BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 57, 2010 (13–30)

Galerkin Approximations of Periodic Solutions of Boussinesq Systems

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Abstract

We consider the periodic initial-value problem for the family of *a*-*b*-*c*-*d* Boussinesq systems, [8], [9], and their completely symmetric analogs, [10]. We approximate their solutions by the standard Galerkin-finite element method using smooth periodic splines for discretizing in space. We prove optimal-order L^2 error estimates for the resulting semidiscretizations. The numerical schemes are usual in computations of cnoidal-wave solutions of these systems, as well as of solitary-wave solutions of systems with negative b and d.

1. Introduction

In this note we consider the so-called *a*-*b*-*c*-*d Boussinesq systems* of water wave theory, [8], [9]

$$\eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0\,, \tag{1.1}$$
$$u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0\,, \tag{1.2}$$

as well as the analogous problem for their *symmetric* relatives, [10]

$$\eta_t + u_x + \tfrac{1}{2}(\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0\,, \tag{1.3}$$
$$u_t + \eta_x + \tfrac{1}{2}\eta\eta_x + \tfrac{3}{2}u u_x + c\eta_{xxx} - d u_{xxt} = 0\,. \tag{1.4}$$

Here η and u are real-valued functions of $x \in \mathbb{R}$ and $t \geq 0$ and a, b, c, d are real constants, which for modelling purposes should satisfy $a + b = \frac{1}{2}(\theta^2 - 1/3)$, $c + d = \frac{1}{2}(1 - \theta^2)$ for some $\theta \in [0, 1]$. Three categories of *a*-*b*-*c*-*d* systems were identified in [8] to have the property that their Cauchy problems are linearly well posed. These were:

(C1) $\quad b \geq 0,\ d \geq 0\,, \quad a \leq 0\,,\ c \leq 0\,,$
(C2) $\quad b \geq 0,\ d \geq 0\,, \quad a = c > 0\,,$
(C3) $\quad b = d < 0\,, \quad a = c > 0\,.$

The Cauchy problem for systems with coefficients belonging to (C1) and (C2) was shown in [9] to be nonlinearly well posed, at least locally in time. The systems of category (C3) were not analyzed in [9] due to the singular nature of the elliptic operator $I - \mu\partial_x^2$, where $\mu = b = d$. In [10] it was shown that the 'completely' symmetric systems of the form (1.3) − (1.4) with $a = c$ and $b, d \geq 0$ are nonlinearly well posed locally in time. Examples of well known Boussinesq systems include the family of the *Bona-Smith systems* ($a = 0$, $b = d = (3\theta^2 - 1)/6$, $c = (2 - 3\theta^2)/3$, $\frac{2}{3} \leq \theta^2 < 1$), cf. [14], [4], [5], whose member with $\theta^2 = 2/3$, i.e. with $a = 0$, $c = 0$, $b = d = 1/6$, is the *BBM-BBM system*, [7], the so-called *'classical'* Boussinesq system ($a = b = c = 0$, $d = 1/3$), [23], [1], [3], and the *KdV-KdV* system ($b = d = 0$, $a = c = 1/6$), [12], [13].

In the present paper we shall consider the periodic initial-value problem for (1.1)−(1.2) and (1.3) − (1.4) on the spatial interval $[0, 1]$ with given initial conditions

$$\eta(x, 0) = \eta_0(x)\,, \quad u(x, 0) = u_0(x)\,, \quad x \in [0, 1]\,, \tag{1.5}$$

where η_0 and u_0 are smooth 1−periodic functions. We shall assume that the periodic initial-value problems for the systems that will be approximated numerically in the sequel possess sufficiently smooth unique solutions at least on a temporal interval $[0, T]$. We shall discretize the periodic initial-value problems in space using the standard Galerkin-finite element method, based on smooth, 1−periodic splines on a uniform mesh in $[0, 1]$. The resulting *semidiscretizations* are represented by initial-value problems for systems of o.d.e.'s. After introducing the relevant notation and the approximation properties of the spline spaces in Section 2, we prove, in Section 3, error estimates for the Galerkin approximations, that are of optimal rate of convergence in L^2, for several types of systems. Specifically, our convergence proofs are valid for the usual systems (1.1) − (1.2) provided their coefficients belong either to a subset of the class (C1) that is defined in Section 3 and includes e.g. the Bona-Smith, the BBM-BBM, and the 'classical' Boussinesq but not the KdV-KdV system, or to a subset of the class (C2) or to the class (C3) provided that $b \neq -\frac{1}{4n^2\pi^2}$, for $n = 1, 2, 3, \ldots$. The analogous Galerkin semidiscretizations of the completely symmetric systems (1.3) − (1.4) with $a = c$, $b, d \geq 0$ are analyzed in Section 4; they all possess an L^2 optimal-order convergence theory.

The error analysis of the semidiscretizations is based on the properties of suitable *quasiinterpolants* of smooth periodic functions in the space of periodic splines, that afford proving some crucial cancellation properties used in the estimation of the truncation error of the semidiscrete approximations. This technique was introduced by Thomée and Wendroff in [24] in the context of approximating periodic solutions of linear p.d.e.'s, and has been utilized to prove optimal-order error estimates for nonlinear dispersive wave equations in [6], [18], [11], for Boussinesq systems of Bona-Smith and 'classical' type in [2] and for completely symmetric systems of KdV-KdV type in [21]. Here it is extended to cover more general cases. To discretize the semidiscrete systems in the temporal variable one may use explicit Runge-Kutta methods in case the systems are not stiff (as e.g. the Bona-Smith systems, [2], [5],) or mildly stiff (as e.g. the 'classical' Boussinesq system, [2], [3]). Systems with purely KdV type dispersion terms are very stiff and should be discretized by time-stepping methods having good nonlinear stability properties, such as the Gauss-Legendre implicit Runge-Kutta schemes, [11], [21], [12], [13]. The error analysis of such full discretizations will not be carried out in this paper; the reader is referred to the papers mentioned above.

In Section 5 we check our fully discrete scheme with explicit, fourth-order Runge-Kutta time-stepping by simulating *cnoidal-wave solutions* for one Bona-Smith system for which it is possible to derive closed formulas for this type of solutions of the periodic initial-value problem. We also construct *generalized solitary-wave* solutions of systems of class (C3) and use the numerical code to study the evolution that ensues from an initial 'heap of water' for this type of systems.

In this paper, we denote by H^k_{per}, for $k \geq 0$ integer, the usual, L^2-based (real) Sobolev spaces of periodic functions on $[0, 1]$ with associated norm $\|\cdot\|_k$. The norm on $L^2 = L^2(0, 1)$ will be denoted by $\|\cdot\| = \|\cdot\|_0$ and the associated inner product by $(\cdot, \cdot)$. The norms on $L^\infty(0, 1)$ and $W^{1,\infty}(0, 1)$ will be denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_{1,\infty}$, respectively.

2. Smooth periodic splines and the quasiinterpolant

Let N be a positive integer, $h = 1/N$, and $x_j = jh$, $j = 0, 1, \ldots, N$. For integer $r \geq 3$ consider the associated $N-$dimensional space of smooth $1-$periodic splines

$$S_h = \{\phi \in C^{r-2}_{per}[0, 1] \, : \, \phi\big|_{[x_j, x_{j+1}]} \in \mathbb{P}_{r-1} \, , \, 0 \leq j \leq N - 1\} \, ,$$

where

$$C^s_{per} = \{f \in C^s[0, 1] \, : \, f^{(k)}(0) = f^{(k)}(1) \, , \, 0 \leq k \leq s\} \, ,$$

and $\mathbb{P}_{r-1}$ are the polynomials of degree at most $r - 1$. In what follows C will denote a generic constant independent of h.

It is well known that S_h has the approximation property that given a sufficiently smooth $1-$periodic function w, there exists a $\chi \in S_h$ such that

$$\sum_{j=0}^{s-1} h^j \|w - \chi\|_j \leq C h^s \|w\|_s \, , \quad 1 \leq s \leq r \, , \tag{2.1}$$

for some constant C independent of h and w. In addition, the following inverse inequalities hold : There exists a constant C independent of h such that

$$\|\chi\|_\beta \leq C h^{-(\beta-\alpha)} \|\chi\|_\alpha \, , \quad 0 \leq \alpha \leq \beta \leq r - 1 \, , \tag{2.2a}$$

$$\|\chi\|_\infty \leq C h^{-1/2} \|\chi\| \, , \tag{2.2b}$$

for all $\chi \in S_h$. Following Thomée and Wendroff, [24], one may construct a basis $\{\phi_j\}_{j=1}^N$ of S_h, with $\mathrm{supp}(\phi_j) = O(h)$, such that for a sufficiently smooth $1-$periodic function w, the associated *quasiinterpolant*

$$Q_h w = \sum_{j=1}^{N} w(x_j)\phi_j \, ,$$

satisfies

$$\|w - Q_h w\| \le C h^r \|w^{(r)}\| \,. \tag{2.3}$$

In addition, it follows from [24] that the basis $\{\phi_j\}_{j=1}^N$ may be chosen so that the following properties hold :

(i) If $\psi \in S_h$, then

$$\|\psi\| \le C h^{-1} \max_{1\le i\le N} |(\psi, \phi_i)| \,. \tag{2.4}$$

(ii) Let w be a sufficiently smooth 1−periodic function and ν, κ integers such that $0 \le \nu, \kappa \le r-1$. Then

$$\big((Q_h w)^{(\nu)}, \phi_i^{(\kappa)}\big) = (-1)^\kappa h w^{(\nu+\kappa)}(x_i) + O(h^{2r+j-\nu-\kappa}) \,, \quad 1 \le i \le N \,, \tag{2.5}$$

where $j = 1$ if $\nu + \kappa$ is even, and $j = 2$ if $\nu + \kappa$ is odd.

(iii) Let f, g be sufficiently smooth 1−periodic functions and ν and κ as in (ii) above. Let

$$\beta_i = \big(f (Q_h g)^{(\nu)}, \phi_i^{(\kappa)}\big) - (-1)^\kappa \big(Q_h[(f g^{(\nu)})^{(\kappa)}], \phi_i\big) \,, \quad 1 \le i \le N \,.$$

Then

$$\max_{1\le i\le N} |\beta_i| = O(h^{2r+j-\nu-\kappa}) \,, \tag{2.6}$$

where j as in (ii).

3. Semidiscretization of the usual a-b-c-d Boussinesq systems

In this section we consider the periodic initial-value problem on $[0, 1]$ of the system (1.1)−(1.2), discretize it in space in S_h and analyze the resulting semidiscrete problem, which is defined as follows : We seek η_h , $u_h \in C^1(0, T; S_h)$, so that for $0 \le t \le T$

$$(\eta_{ht}, \phi) + (u_{hx}, \phi) + ((\eta_h u_h)_x, \phi) - a(u_{hxx}, \phi') + b(\eta_{htx}, \phi') = 0 \,, \quad \forall \phi \in S_h \,, \tag{3.1}$$

$$(u_{ht}, \chi) + (\eta_{hx}, \chi) + (u_h u_{hx}, \chi) - c(\eta_{hxx}, \chi') + d(u_{htx}, \chi') = 0 \,, \quad \forall \chi \in S_h \,, \tag{3.2}$$

where the initial values $\eta_h(0)$, $u_h(0)$ will be suitable approximations in S_h of η_0, u_0, respectively. Here $[0, T]$ is a temporal interval in which smooth solutions of the periodic initial-value problem for (1.1) − (1.2) exist uniquely. We first estimate a suitable truncation error of the system (3.1) − (3.2) as in [2].

Lemma 3.1 *Let $H = Q_h \eta$, $U = Q_h u$ and define ψ, $\zeta \in S_h$ so that for $0 \le t \le T$*

$$(H_t, \phi) + (U_x, \phi) + ((HU)_x, \phi) - a(U_{xx}, \phi') + b(H_{tx}, \phi') = (\psi, \phi) \,, \quad \forall \phi \in S_h \,, \tag{3.3}$$

$$(U_t, \chi) + (H_x, \chi) + (U U_x, \chi) - c(H_{xx}, \chi') + d(U_{tx}, \chi') = (\zeta, \chi) \,, \quad \forall \chi \in S_h \,. \tag{3.4}$$

Then, there is a constant C independent of h and t, such that

$$\|\psi(t)\| + \|\zeta(t)\| \le C h^r \,, \quad \forall t \in [0, T] \,. \tag{3.5}$$

Proof: Applying (2.5) to (3.3) and using the p.d.e. (1.1) yields for $1 \le i \le N$, $0 \le t \le T$

$$\begin{aligned}(\psi,\phi_i) &= h(\eta_t + u_x + a u_{xxx} - b\eta_{txx})(x_i,t) + O(h^{2r-1}) + ((HU)_x,\phi_i)\\ &= -h(\eta u)_x(x_i,t) + O(h^{2r-1}) + ((HU)_x,\phi_i)\\ &= -\big([Q_h(\eta u)]_x,\phi_i\big) + ((HU)_x,\phi_i) + O(h^{2r-1})\,.\end{aligned}$$

Since

$$HU - Q_h(\eta u) = \varepsilon e + \eta u - u\varepsilon - \eta e - Q_h(\eta u)\,,$$

where $\varepsilon := \eta - H$, $e = u - U$, we have, using (2.6), for $1 \le i \le N$

$$\begin{aligned}\big(\psi,\phi_i\big) &= \big((\varepsilon e)_x,\phi_i\big) + \big(u\varepsilon,\phi_i'\big) + \big(\eta e,\phi_i'\big) + \big((\eta u)_x - Q_h\big[(\eta u)_x\big],\phi_i\big)\\ &\quad + \big(Q_h\big[(\eta u)_x\big] - \big[Q_h(\eta u)\big]_x,\phi_i\big) + O(h^{2r-1})\\ &= \big((\varepsilon e)_x,\phi_i\big) - \big((\eta u)_x - Q_h\big[(\eta u)_x\big],\phi_i\big) + O(h^{2r-1})\,.\end{aligned}$$

Therefore, by (2.4) we obtain, using (2.3) and (2.1)

$$\begin{aligned}\|\psi\| &\le C\big(\|\varepsilon\|_1\|e\|_1 + \|(\eta u)_x - Q_h\big[(\eta u)_x\big]\| + h^{2r-2}\big)\\ &\le C\big(\|\varepsilon\|_1\|e\|_1 + h^r\big) \le Ch^r\,.\end{aligned}$$

The analogous estimate for $\|\zeta\|$ follows similarly. □

We will now prove an optimal-order L^2 error estimate for the semidiscrete problem (3.1) – (3.2). For this purpose we will assume that the dispersion coefficients a, b, c, d of the Boussinesq system (1.1) – (1.2) belong to one of the following categories :

(C1.i) $b > 0$, $d > 0$, $a < 0$, $c < 0$. (The 'generic' case.)

(C1.ii) One of b, d is positive and the other is zero, and $a < 0$, $c < 0$.

(C1.iii) $b > 0$, $d > 0$, $a \le 0$, $c \le 0$, $a \cdot c = 0$. (This case includes the Bona-Smith systems for which $b = d > 0$, $a = 0$ and $c < 0$, and the BBM-BBM system $b = d > 0$, $a = c = 0$.)

(C1.iv) One of the b, d is positive and the other is zero, and $a = c = 0$. (This case includes the 'classical' Boussinesq system for which $a = b = c = 0$, $d > 0$.)

(C2.i) $b \ge 0$, $d \ge 0$ with $b + d \ne 0$, $a = c > 0$.

(C3) $b = d < 0$, $a = c > 0$.

Note that the union of (C1.i), (C1.ii), (C1.iii), (C1.iv) is a proper subset of (C1) defined in the Introduction, and that (C2.i) is a proper subset of (C2). In the sequel, we will examine (C3) separately.

Theorem 3.1 *Suppose that a, b, c, d satisfy one of the conditions (C1.i), (C1.ii), (C1.iii), (C1.iv) or (C2.i). Let $\eta_h(0) = Q_h\eta_0$, $u_h(0) = Q_h u_0$. Then, for h sufficiently small, the initial-value problem consisting of* (3.1), (3.2) *and the specified initial values has a unique solution $(\eta_h, u_h) \in \big(C^1(0,T;S_h)\big)^2$ which satisfies*

$$\max_{0\le t\le T}(\|\eta - \eta_h\| + \|u - u_h\|) \le Ch^r\,. \tag{3.6}$$

Proof: Let $\theta := Q_h\eta - \eta_h$, $\xi := Q_h u - u_h$. Then, from (3.1) – (3.2), (3.3) – (3.4) there follows that

$$(\theta_t, \phi) + (\xi_x, \phi) + \big((H\xi + U\theta - \theta\xi)_x, \phi\big) - a(\xi_{xx}, \phi') + b(\theta_{tx}, \phi') = (\psi, \phi)\,, \quad \forall \phi \in S_h\,, \tag{3.7}$$

$$(\xi_t, \chi) + (\theta_x, \chi) + \big((U\xi)_x - \xi\xi_x, \chi\big) - c(\theta_{xx}, \chi') + d(\xi_{tx}, \chi') = (\zeta, \chi)\,, \quad \forall \chi \in S_h\,, \tag{3.8}$$

$$\theta(0) = \xi(0) = 0\,. \tag{3.9}$$

Putting $\phi = \theta$ in (3.7) and $\chi = \xi$ in (3.8) and using periodicity we have

$$\tfrac{1}{2}\tfrac{d}{dt}\big(\|\theta\|^2 + b\|\theta_x\|^2\big) + (\xi_x, \theta) + \big((H\xi)_x, \theta\big) + \tfrac{1}{2}(U_x\theta, \theta) - \tfrac{1}{2}(\xi_x\theta, \theta) - a(\xi_{xx}, \theta_x) = (\psi, \theta)\,, \tag{3.10}$$

$$\tfrac{1}{2}\tfrac{d}{dt}\big(\|\xi\|^2 + d\|\xi_x\|^2\big) + (\theta_x, \xi) + \big((U\xi)_x, \xi\big) - c(\theta_{xx}, \xi_x) = (\zeta, \xi)\,. \tag{3.11}$$

Note that (2.3) and inverse estimates like (2.2a, b) imply that $\|H\|_{1,\infty} + \|U\|_{1,\infty} \le C$ for $0 \le t \le T$.

Let $t_h \in (0, T]$ be the maximal temporal instance for which the initial-value problem (3.7) – (3.9) has unique solution such that

$$\|\theta(t)\|_\infty \le 1 \quad \text{for} \quad 0 \le t \le t_h\,. \tag{3.12}$$

Let first a, b, c, d satisfy condition (C1.i). Multiplying (3.10) by $-c$ and (3.11) by $-a$, adding the resulting equations, and using the Cauchy-Schwartz inequality, (3.12) and (3.5) we obtain for $0 \le t \le t_h$

$$|c|\tfrac{d}{dt}\big(\|\theta\|^2 + b\|\theta_x\|^2\big) + |a|\tfrac{d}{dt}\big(\|\xi\|^2 + d\|\xi_x\|^2\big) \le C\big[(\|\theta\|^2 + b\|\theta_x\|^2) + (\|\xi\|^2 + d\|\xi_x\|^2) + h^{2r}\big]\,,$$

for some constant C independent of h and t_h . Using (3.9) and Gronwall's Lemma we see that

$$\|\theta(t)\|_1 + \|\xi(t)\|_1 \le Ch^r \tag{3.13}$$

holds for $0 \le t \le t_h$, which implies, by Sobolev's inequality, that $\|\theta(t)\|_\infty \le Ch^r$ for $0 \le t \le t_h$. Therefore, if h is sufficiently small, (2.2b) and (3.12) imply that t_h may be taken to be equal to T. Existence-uniqueness of solutions of the initial- value problem (3.1) – (3.2) in $[0, T]$ follows, as does (3.6) in view of (2.3).

In case that (C1.ii) is satisfied with e.g. $b = 0$, $d > 0$, similar considerations imply that

$$\|\theta(t)\| + \|\xi(t)\|_1 \le Ch^r\,, \quad 0 \le t \le t_h\,,$$

and the conclusion of the Theorem follows. If (C1.iii) holds, we argue, cf. [2], as follows. Suppose e.g. that $a = 0$, $c < 0$. We may then write (3.7) in the form

$$A(\theta_t, \phi) + \big((\xi + H\xi + U\theta - \theta\xi)_x, \phi\big) = (\psi, \phi) \quad \forall \phi \in S_h\,,$$

which implies that $\theta_t = R_h\Theta$, where Θ is the solution of the problem $\Theta - b\Theta'' = \psi - (\xi + H\xi + U\theta - \theta\xi)_x$ on $[0, 1]$ with periodic boundary conditions, and R_h is the *elliptic* *projection* onto S_h defined in terms of the bilinear form $A(v,w) := (v,w)+b(v',w')$ on H^1_{per}. It is not hard to see, cf. [2], [5], that R_h is stable on H^2_{per}. This property and elliptic regularity of the above mentioned two-point periodic boundary-value problem yields that

$$\|\theta_t\|_2 \le C(\|\psi\| + \|\xi + H\xi + U\theta - \theta\xi\|_1)\,.$$

Suppose now that $t_h \in (0,T]$ is the maximal time for which e.g. $\|\xi(t)\|_1 \le 1$ for $0 \le t \le t_h$. Then the above estimate and (3.5) yield that

$$\|\theta_t\|_2 \le C(h^r + \|\theta\|_1 + \|\xi\|_1)\,, \quad 0 \le t \le t_h\,. \tag{3.14}$$

Putting now $\chi = \xi_t$ in (3.8) we see that

$$\|\xi_t\|_1 \le C(h^r + \|\theta\|_2 + \|\xi\|_1)\,, \quad 0 \le t \le t_h\,. \tag{3.15}$$

Hence, (3.14), (3.15), and Gronwall's Lemma imply that

$$\|\theta(t)\|_2 + \|\xi(t)\|_1 \le Ch^r \quad \text{for} \quad t \in [0,t_h]\,,$$

and it may be seen again that t_h can be taken equal to T and that the conclusion of the Theorem follows. If $a < 0$, $c = 0$, the roles of θ and ξ are reversed. Finally, if $a = c = 0$, the proof proceeds easily by putting $\phi = \theta$ in (3.7) and $\chi = \xi$ in (3.8), which leads again to an inequality of the type (3.13).
To treat the case (C1.iv) suppose, for example, that $b = 0$, $d > 0$, and $a = c = 0$. Assuming $\|\theta(t)\|_\infty \le 1$ for $0 \le t \le t_h$, adding (3.10) and (3.11), and using (3.5) we obtain

$$\|\theta(t)\| + \|\xi(t)\|_1 \le Ch^r\,, \quad \text{for} \quad 0 \le t \le t_h\,,$$

and the conclusion of the theorem follows as before. In the case (C2.i) assume e.g. that $b = 0$, $d > 0$. Then adding (3.10) and (3.11), using periodicity and the fact that $a = c$ gives, again under the hypothesis that $\|\theta(t)\|_\infty \le 1$ for $0 \le t \le t_h$, that $\|\theta(t)\| + \|\xi(t)\|_1 \le Ch^r$ for $0 \le t \le t_h$ and the proof proceeds analogously. □

Remark: The hypothesis in Theorem 3.1 that the initial values $\eta_h(0)$, $u_h(0)$ are taken as the quasiinterpolants of η_0, u_0, respectively, is convenient in that it allows, in all cases, to apply Gronwall's Lemma with $\theta(0) = \xi(0) = 0$ and obtain (superaccurate in general) estimates like (3.13) etc., from which the L^2 estimates for the errors $\eta - \eta_h$, $u - u_h$ follow. In several cases $\eta_h(0)$ and $u_h(0)$ may be taken, more generally, as optimal-order in L^2 approximations of η_0, u_0, respectively, for example as L^2 projections onto S_h, interpolants etc. Thus, e.g. in case (C1.ii), if $b = 0$, it suffices that $\eta_h(0) - \eta_0 = O(h^r)$ in L^2. Similar observations may be made in the cases (C1.iv) and (C2.i). □

We proceed now to the case (C3). Here we assume that the periodic initial-value problem for the corresponding Boussinesq system is well posed locally in time and that $b = d \ne -1/(4n^2\pi^2)$, for $n = 1, 2, \ldots$, so that the problem

$$w + |b|w'' = 0 \quad \text{on} \quad [0,1]\,,$$

with periodic boundary conditions, has only the trivial solution. From the Fredholm alternative and standard o.d.e. theory it follows that for $f \in L^2(0,1)$ the nonhomogeneous problem

$$-\,|b|w'' - w = f \quad \text{on} \quad [0,1]\,, \tag{3.16}$$

with periodic boundary conditions, has a unique solution $w \in H^2_{per}$ that satisfies the estimates

$$\|w\|_{2-j} \le C\|f\|_{-j}\,, \quad j = 0,1,2\,, \tag{3.17}$$

where the negative norms are defined as usual by

$$\|f\|_{-j} = \sup_{\substack{v \in H^j_{per} \\ v \ne 0}} \frac{(f,v)}{\|v\|_j}\,, \quad j = 1,2\,.$$

We begin the error estimation by stating the following result due to Schatz, [22], including its proof for the convenience of the reader.

Lemma 3.2 *Let* $f \in L^2(0,1)$ *and* w *be the solution of* (3.16) *with periodic boundary conditions, and* $|b| \ne 1/4n^2\pi^2$, $n = 1,2,3,\dots$. *Then, there exist positive constants* h_0 *and* C *such that the discrete problem*

$$|b|(w_h', \phi') - (w_h, \phi) = (f, \phi)\,, \quad \forall \phi \in S_h\,, \tag{3.18}$$

has a unique solution for $h \le h_0$, *for which there hold*

$$\|w - w_h\| \le Ch\|w - w_h\|_1\,, \tag{3.19}$$

and

$$\|w - w_h\|_1 \le C\|w\|_1\,. \tag{3.20}$$

Proof: Assume that (3.18) has a solution $w_h \in S_h$ and let $e = w - w_h$. Then, from (3.16) and (3.18) we have

$$|b|(e', \phi') - (e, \phi) = 0\,, \quad \forall \phi \in S_h\,. \tag{3.21}$$

Let v be the solution of the problem

$$-|b|v'' - v = e \quad \text{on} \quad [0,1]\,,$$

with periodic boundary conditions. Then, cf. (3.17), for some constant $C = C(|b|)$ it follows that

$$\|v\|_2 \le C\|e\|\,. \tag{3.22}$$

Moreover, using (3.21), for each $\chi \in S_h$ we have

$$\begin{aligned}\|e\|^2 &= |b|(e', v') - (e, v) = |b|(e', v' - \chi') - (e, v - \chi) \\ &\le (|b| + 1)\|e\|_1\|v - \chi\|_1\,.\end{aligned}$$

Hence, using (2.1), we obtain, for some constant C_1 independent of h,

$$\|e\| \leq C_1 h \|e\|_1 \,. \tag{3.23}$$

Now, from (3.21) it follows that

$$|b|\|e'\|^2 - \|e\|^2 = |b|(e', w') - (e, w)\,.$$

Therefore,

$$|b|\|e\|_1^2 \leq (1 + |b|)(\|e\|^2 + \|e\|_1\|w\|_1)\,,$$

and by (3.23)

$$|b|\|e\|_1^2 \leq C_1 h^2 (1 + |b|)\|e\|_1^2 + (1 + |b|)\|e\|_1\|w\|_1\,,$$

so that, for h sufficiently small, $|b| - C_1 h^2(1 + |b|) > 0$, and

$$\|e\|_1 \leq C_1 \|w\|_1\,. \tag{3.24}$$

It follows that (3.18) has a unique solution, since if $f = 0$ then $w = 0$, and by (3.24) $w_h = 0$. The required estimates (3.19) and (3.20) are the inequalities (3.23) and (3.24). □

We may now prove a result analogous to that of Theorem 3.1 in the case (C3).

Theorem 3.2 *Suppose a, b, c, d satisfy condition (C3) and that* $|b| \neq 1/4\pi^2 n^2$, $n = 1, 2, 3, \ldots$. *Let* $r \geq 4$ *and* $\eta_h(0) = Q_h \eta_0$, $u_h(0) = Q_h u_0$. *Then, for h sufficiently small, the initial-value problem consisting of* (3.1), (3.2) *and the specified initial values has a unique solution* $(\eta_h, u_h) \in \big(C^1(0, T; S_h)\big)^2$ *that satisfies*

$$\max_{0 \leq t \leq T} (\|\eta - \eta_h\| + \|u - u_h\|) \leq C h^r\,. \tag{3.25}$$

Proof: Let $\theta := Q_h \eta - \eta_h$, $\xi = Q_h u - u_h$. Then (3.7) – (3.9) hold with $b = d < 0$, $a = c > 0$. The initial-value problem (3.7) – (3.9) has a local in time solution for h sufficiently small, essentially because (3.18) has a unique solution for such h. Let $t_h \in (0, T]$ be the maximal temporal instance for which the initial-value problem (3.7) – (3.9) has a unique solution such that

$$\|\xi(t)\|_\infty \leq 1 \quad \text{for} \quad t \in [0, t_h]\,. \tag{3.26}$$

Then, there holds that

$$\tfrac{d}{dt}\big(\|\theta\|^2 + \|\xi\|^2\big) \leq C\big(\|\theta\|_1^2 + \|\xi\|_1^2 + h^{2r}\big)\,, \quad 0 \leq t \leq t_h\,. \tag{3.27}$$

To see this, for $0 \leq t \leq t_h$ let Θ be the solution of the problem

$$\Theta + |b|\Theta_{xx} = -\xi_x - a\xi_{xxx} - (H\xi + U\theta - \xi\theta)_x + \psi\,, \quad \text{on} \quad [0, 1]\,,$$

with periodic boundary conditions, and Ξ be the solution of

$$\Xi + |b|\Xi_{xx} = -\theta_x - a\theta_{xxx} - \big((U\xi)_x - \xi\xi_x\big) + \zeta\,, \quad \text{on} \quad [0, 1]\,,$$

with periodic boundary conditions. Using periodicity to estimate the negative norms of the right-hand sides of the above problems, we have by (3.17) and (3.5)

$$\|\Theta\|_1 \le C(\|\xi\|_2 + \|\theta\| + h^r)\,, \tag{3.28}$$
$$\|\Theta\| \le C(\|\xi\|_1 + \|\theta\| + h^r)\,, \tag{3.29}$$
$$\|\Xi\|_1 \le C(\|\theta\|_2 + \|\xi\| + h^r)\,, \tag{3.30}$$
$$\|\Xi\| \le C(\|\theta\|_1 + \|\xi\| + h^r)\,, \tag{3.31}$$

with a constant C independent of h and t_h. If now $\Theta_h \in S_h$ is the solution of the problem

$$(\Theta_h, \phi) - |b|(\Theta_{hx}, \phi') = -(\xi_x, \phi) + a(\xi_{xx}, \phi') + \big((H\xi + U\theta - \xi\theta)_x, \phi\big) + (\psi, \phi)\,, \quad \forall \phi \in S_h\,,$$

then, by (3.7), $\theta_t = \Theta_h$. Now, by Lemma 3.2 we have $\|\Theta - \Theta_h\| \le Ch\|\Theta\|_1$. Therefore, by (2.2$a$), (3.28) and (3.29) we see that

$$\|\Theta_h\| \le \|\Theta - \Theta_h\| + \|\Theta\| \le C(\|\xi\|_1 + \|\theta\| + h^r)\,,$$

and thus, for $0 \le t \le t_h$

$$\tfrac{1}{2}\tfrac{d}{dt}\|\theta\|^2 = (\theta_t, \theta) = (\Theta_h, \theta) \le C(\|\xi\|_1^2 + \|\theta\|^2 + h^{2r})\,. \tag{3.32}$$

In a similar manner we obtain, using (3.8), (3.30), (3.31) and (2.2a), that for $0 \le t \le t_h$

$$\tfrac{1}{2}\tfrac{d}{dt}\|\xi\|^2 \le C(\|\theta\|_1^2 + \|\xi\|^2 + h^{2r})\,, \tag{3.33}$$

and (3.27) follows by adding (3.32) and (3.33).
The final step of the proof consists in putting $\phi = -\theta$ in (3.7), $\chi = -\xi$ in (3.8), adding and using periodicity to get

$$\begin{aligned}\tfrac{|b|}{2}\tfrac{d}{dt}(\|\theta\|_1^2 + \|\xi\|_1^2) =& \tfrac{1+|b|}{2}\tfrac{d}{dt}(\|\theta\|^2 + \|\xi\|^2) - (H\xi, \theta_x) + \tfrac{1}{2}(U_x\theta, \theta) + (\xi\theta_x, \theta)\\ &+ \tfrac{1}{2}(U_x\xi, \xi) - (\psi, \theta) - (\zeta, \xi)\,,\end{aligned}$$

for $t \in [0, t_h]$. Using (3.26), (3.27) and (3.5) yields

$$\tfrac{d}{dt}(\|\theta\|_1^2 + \|\xi\|_1^2) \le C(\|\theta\|_1^2 + \|\xi\|_1^2 + h^{2r})\,, \quad 0 \le t \le t_h\,,$$

from which, by Gronwall's Lemma and (3.9) we get $\|\theta\|_1 + \|\xi\|_1 \le Ch^r$ in $[0, t_h]$. We conclude, as in Theorem 3.1, that the conclusion of Theorem 3.2 holds. □

4. Semidiscretization of the symmetric Boussinesq systems

We consider now the periodic initial-value problem on $[0, 1]$ of the system (1.3)−(1.4), where $a = c$ and $b, d \ge 0$, which we discretize again in space in S_h for $r \ge 3$. We seek η_h, $u_h \in C^1(0, T; S_h)$, so that for $0 \le t \le T$:

$$(\eta_{ht}, \phi) + (u_{hx}, \phi) + \tfrac{1}{2}\big((\eta_h u_h)_x, \phi\big) - a(u_{hxx}, \phi') + b(\eta_{htx}, \phi') = 0\,, \quad \forall \phi \in S_h\,, \tag{4.1}$$

$$(u_{ht}, \chi) + (\eta_{hx}, \chi) + \tfrac{1}{2}(\eta_h\eta_{hx}, \chi) + \tfrac{3}{2}(u_h u_{hx}, \chi) - a(\eta_{hxx}, \chi') + d(u_{htx}, \chi') = 0\,, \quad \forall \chi \in S_h\,, \tag{4.2}$$

with $\eta_h(0)$ and $u_h(0)$ taken as suitable approximations of η_0, u_0 in S_h. We assume again that smooth solutions of the periodic initial-value problem for (1.3) − (1.4) exist uniquely for $0 \le t \le T$. Taking $\phi = \eta_h$ and $\chi = u_h$ in (4.1), (4.2), adding, and using periodicity, we see that the quantity

$$\|\eta_h\|^2 + \|u_h\|^2 + b\|\eta_{hx}\|^2 + d\|u_{hx}\|^2$$

is a temporal invariant of the o.d.e. system (4.1) − (4.2). The proof of the following result is entirely analogous to that of Lemma 3.1.

Lemma 4.1 *Let $H = Q_h\eta$, $U = Q_h u$, and define $\psi, \zeta \in S_h$ so that for $0 \le t \le T$*

$$(H_t, \phi) + (U_x, \phi) + \tfrac{1}{2}\big((HU)_x, \phi\big) - a(U_{xx}, \phi') + b(H_{tx}, \phi') = (\psi, \phi)\,, \quad \forall \phi \in S_h\,, \tag{4.3}$$

$$(U_t, \chi) + (H_x, \chi) + \tfrac{1}{2}(HH_x, \chi) + \tfrac{3}{2}(UU_x, \chi) - a(H_{xx}, \chi') + d(U_{tx}, \chi') = (\zeta, \chi)\,, \quad \forall \chi \in S_h\,. \tag{4.4}$$

Then, there is a constant C, independent of h and t, such that

$$\|\psi(t)\| + \|\zeta(t)\| \le Ch^r\,, \quad \forall t \in [0, T]\,. \tag{4.5}$$

The main result of this section is

Theorem 4.1 *Let $\eta_h(0) = Q_h\eta_0$, $u_h(0) = Q_h u_0$, and $a = c$, $b, d \ge 0$. Then, for h sufficiently small, the initial-value problem consisting of (4.1), (4.2), and the specified initial values has a unique solution $(\eta_h, u_h) \in \big(C^1(0, T; S_h)\big)^2$ that satisfies*

$$\max_{0 \le t \le T}(\|\eta - \eta_h\| + \|u - u_h\|) \le Ch^r\,. \tag{4.6}$$

Proof: We proceed as in the proof of the analogous result in Section 3. Let $\theta = Q_h\eta - \eta_h$, $\xi = Q_h u - u_h$. Then it follows that

$$(\theta_t, \phi) + (\xi_x, \phi) + \tfrac{1}{2}\big((H\xi + U\theta - \theta\xi)_x, \phi\big) - a(\xi_{xx}, \phi') + b(\theta_{tx}, \phi') = (\psi, \phi)\,, \quad \forall \phi \in S_h, \tag{4.7}$$

$$(\xi_t, \chi) + (\theta_x, \chi) + \tfrac{1}{2}\big((H\theta)_x - \theta\theta_x, \chi) + \tfrac{3}{2}\big((U\xi)_x - \xi\xi_x, \chi) - a(\theta_{xx}, \chi') + d(\xi_{tx}, \chi') = (\zeta, \chi)\,,\ \forall \chi \in S_h, \tag{4.8}$$

$$\theta(0) = \xi(0) = 0. \tag{4.9}$$

Putting $\phi = \theta$ in (4.7) and $\chi = \xi$ in (4.8) and adding, using periodicity, we see that

$$\tfrac{1}{2}\tfrac{d}{dt}(\|\theta\|^2 + b\|\theta_x\|^2) + \tfrac{1}{2}\tfrac{d}{dt}(\|\xi\|^2 + d\|\xi_x\|^2) + \tfrac{1}{2}(H_x\theta, \xi) + \tfrac{1}{2}(\xi_x\theta, \theta) + \tfrac{1}{4}(U_x\theta, \theta) + \tfrac{3}{4}(U_x\xi, \xi) = (\psi, \theta) + (\zeta, \xi). \tag{4.10}$$

As usual, we use the fact that $\|H\|_{1,\infty} + \|U\|_{1,\infty} \leq C$ for $0 \leq t \leq T$, and assume that $t_h \in [0, T]$ is the maximal temporal instance for which $\|\xi_x\|_\infty \leq 1$ for $0 \leq t \leq t_h$. Then, (4.10) and (4.5) give for $0 \leq t \leq t_h$

$$\tfrac{1}{2}\tfrac{d}{dt}\left[\|\theta\|^2 + b\|\theta_x\|^2 + \|\xi\|^2 + d\|\xi_x\|^2\right] \leq C\big(\|\theta\|^2 + \|\xi\|^2 + h^{2r}\big)\,.$$

Hence, Gronwall's Lemma and (4.9) yield

$$\|\theta\|^2 + b\|\theta_x\|^2 + \|\xi\|^2 + d\|\xi_x\|^2 \leq Ch^{2r}\,, \quad 0 \leq t \leq t_h\,.$$

Arguing now as in Theorem 3.1 we obtain the desired result. □

Remarks: (i) Again, if e.g. $b = 0$, we may take $\eta_h(0)$ as an optimal-order in L^2 approximation to η_0 in S_h; if $d = 0$ we may do the same for $u_h(0)$.

(ii) Note that the result (4.6) holds even if both dispersion coefficients b and d are equal to zero, e.g. in the case of the symmetric KdV-KdV system with $a = 1/6$, [12], [13]. In the case $a = 0$, the semidiscrete scheme provides an optimal-order (r can be taken now greater or equal to 2), L^2-conservative approximation in S_h of the solution of the periodic initial-value problem for the *symmetric shallow water equations*

$$\begin{aligned} &\eta_t + u_x + \tfrac{1}{2}(\eta u)_x = 0\,, \\ &u_t + \eta_x + \tfrac{1}{2}\eta\eta_x + \tfrac{3}{2}uu_x = 0\,, \end{aligned} \tag{4.11}$$

provided, of course, that a smooth solution of (4.11) exists for $0 \leq t \leq T$.

5. Numerical examples

The semidiscrete schemes analyzed in the preceding sections may be discretized in the temporal variable by a time-stepping method to provide full discretizations of the periodic initial-value problem for the Boussinesq systems. In this section we show the results of some numerical experiments that we performed for two types of systems for which the corresponding semidiscretizations are not stiff. This permitted using an explicit time-stepping method, and we consequently employed in all examples shown the classical, explicit four-stage Runge-Kutta method (analyzed in [2], [5]), coupled with cubic splines in space.

5.1. Cnoidal waves of the Bona-Smith systems

Cnoidal-wave solutions of the periodic initial-value problem for the KdV equation are well known, cf. [25]. The existence of such solutions for the *a-b-c-d* Boussinesq system (1.1) – (1.2) has been studied in [16]. In this paragraph we derive a family of cnoidal-wave solutions in closed form for some of these systems. These exact solutions are then used to test the accuracy of the fully discrete Galerkin scheme.

To derive the exact solutions, we follow the procedure used in [15] to construct exact solitary waves. Assume that $\eta(\xi)$, $u(\xi)$ are travelling-wave solutions of (1.1) – (1.2) with $\xi = x - c_s t$, such that $u(\xi) = B\eta(\xi)$. Consistency of the Boussinesq system with this type of solutions yields, [15], $B^2 = \frac{2(b-c-2d)}{b-a-2d}$, $c_s = \frac{2-B^2}{B}$ (provided that $b-a-2d \neq 0$, $B > 0$) and that the o.d.e.

$$\alpha\eta' + \beta(\eta^2)' + \gamma\eta''' = 0\,, \tag{5.1}$$

holds, where $' = d/d\xi$ and $\alpha = B(B-c_s)$, $\beta = B^2$, $\gamma = B(aB+bc_s)$. Integrating (5.1) twice, looking as usual, cf. [25], for bounded solutions that oscillate between zero and a maximum value denoted by η_0, we may derive the following family of cnoidal-wave solutions of (1.1) – (1.2):

$$\eta(\xi) = \eta_0 cn^2[\lambda\xi; m]\,, \quad u(\xi) = B\eta(\xi)\,, \tag{5.2}$$

where $\eta_0 = (3-3B^2+\sqrt{9+6(2A-3)B^2+9B^4})/2B^2$, $\kappa = \sqrt{9+6(2A-3)B^2+9B^4}\,/B^2$, $m = (\eta_0/\kappa)^{1/2}$, $\lambda = (\beta\kappa/6\gamma)^{1/2}$, all quantities under the square roots and η_0 assumed to be positive. These solutions have period $\frac{2}{\lambda}K(m)$, where $K(m)$ is the complete elliptic integral of the first kind, given by $K(m) = \int_0^{\pi/2}(1-m^2\sin^2\tau)^{-1/2}d\tau$. The solutions depend on a free parameter A, the constant of the first integration of (5.1) with respect to ξ. Hence, given a suitable *a-b-c-d* system, one positive value of c_s and B is determined by this procedure and one may find a family of cnoidal waves depending on A. As $A \to 0$, since $(\kappa-\eta_0)\eta_0 = 3A/\beta$, we have $m \to 1$, and the solution (5.2) tends to a solitary-wave solution of sech^2 type, cf. [15].

In the case of Bona-Smith systems, where

$$a = 0\,, \quad c = (2-3\theta^2)/3\,, \quad b = d = (3\theta^2-1)/6\,, \quad 2/3 \le \theta^2 \le 1\,,$$

one may derive from the above formulas the family of cnoidal waves, corresponding to the following values of the constants and provided $\theta^2 \in (7/9, 1)$:

$$\begin{aligned}
\alpha &= \tfrac{8}{3\theta^2-1} - 6\,, \quad \beta = \tfrac{4}{3\theta^2-1} - 2\,, \quad \gamma = 2\theta^2 - 4/3\,,\\
B &= \sqrt{\tfrac{2(1-\theta^2)}{\theta^2-1/3}}\,, \quad c_s = \tfrac{4(\theta^2-2/3)}{\sqrt{2(1-\theta^2)(\theta^2-1/3)}}\,,\\
\eta_0 &= \tfrac{-(7-9\theta^2)+\sqrt{(7-9\theta^2)^2-8A(1-4\theta^2+3\theta^4)}}{4(1-\theta^2)}\,,\\
\lambda :&= \sqrt{\tfrac{\beta\kappa}{6\gamma}} = \left(\tfrac{3\sqrt{(7-9\theta^2)^2-8A(1-4\theta^2+3\theta^4)}}{4(2-9\theta^2+9\theta^4)}\right)^{1/2}\,,\\
\kappa &= \tfrac{\sqrt{(7-9\theta^2)^2-8A(1-4\theta^2+3\theta^4)}}{2(1-\theta^2)}\,.
\end{aligned} \tag{5.3}$$

In Figure 1 we show profiles of the η component of these cnoidal waves (over one period) for various values of $A \in [0, 1/2]$ in the case of the Bona-Smith system for $\theta^2 = 9/11$. These profiles were computed using the formulas (5.2) and (5.3). As A tends to zero, the height η_0 of the cnoidal wave decreases and its period increases. In the limit $A = 0$ we obtain the solitary wave of sech^2 type given in [17]. The speed c_s of all these waves is equal to $5\sqrt{3}/6 \backsimeq 1.4434$. In Figure 1 they are depicted over one

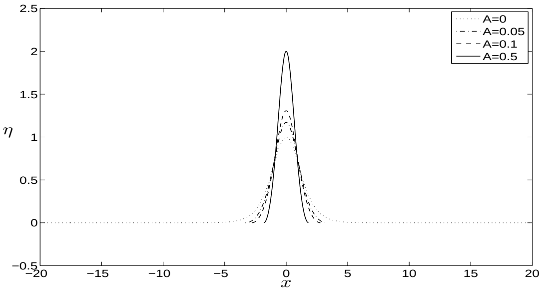


Figure 1: Profiles of η-cnoidal waves for various values of A; Bona-Smith system, $\theta^2 = 9/11$

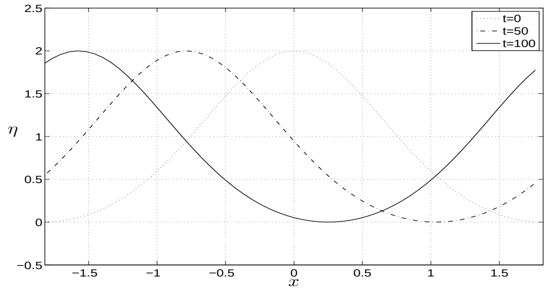


Figure 2: Evolution of an η-cnoidal wave, $A = 0.5$; Bona-Smith system, $\theta^2 = 9/11$

period $[-L, L]$, where

$$\begin{aligned} A &= 0.5\,, & L &= 1.82390 \\ A &= 0.1\,, & L &= 2.79656 \\ A &= 0.05\,, & L &= 3.31353 \\ A &= 0\,, & L &= \infty\,. \end{aligned}$$

Using these profiles as initial values we integrated in time the periodic initial-value problem on $[-L, L]$ for the Bona-Smith system with $\theta^2 = 9/11$ using our fully discrete numerical method. Three profiles of the η component of the numerical solution in the case $A = 0.5$ are shown at $t = 0$, 50 and 100 in Figure 2. (This simulation was done with 240 spatial mesh intervals and a timestep $\Delta t = 10^{-2}$.) At $T = 10$ (i.e. after about four revolutions) the amplitude of the wave $\eta_0 = 2$ was conserved to 9 digits, and so was also the Hamiltonian of the problem $H := \int_{-L}^{L}\big(\eta^2 + |c|\eta_x^2 + (1+\eta)u^2\big)dx$. The speed c_s was conserved to 8 digits and the shape and phase errors (see e.g. [17] for the relevant definitions) were equal to about 0.405×10^{-8} and 0.464×10^{-8} respectively. We obtained similar results for other values of A.

5.2. **Solitary waves of Boussinesq systems of type (C3)**

In this paragraph we study numerically some issues related to solitary-wave solutions of *a*-*b*-*c*-*d* Boussinesq systems of type (C3), i.e. when $b = d < 0$ are such that the homogeneous problem $w + |b|w'' = 0$ on $[-L, L]$ with periodic boundary conditions at $x = \pm L$ has only the trivial solution, and $a = c > 0$. As was mentioned in the Introduction, these systems are linearly well posed, [8], but the question of their nonlinear local well-posedness remains open.

For values of the speed c_s close to one, a local theory, [21], [20], shows that some of these systems possess solitary-wave type solutions. In fact, for a range of values of b and a, specifically when $b < -a$ and the spectrum of the matrix of the linearized solitary-wave equations of the system consists of two pairs of imaginary eigenvalues (i.e. we are in region No. 4 of Figure 3.3 of [19]), the local theory predicts the existence of *generalized solitary waves* (gsw's) for this type of systems, just as in the case of the KdV-KdV systems of [12] and [13]. These are travelling wave solutions

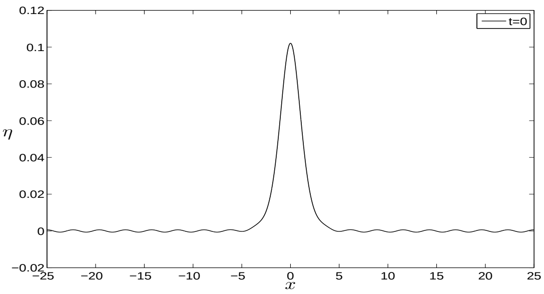


Figure 3: Generalized solitary wave (η profile) of the (C3) Boussinesq system with $a = 1/5$, $b = -1/6$.

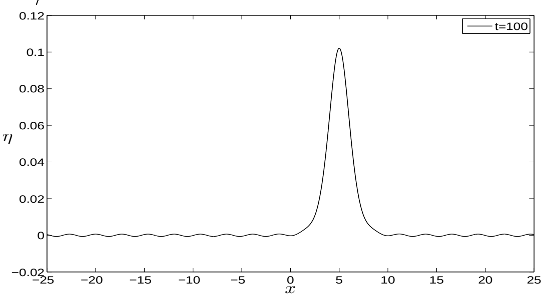
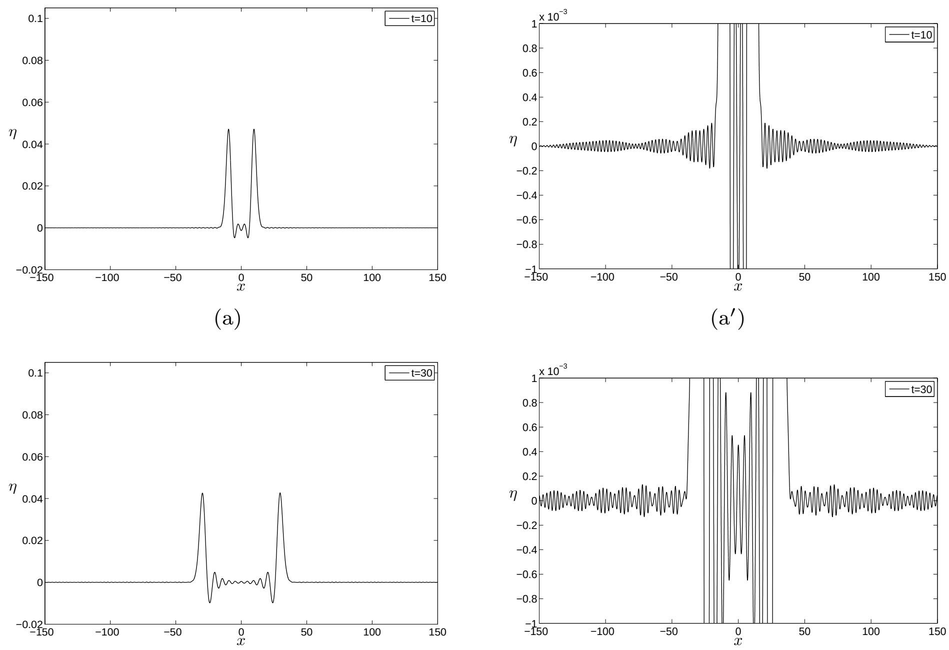


Figure 4: Numerical solution of (C3) Boussinesq system with $a = 1/5$, $b = -1/6$, $t = 100$

with a single hump that are symmetric about their crest and decay to small-amplitude periodic structures (instead of to zero) as their spatial variable becomes large.

As an example, we constructed such gsw's in the case of the system with $a = \frac{1}{5}$,



(a) (a′) (b) (b′)

Figure 5: Evolution from the Gaussian $\eta_0 = 0.1\exp(-x^2/5)$, $u_0 = 0$; η-profiles of the (C3) Boussinesq system with $a = 1/5$, $b = -1/9$. (a′) is a magnification of (a), and (b′) of (b).

$b = -\frac{1}{6}$. For this purpose, we derived the o.d.e. system for $[\eta, \eta', u, u']^T$ in the standard way, i.e. by assuming travelling wave solutions for (1.1) – (1.2), integrating once and converting to a first-order o.d.e. system, on which we imposed periodic boundary conditions at the endpoints of the interval $[-25, 25]$. Taking $c_s = 1.05$ and as initial guess for all components of the system the function $0.2\text{sech}^2 x$, we employed the collocation routine bvp4c of MATLAB as in [12] to compute (with relative tolerance less than 10^{-10}) the η-gsw profile shown in Figure 3. As a check of its travelling-wave property we integrated forward in time with our evolution numerical code the periodic initial-value problem (using $h = 0.01$ on $[-25, 25]$ and $\Delta t = h$) with initial profile (η_0, u_0) given by the η- and u- components of this gsw. The wave propagated with speed $c_s = 1.05$ with no perceptible change of shape. In Figure 4 we show its η-profile at $t = 100$. During this evolution, the Hamiltonian $H = \int_{-L}^{L}[\eta^2 + (1+\eta)u^2 - a(\eta_x^2 + u_x^2)]dx$ of this problem was preserved to 9 digits.

In another series of numerical experiments we tried to study the evolution that ensues from a 'heap-of-water' initial profile of the form $\eta_0(x) = A\exp(-kx^2)$, $u_0(x) = 0$ for such a Boussinesq system. This type of initial profile for the system with $a = 1/5$, $b = -1/6$ produced numerical solutions that blew up soon. An apparently stable evolution was produced by the system with $a = 1/5$, $b = -1/9$ with a Gaussian for η_0 with $A = 0.1$, $k = 1/5$. (We took $h = 0.01$, $\Delta t = h/10$ on $[-150, 150]$.) Figure 5 shows the η-profile of the emerging solution at $t = 10$ and at $t = 30$. We observe that by $t = 10$ the initial profile has separated into two symmetric wavetrains travelling to opposite directions (Figure 5(a)). Figure 5(a′) shows a magnification of the profile in 5(a). Small oscillations radiate in front of the larger pulses as in the case of the KdV-KdV systems, [13]. These oscillations decay in amplitude as $|x|$ grows. By $t = 30$ (Figures 5(b) and 5(b′)) these oscillations have wrapped around the boundary due to periodicity and are interacting, apparently stabilizing in mean amplitude. We are not sure if the leading, larger pulses of the wavetrains are (radiating) generalized solitary waves that will eventually separate from their dispersive tails; they could be linear dispersive structures due to the small size of the initial data.

Acknowledgment: D. Mitsotakis was supported by the Marie Curie Fellowship No. PIEF-GA-2008-219399 of the European Commission.

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