma_{\alpha\beta}\,\dot\gamma_{\mu\nu} - \gamma^{\alpha\beta}\,\gamma^{\mu\nu}\,\dot\gamma_{\alpha\beta}\,\dot\gamma_{\mu\nu} = 0 \tag{18}$$

$$\ddot\gamma_{\alpha\beta} - \gamma^{\rho\tau}\,\dot\gamma_{\rho\alpha}\,\dot\gamma_{\tau\beta} = 0 \tag{19}$$

The last equation can be integrated if multiplied by $\gamma^{\alpha\sigma}$:

$$\gamma^{\alpha\sigma}\,\dot\gamma_{\alpha\beta} = \vartheta^\sigma{}_\beta, \quad \vartheta^\sigma{}_\beta = \text{const.} \tag{20}$$

Multiplying the latter by $\gamma_{\sigma\rho}$ we obtain the linear system

$$\dot\gamma_{\beta\rho} = \vartheta^\sigma{}_\beta\,\gamma_{\sigma\rho} \quad \text{or, in matrix form,} \quad \dot\gamma = \vartheta^T\,\gamma \tag{21}$$

Thus, (18) becomes the following restriction on $\vartheta^\alpha{}_\beta$'s:

$$\vartheta^\alpha{}_\beta\vartheta^\beta{}_\alpha - (\vartheta^\alpha{}_\alpha)^2 = 0 \tag{22}$$

The general solution of the linear system (21) is of the form

$$\gamma_{\alpha\beta} = \exp\left(\vartheta^T\,t\right)^\rho{}_\alpha\,c_{\rho\beta}, \quad \exp\left(\vartheta^T\,t\right)^\rho{}_\alpha\,c_{\rho\beta} = \exp\left(\vartheta^T\,t\right)^\rho{}_\beta\,c_{\rho\alpha}, \quad c_{\alpha\beta} = \text{const.} \tag{23}$$

In order to find the analytic form of this solution, it is necessary to calculate the matrix exponential of the 3×3 matrix ϑ. To this purpose, we first note that the metric $\gamma_{\alpha\beta}$ transforms, by using $GL_3(\mathbb{R})$, as

$$\gamma_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\,\Lambda^{\nu}{}_{\beta}\,\widetilde{\gamma}_{\mu\nu} \Rightarrow \gamma = \Lambda^T\,\widetilde{\gamma}\,\Lambda, \quad \Lambda \in GL_3(\mathbb{R}) \tag{24}$$

so we have for the linear system (21)

$$\Lambda^T\,\dot{\widetilde{\gamma}}\,\Lambda = \vartheta^T\,\Lambda^T\,\widetilde{\gamma}\,\Lambda \Rightarrow \dot{\widetilde{\gamma}} = \widetilde{\vartheta}\,\widetilde{\gamma}, \qquad \widetilde{\vartheta} = \Lambda^{-T}\,\vartheta^T\,\Lambda^T \tag{25}$$

Now, Λ is an arbitrary matrix, hence it can be used to simplify the matrix ϑ. The degree of simplification depends on the eigenvalues of the matrix ϑ which may be (i) three real, distinct eigenvalues, (ii) one real and two complex conjugate eigenvalues, (iii) three real eigenvalues two of which are repeated and, (iv) three real, repeated eigenvalues. If the matrix ϑ is the zero matrix, then $\gamma_{\alpha\beta} =$ const. and the metric thus obtained describes Minkowski flat spacetime .

3.2. **Three real distinct eigenvalues**

In this case, we can choose the matrix Λ in such a way as to diagonalize the matrix ϑ, i.e.

$$\widetilde{\vartheta} = diag(p_1, p_2, p_3) \Rightarrow \exp\widetilde{\vartheta}\,t = diag(e^{p_1\,t}, e^{p_2\,t}, e^{p_3\,t}). \tag{26}$$

with the constants (p_1, p_2, p_3) related through (22), i.e.

$$p_1^2 + p_2^2 + p_3^2 = (p_1 + p_2 + p_3)^2 \Rightarrow p_1\,p_2 + p_1\,p_3 + p_2\,p_3 = 0 \tag{27}$$

Since the constants (p_1, p_2, p_3) are different, none of them vanishes. Thus, dividing by, let's say, p_1^2, we can eliminate it, so

$$\frac{p_2}{p_1} + \frac{p_3}{p_1} + \frac{p_2}{p_1}\frac{p_3}{p_1} = 0 \Rightarrow \alpha + \beta + \alpha\,\beta = 0, \quad \alpha = \frac{p_2}{p_1},\, \beta = \frac{p_3}{p_1} \tag{28}$$

Making full use of (23) and a final scaling of (x, y, z) we can set $c_{\alpha\beta} = p_1^{-2} diag(1, 1, 1)$. The change $\tau = p_1\,t$, results in the following final form of the metric:

$$d\,s^2 = -e^{(1+\alpha+\beta)\,\tau}\,d\,\tau^2 + e^{\tau}\,d\,x^2 + e^{\alpha\,\tau}\,d\,y^2 + e^{\beta\,\tau}\,d\,z^2 \tag{29}$$

where (α, β) satisfy (28). p_1^{-2} does not appear in (29) due to the homothety field

$$H = 2\,(\alpha+1)\,\partial_\tau + \alpha^2\,x\,\partial_x + y\,\partial_y + (\alpha+1)^2\,z\,\partial_z. \tag{30}$$

The above metric was first given, although in a different form, by Kasner [18]. The metric is particularly interesting for the values $(\alpha, \beta) = (1, -1/2)$ or $(\alpha, \beta) = (-1/2, 1)$. In addition to the three Killing fields (17), there is now a fourth one in the form

$$\xi_4 = y\,\partial_x - x\,\partial_y. \tag{31}$$

The non permitted pair of values $(\alpha, \beta) = (0, 0)$ leads the metric (29) to its Minkowski flat form, so we can include these values into the domain of the constants (α, β).

3.3. One real and two complex conjugate eigenvalues

In this case, the matrix ϑ cannot be diagonalized, but we can choose the matrix Λ in such a way as to bring the matrix ϑ into its "rational normal form", i.e.,

$$\widetilde{\vartheta} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -(p_2^2 + p_3^2) & 2\,p_2 \end{pmatrix} \tag{32}$$

with eigenvalues $(p_1, p_2 + \mathrm{i}\,p_3, p_2 - \mathrm{i}\,p_3)$; (p_1, p_2, p_3) are related through (22):

$$2\,p_1\,p_2 + p_2^2 + p_3^2 = 0 \;\Rightarrow\; \beta = -\frac{2\,\lambda}{\lambda^2+1}, \quad \beta \equiv \frac{p_3}{p_1}, \quad \lambda \equiv \frac{p_2}{p_3} \tag{33}$$

Full use of (23) and the freedom of using the automorphisms result, after the transformation $\tau = \beta\,p_1\,t$, in the following metric:

$$\begin{aligned} d\,s^2 = e^{(2\,\lambda+\beta^{-1})\,\tau}\,d\,\tau^2 + e^{\tau/\beta}\,d\,x^2 - e^{\lambda\,\tau}\,\sin\tau\,d\,y^2 + \\ + 2\,e^{\lambda\,\tau}\,\cos\tau\,d\,y\,d\,z + e^{\lambda\,\tau}\,\sin\tau\,d\,z^2 \end{aligned} \tag{34}$$

which possesses a homothety generated by the field

$$H = -4\lambda\,\partial_\tau - 4\,\lambda^2\,x\,\partial_x + \big(y\,(-\lambda^2+1) + 2\,\lambda\,z\big)\,\partial_y - \big(z\,(\lambda^2-1) + 2\,\lambda\,y\big)\,\partial_z \tag{35}$$

This metric was first given, although obtained in a different way, by Harrison [19]. There are special values of the constant λ for which we have a fourth Killing field:

$$\lambda = \pm\frac{\sqrt{3}}{3} \Rightarrow \xi_4 = 6\,\partial_\tau \pm 2\,\sqrt{3}\,x\,\partial_x \mp (\sqrt{3}\,y \pm 3\,z)\,\partial_y + (3\,y \mp \sqrt{3}\,z)\,\partial_z$$

while there is **no** homothety. Finally, it is worth noting that in the metric (34) the hypersurface $t = \text{const.}$ is spacelike.

3.4. Three real (two repeated) eigenvalues

In this case, the matrix ϑ can be brought in its Jordan normal form by a proper choice of the matrix Λ, i.e.

$$\widetilde{\vartheta} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 1 \\ 0 & 0 & p_2 \end{pmatrix} \Rightarrow \exp(\widetilde{\vartheta}\,t) = \begin{pmatrix} e^{p_1\,t} & 0 & 0 \\ 0 & e^{p_2\,t} & e^{p_2\,t}\,t \\ 0 & 0 & e^{p_2\,t} \end{pmatrix} \tag{36}$$

where the constants (p_1, p_2) are related through (22):

$$p_2\,(p_2 + 2\,p_1) = 0 \tag{37}$$

Here too, the constant p_1 is not zero, so dividing the previous relation by p_1^2 yields

$$\lambda\,(\lambda+2) = 0 \Rightarrow \lambda = 0 \,\vee\, \lambda = -2, \quad \frac{p_2}{p_1} = \lambda \tag{38}$$

With the help of the automorphisms and the transformation $\tau = p_1\, t$ we get the metric

$$d\,s^2 = e^{(2\,\lambda+1)\,\tau}\, d\,\tau^2 + e^{\tau}\, d\,x^2 + \tau\, e^{\lambda\,\tau}\, d\,y^2 + 2\, e^{\lambda\,\tau}\, d\,y\, d\,z \tag{39}$$

This metric is also a member of the Harrison class [19] and admits a homothety produced by the field

$$H = 2\,\partial_\tau + 2\,x\,\lambda\,\partial_x + (\lambda+1)\,y\,\partial_y + (z\,(\lambda+1) - y)\,\partial_z \tag{40}$$

In this case too, the hypersurface $t =$ const. is spacelike. Furthermore, for the value $\lambda = 0$ the metric (39) describes a pp-wave, since the Killing field $u = \xi_3 = \partial_z$ has zero covariant derivative and zero measure:

$$\lambda = 0 \Rightarrow u^\alpha\, u_\alpha = 0 \quad \text{and} \quad u^\alpha{}_{;\beta} = 0 \tag{41}$$

3.5. **Three real repeated eigenvalues**

In this case, the Jordan normal form of the matrix ϑ is

$$\widetilde{\vartheta} = \begin{pmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{pmatrix} \Rightarrow \exp(\widetilde{\vartheta}\, t) = e^{p\,t} \begin{pmatrix} 1 & t & \frac{1}{2}\,t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \tag{42}$$

so (22) gives $p = 0$. Again, full use of (23) and the automorphism matrices, leads to:

$$d\,s^2 = d\,t^2 + 2\,t^2\, d\,x^2 + d\,y^2 - 4\,t\, d\,x\, d\,y + 4\, d\,x\, d\,z \tag{43}$$

which has the following properties:

— There is a homothety, with the corresponding field

$$H = t\,\partial_t + y\,\partial_y + 2\,z\,\partial_z \tag{44}$$

— It is a pp-wave metric since, for the Killing field $u = \xi_3 = \partial_z$, we have

$$u^\alpha\, u_\alpha = 0 \quad \text{and} \quad u^\alpha{}_{;\beta} = 0 \tag{45}$$

— The hypersurface $t =$ const. is spacelike.

At this point, the question arises whether the two metrics (39) with $\lambda = 0$ and (43) are actually the same since they both are pp-wave metrics, they have three Killing fields one of which is timelike, and they both admit a homothety. The classical way to answer this question is by turning to the scalar invariants that can be constructed by the Riemann tensor and its covariant derivatives. This approach is not applicable to our case, since the fact that the metrics are pp-waves implies the vanishing of all these scalar invariants. An alternative approach is to find a tensor that is identically zero for one of the metrics and non zero for the other. This would mean that there is no coordinate transformation relating the two metrics. If we consider the tensor

$$\Pi_{\alpha\beta\gamma\delta\epsilon} = R^{\kappa\ \ \lambda}_{\ \alpha\ \beta}\, R_{\kappa\gamma\delta\epsilon;\lambda}, \tag{46}$$

we observe that it vanishes identically for the metric (43) but not for the metric (39) with $\lambda = 0$, i.e., the two metrics are different.

4. Bianchi Type II

Here the non vanishing structure constants, basis 1-forms and Killing fields are:

$$C^1{}_{23} = -C^1{}_{32} = \frac{1}{2} \tag{47}$$

$$\boldsymbol{\sigma}^1 = z\,\boldsymbol{d}x + \boldsymbol{d}y, \quad \boldsymbol{\sigma}^2 = \boldsymbol{d}z, \quad \boldsymbol{\sigma}^3 = \boldsymbol{d}x \tag{48}$$

$$\xi_1 = \partial_x, \quad \xi_2 = -x\,\partial_y + \partial_z, \quad \xi_3 = \partial_y \tag{49}$$

From (9) we have for the automorphisms $\Lambda^\alpha{}_\beta$ and the triplett $P^\alpha(t)$

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} e^{s_1+s_2} - s_3\,s_4 & x(t) & y(t) \\ 0 & e^{s_1} & s_3 \\ 0 & s_4 & e^{s_2} \end{pmatrix},\ P^\alpha(t) = \left(P(t), \frac{e^{s_1}\,\dot{y} - s_3\,\dot{x}}{e^{s_1+s_2} - s_3\,s_4}, \frac{s_4\,\dot{y} - e^{s_2}\,\dot{x}}{e^{s_1+s_2} - s_3\,s_4}\right)$$

After $P(t)$, $x(t)$ and $y(t)$ are used to eliminate the shift N^α, the residual "rigid" symmetry is described by

$$M^\alpha{}_\beta = \begin{pmatrix} e^{s_1+s_2} - s_3\,s_4 & s_5 & s_6 \\ 0 & e^{s_1} & s_3 \\ 0 & s_4 & e^{s_2} \end{pmatrix} \tag{50}$$

The corresponding motions $\tilde{\gamma}_{\alpha\beta} = M^\mu{}_\alpha M^\nu{}_\beta \gamma_{\mu\nu}$ are generated by the vector fields

$$X_A = \left(\frac{\partial \tilde{\gamma}_{\alpha\beta}}{\partial s_A}\right)_{s=0} \frac{\partial}{\partial \gamma_{\alpha\beta}} \tag{51}$$

where $A = \{1, 2, 3, 4, 5, 6\}$ and

$$\begin{aligned}
X_1 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + 2\,\gamma_{12}\,\partial_{\gamma_{12}} + \gamma_{13}\,\partial_{\gamma_{13}} + 2\,\gamma_{22}\,\partial_{\gamma_{22}} + \gamma_{23}\,\partial_{\gamma_{23}} \\
X_2 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + \gamma_{12}\,\partial_{\gamma_{12}} + 2\,\gamma_{13}\,\partial_{\gamma_{13}} + \gamma_{23}\,\partial_{\gamma_{23}} + 2\,\gamma_{33}\,\partial_{\gamma_{33}} \\
X_3 &= \gamma_{12}\,\partial_{\gamma_{13}} + \gamma_{22}\,\partial_{\gamma_{23}} + 2\,\gamma_{23}\,\partial_{\gamma_{33}} \\
X_4 &= \gamma_{13}\,\partial_{\gamma_{12}} + 2\,\gamma_{23}\,\partial_{\gamma_{22}} + \gamma_{33}\,\partial_{\gamma_{23}} \\
X_5 &= \gamma_{11}\,\partial_{\gamma_{12}} + 2\,\gamma_{12}\,\partial_{\gamma_{22}} + \gamma_{13}\,\partial_{\gamma_{23}} \\
X_6 &= \gamma_{11}\,\partial_{\gamma_{13}} + \gamma_{12}\,\partial_{\gamma_{23}} + 2\,\gamma_{13}\,\partial_{\gamma_{33}}
\end{aligned} \tag{52}$$

The generators X_5, X_6 commute with each other, as well as with Y_2, so we can bring them into their normal form by the following transformation of the $\gamma_{\alpha\beta}$:

$$\begin{cases} \gamma_{11} = e^{u_1(t)},\, \gamma_{12} = e^{u_1 t}\, u_2(t),\, \gamma_{13} = e^{u_1(t)}\, u_3(t) \\ \gamma_{22} = e^{u_1(t)}\,(u_2^2(t) + u_4(t)) \\ \gamma_{23} = e^{u_1(t)}\,(u_2(t)\, u_3(t) + u_5(t)),\, \gamma_{33} = e^{u_1(t)}\,(u_3^2(t) + u_6(t)) \end{cases} \tag{53}$$

Before we proceed to the solution of the Einstein equations, we have to find the values allowed for the functions $u_i, i = 1, \ldots, 6$. The determinant of the matrix $\gamma_{\alpha\beta}$, is

$$det[\gamma_{\alpha\beta}] = e^{3\,u_1}\,\left(u_4\, u_6 - u_5^2\right), \tag{54}$$

therefore $u_4\, u_6 - u_5^2 > 0$.

Starting from the equations $E_2 = 0$, $E_3 = 0$, we get the following system for the variables $\dot{u}_2$, $\dot{u}_3$:

$$E_2 = 0 \Rightarrow u_5\, \dot{u}_2 - u_4\, \dot{u}_3 = 0 \tag{55a}$$

$$E_3 = 0 \Rightarrow u_6\, \dot{u}_2 - u_5\, \dot{u}_3 = 0. \tag{55b}$$

This system admits non zero solutions only if $u_4\, u_6 - u_5^2 = 0$, a condition forbidden by our restrictions. Thus, we have

$$u_2(t) = k_2, \quad u_3(t) = k_3 \tag{56}$$

Now, by using the automorphism matrix

$$M^\alpha{}_\beta = \begin{pmatrix} 1 & -k_2 & -k_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{57}$$

the matrix $\gamma_{\alpha\beta}$ is brought to block diagonal form, and we have the following theorem:

Theorem 4.1 The general solution of Bianchi Type II corresponds to a block diagonal form in the basis of the 1-forms $\boldsymbol{\sigma}^\alpha$.

As we have now proved that the matrix $\gamma_{\alpha\beta}$ is in block diagonal form with no loss of generality, we can repeat the former procedure for the calculation of the generators, i.e., we can start from the matrix

$$\gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{11} & 0 & 0 \\ 0 & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{23} & \gamma_{33} \end{pmatrix} \tag{58}$$

with the corresponding automorphism matrix

$$M^\alpha{}_\beta = \begin{pmatrix} e^{s_1+s_2} - s_3\, s_4 & 0 & 0 \\ 0 & e^{s_1} & s_3 \\ 0 & s_4 & e^{s_2} \end{pmatrix}. \tag{59}$$

The generators thus obtained are

$$\begin{aligned}
X_1 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + 2\,\gamma_{22}\,\partial_{\gamma_{22}} + \gamma_{23}\,\partial_{\gamma_{23}} \\
X_2 &= 2\,\gamma_{11}\,\partial_{\gamma_{11}} + \gamma_{23}\,\partial_{\gamma_{23}} + 2\,\gamma_{33}\,\partial_{\gamma_{33}} \\
X_3 &= \gamma_{22}\,\partial_{\gamma_{23}} + 2\,\gamma_{23}\,\partial_{\gamma_{33}} \\
X_4 &= 2\,\gamma_{23}\,\partial_{\gamma_{22}} + \gamma_{33}\,\partial_{\gamma_{23}}.
\end{aligned} \tag{60}$$

The generators Y_2, $X_1 + X_2$, X_3 commute with each other, so they can be brought to their normal form by applying the transformation

$$\gamma_{\alpha\beta} = \begin{pmatrix} e^{u_1+4\,u_6} & 0 & 0 \\ 0 & e^{u_1+2\,u_6}\,u_4 & e^{u_1+2\,u_6}\,u_4\,u_5 \\ 0 & e^{u_1+2\,u_6}\,u_4\,u_5 & e^{u_1+2\,u_6}(u_4\,u_5^2+1) \end{pmatrix}. \tag{61}$$

In this parameterization, the only variable the second derivative of which appears in the field equations is u_4, since the generators have been transformed to

$$Y_2 = \frac{\partial}{\partial u_1},\; X_3 = \frac{\partial}{\partial u_5},\; X_1 + X_2 = \frac{\partial}{\partial u_6} \tag{62}$$

To further simplify the present situation, we shall also use the generator $Y_1 = \frac{\partial}{\partial t}$. The variable u_4, that does not appear in the generators Y_2, $X_1 + X_2$, X_3, can be utilized in order to parameterize time t, i.e., we can apply the transformation

$$t \to u_4(t) = s,\; u_1(t) \to u_1(s),\; u_5(t) \to u_5(s),\; u_6(t) \to u_6(s) \tag{63}$$

Of course, this transformation can be applied only in the case where $u_4(t)$ is not a constant function. Hence, we must distinguish between two cases: either the function $u_4(t)$ is constant or it is not.

4.1. The case $u_4(t) =$ const.

Let $u_4(t) = k_4$. Then, the linear constraints (4) are identically satisfied, while the quadratic constraint (3) defines the lapse function $N^2(t)$ as

$$N^2 = k_4\,e^{u_1}\left(3\,\dot u_1^2 - k_4\,\dot u_5^2 + 16\,\dot u_1\,\dot u_6 + 20\,\dot u_6^2\right) \tag{64}$$

Substituting the latter into the dynamic equations (5), we can solve them for the variables $\ddot u_1$, $\ddot u_5$, $\ddot u_6$. Starting from equation $E_{11} = 0$, we notice that the coefficient of $\ddot u_5$ is proportional to the quantity

$$\dot u_5\,(\dot u_1 + 4\,\dot u_6). \tag{65}$$

Therefore, in order to solve for $\ddot u_5$ we must be sure that this coefficient is not zero. The case $\dot u_1 + 4\,\dot u_6 = 0$ leads to a zero lapse, so only the case $u_5(t) = k_5$ is left.

Assuming that $u_5(t) \neq k_5$, we solve equation $E_{11} = 0$ for $\ddot{u}_5$ and, substituting in equation $E_{22} = 0$, we can solve the latter for $\ddot{u}_1$. The coefficient of $\ddot{u}_1$ is the quantity $k_4\, \dot{u}_6$ which cannot vanish since, for $k_4 = 0$, we have $det(\gamma_{\alpha\beta}) = 0$ while, for $u_6(t) = k_6$, we obtain a zero lapse. Now, substituting in $E_{23} = 0$ we get

$$k_4\, \dot{u}_5\, (\dot{u}_1 + 3\, \dot{u}_6) = 0 \Rightarrow u_1 = k_1 - 3\, u_6. \tag{66}$$

However, this leads to $u_5 = k_5$, which is impossible in this “branch” of the solution.

If we start with $u_5 = k_5$, then the coefficient $\ddot{u}_1$ in equation $E_{11} = 0$ is proportional to the quantity

$$\dot{u}_6\, (\dot{u}_1 + 3\, \dot{u}_6) \tag{67}$$

The case $u_6 = k_6$ leads to a zero lapse, so the case remained to be considered before we solve equation $E_{11} = 0$ for the variable $\ddot{u}_1$ is

$$u_6(t) = k_6 - \frac{u_1(t)}{3}. \tag{68}$$

With this value for the function $u_6(t)$ the rest of the equations are satisfied identically, but now the lapse function takes the value

$$N^2 = -\frac{1}{9} e^{u_1(t)}\, k_4\, \dot{u}_1^2, \tag{69}$$

i.e., we have ended up with a metric of **Euclidean** signature. Using an appropriate automorphism matrix, and choosing a gauge in which $u_1(t) = 3\, \ln t$, we get the metric

$$d\,s^2 = t\,\boldsymbol{d}t^2 + \frac{1}{t}\,(\boldsymbol{\sigma}^1)^2 + t\,(\boldsymbol{\sigma}^2)^2 + t\,(\boldsymbol{\sigma}^3)^2 \tag{70}$$

This metric admits an extra Killing field and a homothety:

$$\xi_4 = -2\,z\,\partial_x + (z^2 - x^2)\,\partial_y + 2\,x\,\partial_z, \qquad H = t\,\partial_t + x\,\partial_x + 2\,y\,\partial_y + z\,\partial_z \tag{71}$$

Since we have examined the case where the coefficient of $\ddot{u}_1$ in $E_{11} = 0$ vanishes, we can solve this equation for this function, so

$$\ddot{u}_1 = \frac{1}{\dot{u}_6}\left(3\,\dot{u}_1^3 + 25\,\dot{u}_1^2\,\dot{u}_6 + (68\,\dot{u}_6^2 + \ddot{u}_6)\,\dot{u}_1 + 60\,\dot{u}_6^3\right), \tag{72}$$

hence the rest of the equations are also satisfied. In order to simplify the latter equation we choose a gauge in which $u_6(t) = 3\,t$, thus obtaining the autonomous first order equation for $\dot{u}_1$

$$\ddot{u}_1 = \dot{u}_1^3 + 25\,\dot{u}_1^2 + 204\,\dot{u}_1 + 540 \tag{73}$$

This equation can be integrated with the help of the transformation

$$t = y(\xi), \quad \dot{u}_1 = \xi, \quad \ddot{u}_1 = \frac{1}{y'(\xi)} \tag{74}$$

which brings the equation in the following form:

$$y'(\xi) = \frac{1}{(\xi+10)(\xi+9)(\xi+6)} \Rightarrow$$
$$y(\xi) = k_1 + \frac{1}{4}\ln(\xi+10) - \frac{1}{3}\ln(\xi+9) + \frac{1}{12}\ln(\xi+6) \tag{75}$$

Using an appropriate automorphism matrix and redefining k_4 and ξ as $k_4 = \frac{8\sqrt{3}}{27}\mu^2 e^{-k_2}$ and $\xi = -6\,\frac{5\,e^{2\,\psi}-1}{3\,e^{2\,\psi}-1}$ we obtain the metric

$$d\,s^2 = \mu^2\left(-e^{2\,\psi}\cosh\psi\,\boldsymbol{d}\,\psi^2 + \mathrm{sech}\psi\,(\boldsymbol{\sigma}^1)^2 + e^{\psi}\cosh\psi\,(\boldsymbol{\sigma}^2)^2 + e^{\psi}\cosh\psi\,(\boldsymbol{\sigma}^3)^2\right) \tag{76}$$

with an extra Killing field and no homothety (μ being essential):

$$\xi_4 = -2\,z\,\partial_x + (z^2 - x^2)\,\partial_y + 2\,x\,\partial_z. \tag{77}$$

4.2. The case $u_4(t) \neq$ const.

Let us assume that $u_4(t) = t$. Then the linear constraints (4) are satisfied identically, while the quadratic constraint (3) defines the following lapse function $N^2(t)$:

$$N^2 = e^{u_1}\left(3\,t\,\dot{u}_1^2 - t^2\,\dot{u}_5^2 + 2\,(8\,t\,\dot{u}_6 + 1)\dot{u}_1 + 2\,(10\,t\,\dot{u}_6 + 3)\,\dot{u}_6\right) \tag{78}$$

Substituting this in equation $E_{11} = 0$, we find that the coefficient of $\ddot{u}_5$ is proportional to the quantity $\dot{u}_5\,(\dot{u}_1 + 4\,\dot{u}_6)$ which cannot vanish since it leads to a zero lapse. Solving equation $E_{11} = 0$ for $\ddot{u}_5$ and then substituting in equation $E_{22} = 0$, we find that the coefficient of $\ddot{u}_6$ is proportional to the quantity $t\,\dot{u}_1 + 2$ which cannot vanish because in such a case we obtain again a zero lapse. Thus, we can solve for $\ddot{u}_6$ and substitute in the rest of the equations. Equation $E_{23} = 0$ includes the function $\ddot{u}_1$ the coefficient of which is proportional to the quantity

$$6\,t\,\dot{u}_6 + 2\,t\,\dot{u}_1 + 1, \tag{79}$$

the vanishing of which leads, as we shall see, to a metric of Euclidean signature. Assuming that the quantity (79) is not zero, we get the following system of differential equations:

$$\ddot{u}_5 = -\frac{2\,\dot{u}_5}{t}(t^3\,\dot{u}_5^2 + 1) \tag{80a}$$

$$\ddot{u}_6 = \frac{1}{2\,t}\left(-6\,t\,\dot{u}_1^2 + t^2\,\dot{u}_5^2(-4\,t\,\dot{u}_6 + 3) - 4\,\dot{u}_1(8\,t\,\dot{u}_6 + 1) - 2\,\dot{u}_6(20\,t\,\dot{u}_6 + 7)\right) \tag{80b}$$

$$\ddot{u}_1 = \frac{1}{t}\left(9\,t\,\dot{u}_1^2 - 5\,t^2\,\dot{u}_5^2 + 6\,\dot{u}_6(10\,t\,\dot{u}_6 + 3) + (48\,t\,\dot{u}_6 - 2\,t^3\,\dot{u}_5^2 + 5)\,\dot{u}_1\right) \tag{80c}$$

We shall first consider the case of a vanishing quantity (79):

$$6\,t\,\dot{u}_6 + 2\,t\,\dot{u}_1 + 1 = 0 \Rightarrow u_6 = k_6 - \frac{1}{6}\ln t - \frac{1}{3}u_1 \tag{81}$$

Solving equation $E_{11}=0$ for $\ddot{u}_5$ and substituting in $E_{22}=0$, we obtain

$$\dot{u}_5 = 0 \Rightarrow u_5 = k_5 \tag{82}$$

that brings equation $E_{33}=0$ into the form

$$3\,t^2\,\ddot{u}_1 + t^2\,\dot{u}_1^2 + 7\,t\,\dot{u}_1 + 4 = 0. \tag{83}$$

The latter is a Riccati equation for $w_1(t) = \dot{u}_1(t)$

$$\dot{w}_1 = -\frac{1}{3}\,w_1^2 - \frac{7}{3\,t}\,w_1 - \frac{4}{3\,t^2}, \tag{84}$$

which is easily solvable since a partial solution of it is already known: $w_1 = -\frac{2}{t}$. With the transformation

$$w_1(t) = -\frac{2}{t} + \frac{1}{h(t)}, \tag{85}$$

the equation (84) takes the linear form

$$\dot{h} = \frac{1}{t}\,h + \frac{1}{3} \Rightarrow h(t) = \frac{t}{3}\,\ln t + k_1\,t. \tag{86}$$

By integrating, we finally have for $u_1(t)$

$$u_1 = k_2 - 2\,\ln t + 3\,\ln(\ln t + 3\,k_1). \tag{87}$$

Redefining the constants k_2, k_6 as:

$$k_2 = \ln\mu^2 - 3\,k_1, \qquad k_6 = \frac{1}{24}\,(3\,\ln\kappa^2 - \ln\mu^2 + 3\,k_1), \tag{88}$$

using an appropriate automorphism matrix and putting $t = e^{\xi - 3\,k_1}$, we obtain the following metric of **Euclidean** signature

$$d\,s^2 = \mu^2\left(e^{-\xi}\,\xi\,\boldsymbol{d}\,\xi^2 + \frac{1}{\xi}\,(\boldsymbol{\sigma}^1)^2 + \xi\,(\boldsymbol{\sigma}^2)^2 + e^{-\xi}\,\xi\,(\boldsymbol{\sigma}^3)^2\right). \tag{89}$$

This metric does **not** admit a homothety, so the constant μ cannot be absorbed.

Now, returning to the system (80) and multiplying the first equation by $\dot{u}_5$, we have

$$\begin{aligned}\dot{u}_5\,\ddot{u}_5 = -\frac{2\,\dot{u}_5^2}{t}(t^3\,\dot{u}_5^2+1) \Rightarrow \frac{d\,\dot{u}_5^2}{d\,t} = -\frac{4\,\dot{u}_5^2}{t}(t^3\,\dot{u}_5^2+1) \Rightarrow \\ \dot{y} = -\frac{4\,y}{t}(t^3\,y+1), \quad y = \dot{u}_5^2,\end{aligned} \tag{90}$$

i.e. a Ricatti equation for $y(t)$. A partial solution of this equation is $y_1 = -\frac{1}{4\,t^3}$, hence we can reduce it to a linear differential equation by applying the transformation

$$y = -\frac{1}{4\,t^3} + \frac{1}{h} \tag{91}$$

and get

$$\dot{h} = \frac{2}{t}\,h + 4\,t^2 \Rightarrow h = -\frac{4}{k_{51}}\,t^2 + 4\,t^3 \Rightarrow u_5 = k_{52} \pm \sqrt{k_{51} - \frac{1}{t}}. \tag{92}$$

Consequently, the last two equations of the system (80) become

$$\ddot{u}_1 = \langle \dot{u}_1 |\, A_1 \,| \dot{u}_6 \rangle, \quad \ddot{u}_6 = \langle \dot{u}_1 |\, A_2 \,| \dot{u}_6 \rangle, \tag{93}$$

where use has been made of the notation $\langle \dot{u}_i | = \big(1\; \dot{u}_i\; \dot{u}_i^2\big)$ and $| \dot{u}_i \rangle = \langle \dot{u}_i |^t$ with the 3×3 matrices A_1, A_2 given as

$$A_1 = \begin{pmatrix} -\dfrac{5}{4\,t^2\,(k_{51}\,t-1)} & \dfrac{18}{t} & 60 \\ \dfrac{10\,k_{51}\,t - 11}{2\,k_{51}\,t^2 - 2\,t} & 48 & 0 \\ 9 & 0 & 0 \end{pmatrix} \tag{94}$$

$$A_2 = \begin{pmatrix} \dfrac{3}{8\,t^2\,(k_{51}\,t-1)} & -\dfrac{14\,k_{51}\,t - 13}{2\,k_{51}\,t^2 - 2\,t} & -20 \\ -\dfrac{2}{t} & -16 & 0 \\ -3 & 0 & 0 \end{pmatrix}. \tag{95}$$

The latter system is of polynomial form in the variables $\dot{u}_1$, $\dot{u}_6$, hence we can simplify it based on the transformation

$$\dot{u}_1 = \frac{1}{2\,\sqrt{t\,(k_{51}\,t-1)}}\,(y_1 + 5\,y_2) - \frac{1}{t} \tag{96a}$$

$$\dot{u}_6 = -\frac{1}{4\,\sqrt{t\,(k_{51}\,t-1)}}\,(y_1 + 3\,y_2) + \frac{1}{4\,t}. \tag{96b}$$

Thus, we obtain the system

$$\dot{y}_1 = -\frac{4\,y_1\,y_2 + k_{51}}{4\,\sqrt{t\,(k_{51}\,t-1)}}, \qquad \dot{y}_2 = -\frac{4\,y_1\,y_2 + k_{51}}{4\,\sqrt{t\,(k_{51}\,t-1)}} \tag{97}$$

i.e.,

$$\dot{y}_2 = \dot{y}_1 \Rightarrow y_2 = y_1 + k_2\,\sqrt{k_{51}}, \qquad \dot{y}_1 = -\frac{4\,y_1^2 + 4\,k_2\,\sqrt{k_{51}}\,y_1 + k_{51}}{4\,\sqrt{t\,(k_{51}\,t-1)}}. \tag{98}$$

The previous Riccati equation is simplified by applying the transformation

$$t \mapsto \frac{1}{k_{51}}\,\cosh^2(2\,\tau),\; y_1(t) \mapsto \sqrt{k_{51}}\,y_1(\tau),\; k_2 \mapsto -\cosh\mu, \tag{99}$$

so we get the equation

$$\dot{y}_1 + 4\,y_1^2 - 4\,\cosh\mu\,y_1 + 1 = 0 \Rightarrow$$
$$y_1 = \frac{1}{2}\left[\cosh\mu - \sinh\mu\,\tanh\big(2\,(k_1-\tau)\,\sinh\mu\big)\right]. \tag{100}$$

Substituting in $y_1(\tau)$ in (96), we obtain

$$u_1 = c_1 - 4\,\tau\,\cosh\mu + 3\,\ln(\cosh(2\,(k_1-\tau)\,\sinh\mu)) + \ln\frac{\tanh 2\,\tau}{\sinh 4\,\tau} \tag{101a}$$
$$u_6 = c_2 + \tau\,\cosh\mu - \ln(\cosh(2\,(k_1-\tau)\,\sinh\mu)) - \frac{1}{4}\ln\frac{\tanh 2\,\tau}{\sinh 4\,\tau} \tag{101b}$$

Redefining the constants $\mu = 2\,\sigma$, $k_{51} = \exp(c_1 - 4\,k_1\,\cosh 2\,\sigma)/2\,\kappa^2$, choosing the gauge $\tau = k_1 - \xi\,\mathrm{csch}2\,\sigma/2$ and using appropriate automorphism matrices we arrive at the form:

$$\begin{aligned} d\,s^2 = \kappa^2\Big(&e^{2\,\xi\,\coth 2\,\sigma}\,\cosh\xi\,\boldsymbol{d}\,\xi^2 + \mathrm{sech}\xi\,(\boldsymbol{\sigma}^1)^2 \\ &+ e^{\xi\,\coth\sigma}\,\cosh\xi\,(\boldsymbol{\sigma}^2)^2 + e^{\xi\,\tanh\sigma}\,\cosh\xi\,(\boldsymbol{\sigma}^3)^2\Big) \end{aligned} \tag{102}$$

This solution was first obtained by Taub [20]. It does not admit a homothety, hence the constant κ cannot be absorbed. It is noteworthy, that in the limiting case $\sigma \to +\infty$, this metric reduces to metric (76) which possesses four Killing fields.

5. Conclusions

In this work, the methodology of Lie group symmetry analysis for nonlinear differential equations has been utilized in order to uncover the entire space of solutions to Einstein's field equations in the case of the cosmological models of Bianchi Type I and II. The symmetries used are the automorphisms of the Lie algebra of the corresponding three-dimensional isometry group acting on the hyper-surfaces of simultaneity as well as the scaling and the time reparametrization symmetries. The power of the method lies in the fact that it constitutes a semi-algorithm which, if successfully implemented, leads to the acquisition of the entire space of solutions. The general solutions, not considered as such at the time of their first derivation and containing one or two essential constants, were produced with the aid of various simplifying ansätze in a time scale of half a century: Kasner (1921), Taub (1951), and Harrison (1959). In the present work, they are comprehensively re-obtained along with solutions not attributed to any one else and which, to the best of our knowledge, are new: The Bianchi Type I metric (43) and the Bianchi Type II metrics (70) (of Euclidean signature), (76), and (89) (of Euclidean signature). We should point out that the production of metrics with Euclidean signature may, at first sight, strike as odd, since our starting point is a line element of Lorentzian signature. However, this is made possible by allowing the lapse to be determined through the quadratic constraint equation instead of prescribing it by an *ab initio* choice of time gauge.

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