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Novel Integral Representations for Harmonic Functions in the Interior of a Sphere

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Abstract

Solutions of the Laplace equation in terms of novel integral representations valid in the interior of a sphere are obtained by means of the generalized transform introduced by Fokas and his collaborators. The formulae obtained have the advantage of being uniformly convergent on the boundary. Furthermore, it is possible to solve problems with more complicated boundary conditions which demand the solution of a Riemann-Hilbert problem. This is achieved via the so called global relation, i.e. an integral relation connecting the Dirichlet boundary values with the Neumann boundary values. On the other hand, the Dirichlet-to-Neumann correspondence, needed to evaluate the unknown boundary data, for simple boundary data is evaluated by algebraic manipulations of the global relation alone. In the present work we only consider simple boundary conditions, corresponding to the Dirichlet and Neumann problems.

1. Introduction

Within the last decade a generalized transform has been developed by Fokas and his collaborators [1]. The novelty of this transformation is focussed on the fact that it is a transform that meets the particular analytical and geometrical characteristics of the problem at hand. A crucial part of the theory concerns the manipulation of the so-called *global relation*, which is an integral relation connecting the boundary values of the solution (Dirichlet data) with the normal derivative of the solution on the boundary (Neumann data).

Let S be a sphere centered at the origin and radius a. We want to find harmonic functions q_D, q_N that solve the interior Dirichlet and Neumann boundary value problems, respectively. Denote the Dirichlet data on the boundary by g_D, the Neumann data on the boundary by g_N and assume that

$$\frac{\partial}{\partial \phi}\, g_D(\mathbf{r}) = \frac{\partial}{\partial \phi}\, g_N(\mathbf{r}) = 0, \quad r = a,$$

where r denotes the radial spherical coordinate. Using the spherical coordinates (r, θ, ϕ), utilizing the fact that g_D and g_N are ϕ-independent, and introducing the variable

$$\zeta = \cos\theta, \quad \theta \in (0, \pi),$$

Laplace's equation reads

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial \zeta^2} - \frac{2\zeta}{r^2}\frac{\partial}{\partial \zeta}\right) q(r,\zeta) = 0\ , \tag{1.1}$$

2. The Global Relation

Let $q(r,\zeta)$ satisfy the Laplace equation (1.1) and $\overline{q}(r,\zeta)$ satisfies the formal adjoint of equation (1.1)

$$\left(\frac{\partial^2}{\partial r^2} - \frac{2}{r}\frac{\partial}{\partial r} + \frac{2}{r^2} + \frac{1-\zeta^2}{r^2}\frac{\partial^2}{\partial \zeta^2} - \frac{2\zeta}{r^2}\frac{\partial}{\partial \zeta}\right) \overline{q}(r,\zeta) = 0\ . \tag{2.1}$$

Multiply (1.1) by $\overline{q}(r,\zeta)$, (2.1) by $q(r,\zeta)$ and subtracting the resulting equations yields, after some algebraic manipulations, the divergence form

$$\frac{\partial}{\partial r}\left(\overline{q}\,\frac{\partial q}{\partial r} + \left(\frac{2}{r}\,\overline{q} - \frac{\partial \overline{q}}{\partial r}\right) q\right) + \frac{\partial}{\partial \zeta}\left(\frac{1-\zeta^2}{r^2}\left(\overline{q}\,\frac{\partial q}{\partial \zeta} - q\,\frac{\partial \overline{q}}{\partial \zeta}\right)\right) = 0\ . \tag{2.2}$$

Applying Green's theorem to a closed subdomain of Ω, we obtain

$$\oint_C \left[\left(\overline{q}\,\frac{\partial q}{\partial r} + \left(\frac{2}{r}\,\overline{q} - \frac{\partial \overline{q}}{\partial r}\right) q\right) d\zeta - \frac{1-\zeta^2}{r^2}\left(\overline{q}\,\frac{\partial q}{\partial \zeta} - q\,\frac{\partial \overline{q}}{\partial \zeta}\right) dr\right] = 0\ , \tag{2.3}$$

where C is the boundary of the subdomain.

3. The Dirichlet-to-Neumann Correspondence

Replace $\overline{q}(r,\zeta)$ in (2.1) by $\overline{R}(r)\overline{Z}(\zeta)$, to obtain

$$r^2\frac{d^2\overline{R}}{dr^2} - 2r\frac{d\overline{R}}{dr} - (\nu-1)(\nu+2)\overline{R} = 0, \quad \nu \in \mathbb{C} \tag{3.1}$$

for the $\overline{R}(r;\nu)$ function, and the equation

$$(1-\zeta^2)\frac{d^2\overline{Z}}{d\zeta^2} - 2\zeta\frac{d\overline{Z}}{d\zeta} + \nu(\nu+1)\overline{Z} = 0, \quad \nu \in \mathbb{C} \tag{3.2}$$

for the $\overline{Z}(\zeta;\nu)$ function. Hence, the $\zeta-$dependence remain the same, as in the Laplace's equation, while the $r-$dependence is replaced by

$$\overline{R}_1(r;\nu) = r^{\nu+2}, \qquad \overline{R}_2(r;\nu) = r^{-\nu+1}, \quad \nu \in \mathbb{C}. \tag{3.3}$$

Equation (2.1) accepts solutions of the form

$$\overline{q}(r,\zeta;\nu) = \overline{R}(r;\nu)X_\nu(\zeta) \tag{3.4}$$

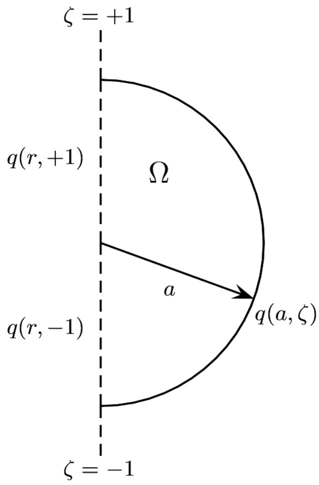

Figure 1: The domain $\Omega = \{0 \le r \le a,\ -1 < \zeta < +1\}$

where $\overline{R}(r;\nu)$ are given by (3.3) and $X_\nu(\zeta)$ stands for any Legendre function. The solution $\overline{q}(r,\zeta;\nu)$ remains bounded in the neighborhood of $r=0$ for $\mathrm{Re}\,\nu \in [-2,+\infty)$ when the functions $\overline{R}_1(r;\nu)$ are chosen and for $\mathrm{Re}\,\nu \in (-\infty,+1]$ when the functions $\overline{R}_2(r;\nu)$ are chosen. Thus, applying (2.3) in the domain Ω, as shown in Figure 1, it is straightforward to show that

$$\overline{R}(a;k)\,\mathfrak{N}(k) + \left(\frac{2}{a}\,\overline{R}(a;k) - \frac{d\overline{R}(a;k)}{dr}\right)\mathfrak{D}(k) = -\frac{2}{\pi}\sin\pi k \int_0^a q(r,-1)\,\overline{R}(r;k)\,\frac{dr}{r^2}\,, \tag{3.5}$$

where the Legendre transforms of the boundary data $\mathfrak{D}(k)$ and $\mathfrak{N}(k)$ are given by

$$\mathfrak{D}(k) = \int_{-1}^{+1} g_{D}(\zeta)P_k(\zeta)d\zeta\,, \tag{3.6}$$

$$\mathfrak{N}(k) = \int_{-1}^{+1} g_{N}(\zeta)P_k(\zeta)d\zeta\,. \tag{3.7}$$

4. A Novel Integral Representation

Applying the global relation (2.3) in the subdomain $\Omega_1(\zeta) = \{0 \le r \le a,\ \zeta \le t < +1\}$, depicted in Figure 2 with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\,P_\nu(\zeta)$ we obtain

$$\int_0^a (1-\zeta^2)\left(P_\nu(\zeta)\,\frac{\partial q(r,\zeta)}{\partial\zeta} - \frac{dP_\nu(\zeta)}{d\zeta}\,q(r,\zeta)\right)\overline{R}(r;\nu)\,\frac{dr}{r^2}$$
$$= \int_\zeta^{+1}\left[\overline{R}(a;\nu)\,g_N(t) + \left(\frac{2}{a}\overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{dr}\right)g_D(t)\right]P_\nu(t)dt, \quad \nu\in\mathbb{C}. \tag{4.1}$$

Analogous, applying the global relation (2.3) in the subdomain $\Omega_2(\zeta) = \{0 \leq r \leq a,\ -1 < t \leq \zeta\}$, depicted in Figure 2, with $\overline{q}(r,\zeta;\nu)$ replaced by $\overline{R}(r;\nu)\, Q_\nu(\zeta)$

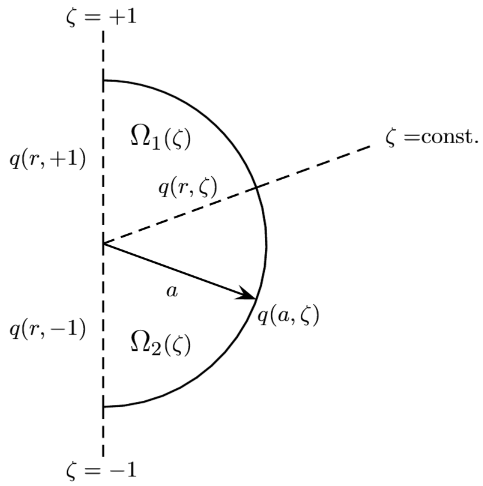

Figure 2: The subdomains $\Omega_1(\zeta) = \{0 \leq r \leq a,\ \zeta \leq t < +1\}$ and $\Omega_2(\zeta) = \{0 \leq r \leq a,\ -1 < t \leq \zeta\}$.

$$\begin{aligned}
&\int_0^a (1-\zeta^2)\left(Q_\nu(\zeta)\, \frac{\partial q(r,\zeta)}{\partial \zeta} - \frac{dQ_\nu(\zeta)}{d\zeta}\, q(r,\zeta) \right) \overline{R}(r;\nu) \frac{dr}{r^2} \\
&= - \int_{-1}^{\zeta} \left[\overline{R}(a;\nu)\, g_N(t) + \left(\frac{2}{a}\, \overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{dr} \right) g_D(t) \right] Q_\nu(t) dt \\
&\quad - \cos(\pi\nu) \int_0^a q(r,-1)\, \overline{R}(r;\nu) \frac{dr}{r^2}, \quad \nu \in \mathbb{C} - \{-1,-2,\dots\} \, . \qquad (4.2)
\end{aligned}$$

Multiply (4.1) by $Q_\nu(\zeta)$ and (4.2) by $P_\nu(\zeta)$ and adding them in order to eliminate the unknown function $\frac{\partial q(r,\zeta)}{\partial \zeta}$

$$\begin{aligned}
&\int_0^a q(r,\zeta)\, \overline{R}(r;\nu) \frac{dr}{r^2} \\
&= \mathfrak{P}_\nu(\zeta) \left[\overline{R}(a;\nu)\, g_N(t) + \left(\frac{2}{a}\, \overline{R}(a;\nu) - \frac{d\overline{R}(a;\nu)}{dr} \right) g_D(t) \right] \\
&\quad + \cos(\pi\nu)\, P_\nu(\zeta) \int_0^a q(r,-1)\, \overline{R}(r;\nu) \frac{dr}{r^2} \, , \qquad (4.3)
\end{aligned}$$

where

$$\mathfrak{P}_\nu(\zeta) \equiv P_\nu(\zeta) \int_{-1}^{\zeta} dt\; Q_\nu(t) + Q_\nu(\zeta) \int_{\zeta}^{1} dt\; P_\nu(t), \quad \nu \in \mathbb{C} \, , \qquad (4.4)$$

is an integral operator, which we will refer to as the *Legendre Integral Operator.*

4.1. **The Half-plane Re $\nu \in [-2, +\infty)$**

In the half-plane $\mathrm{Re}\,\nu \in [-2, +\infty)$, $\overline{q}(r, \zeta; \nu)$ remains bounded in the vicinity of $r = 0$ if $\overline{R}(r; \nu)$ is replaced by $r^{\nu+2}$. Thus (4.3) becomes the Mellin transform for the function $r\, q(r, \zeta)$,

$$\int_0^a r\, q(r,\zeta)\, r^{\nu-1}\, dr = a^{\nu+1}\, \mathfrak{P}_\nu(\zeta)\left(a\, g_N(t) - \nu\, g_D(t)\right) + \cos(\pi\nu)\, P_\nu(\zeta) \int_0^a q(r,-1)\, r^\nu\, dr, \quad \mathrm{Re}\,\nu \geq 0\,, \tag{4.5}$$

where the restriction on ν is due the fact that r^ν must remain bounded as r tends to zero. The inverse transform formula then implies

$$q(r,\zeta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{a}{r}\right)^{\nu+1} \mathfrak{P}_\nu(\zeta)\Big(a\, g_N(t) - \nu\, g_D(t)\Big) d\nu + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-\nu-1}\, \cos(\pi\nu)\, P_\nu(\zeta) \left[\int_0^a q(r,-1)\, r^\nu\, dr\right] d\nu, \tag{4.6}$$

Since

$$P_\nu(\zeta) = P_{-\nu-1}(\zeta) \tag{4.7}$$

for all values of ν and

$$Q_{-\nu-1}(\zeta) = Q_\nu(\zeta) - \pi\cot(\pi\nu) P_\nu(\zeta) \tag{4.8}$$

for every $\nu \in \mathbb{C}$ except integral values [2, p.599], replacing ν with $-\nu - 1$ in (4.5), we obtain the formula

$$a^\nu \int_0^a q(r,\zeta)\, r^{-\nu-1}\, dr = \mathfrak{P}_\nu(\zeta)\left(a\, g_N(t) + (\nu+1)\, g_D(t)\right) - a^\nu\, \cos(\pi\nu)\, P_\nu(\zeta) \int_0^a q(r,-1)\, r^{-\nu-1}\, dr, \quad \mathrm{Re}\,\nu < -1\,, \quad \nu \neq -2, -3, \dots\,. \tag{4.9}$$

Utilizing (4.9) to eliminate the unknown boundary data from (4.6), we obtain the following equations

$$\begin{aligned} q(r,\zeta) = &-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{a}{r}\right)^{\nu+1} (2\nu+1)\left(\mathfrak{P}_\nu(\zeta)\, g_D(t)\right) d\nu \\ &+ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{a}{r}\right)^{\nu+1} a^\nu \left[\int_0^a q(\tau,\zeta)\, \tau^{-\nu-1}\, d\tau\right] d\nu \\ &+ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{a}{r}\right)^{\nu+1} a^\nu\, \cos(\pi\nu)\, P_\nu(\zeta) \left[\int_0^a q(\tau,-1)\, \tau^{-\nu-1}\, d\tau\right] d\nu \\ &+ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-\nu-1}\, \cos(\pi\nu)\, P_\nu(\zeta) \left[\int_0^a q(\tau,-1)\, \tau^\nu\, d\tau\right] d\nu, \end{aligned} \tag{4.10}$$

if Dirichlet boundary values are described, or

$$\begin{aligned} q(r,\zeta) =& \frac{a}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{2\nu+1}{\nu+1}\Big(\mathfrak{P}_\nu(\zeta)\, g_N(t)\Big)\, d\nu \\ &- \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{\nu}{\nu+1}\, a^\nu\left[\int_0^a q(\tau,\zeta)\,\tau^{-\nu-1}\,d\tau\right] d\nu \\ &- \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{r}\right)^{\nu+1}\frac{\nu}{\nu+1}\, a^\nu\,\cos(\pi\nu)\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,d\tau\right] d\nu \\ &+ \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} r^{-\nu-1}\,\cos(\pi\nu)\,P_\nu(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,d\tau\right] d\nu, \end{aligned} \tag{4.11}$$

for Neumann data.
By a deformation of contour process to the left complex $\nu-$plane the three last integrals on the rhs of equations (4.10), (4.11) vanish and we finally obtain

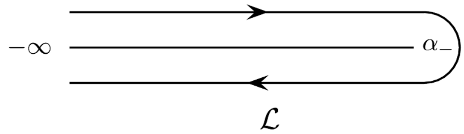

Figure 3: The contour $\mathcal{L}$.

$$q(r,\zeta) = \frac{1}{2\pi i}\int_{\mathcal{L}}\left(\frac{a}{r}\right)^{\nu+1}(2\nu+1)\Big(\mathfrak{P}_\nu(\zeta)\, g_D(t)\Big)\, d\nu, \tag{4.12}$$

if Dirichlet boundary values are described, or

$$q(r,\zeta) = -\frac{a}{2\pi i}\int_{\mathcal{L}}\left(\frac{a}{r}\right)^{\nu+1}\frac{2\nu+1}{\nu+1}\Big(\mathfrak{P}_\nu(\zeta)\, g_N(t)\Big)\, d\nu, \tag{4.13}$$

for Neumann data, respectively.

4.2. **The half-plane Re** $\nu \in (-\infty, +1]$

Replace $\overline{R}(r;\nu)$ with $r^{-\nu+1}$ in order that the solution remains bounded in the neighborhood of $r=0$. Equation (4.3) then reads

$$\begin{aligned} \int_0^a q(r,\zeta)\, r^{-\nu-1}\, dr =& a^{-\nu}\,\mathfrak{P}_\nu(\zeta)\Big(a\, g_N(t) + (\nu+1)\, g_D(t)\Big) \\ &+ \cos(\pi\nu)\, P_\nu(\zeta)\int_0^a q(r,-1)\, r^{-\nu-1}\, dr, \\ &\mathrm{Re}\,\nu < -1,\ \nu \neq -2,-3,\dots\,. \end{aligned} \tag{4.14}$$

The inverse formula then implies

$$q(r,\zeta) = \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\frac{r}{a}\right)^{\nu}\mathfrak{P}_{\nu}(\zeta)\Big(a\,g_N(t)+(\nu+1)\,g_D(t)\Big)d\nu$$
$$+\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} r^{\nu}\,\cos(\pi\nu)\,P_{\nu}(\zeta)\left[\int_0^a q(r,-1)\,r^{-\nu-1}\,dr\right]d\nu, \qquad (4.15)$$

Replacing in (4.14) $-\nu-1$ with ν we obtain

$$\int_0^a r\,q(r,\zeta)\,r^{\nu-1}\,dr = a^{\nu+1}\,\mathfrak{P}_{\nu}(\zeta)\,\Big(a\,g_N(t)-\nu\,g_D(t)\Big)$$
$$-\cos(\pi\nu)\,P_{\nu}(\zeta)\int_0^a q(r,-1)\,r^{\nu}\,dr,$$
$$\operatorname{Re}\nu>0,\ \nu\neq 1,2,\dots\,, \qquad (4.16)$$

Eliminate the unknown boundary data in (4.15) with the aid of (4.16), to derive the following equations.

$$q(r,\zeta) = \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\frac{r}{a}\right)^{\nu}(2\nu+1)\,\Big(\mathfrak{P}_{\nu}(\zeta)\,g_D(t)\Big)\,d\nu$$
$$+\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\frac{r}{a}\right)^{\nu}a^{-\nu-1}\left[\int_0^a q(\tau,\zeta)\,\tau^{\nu}\,d\tau\right]d\nu$$
$$+\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\frac{r}{a}\right)^{\nu}a^{-\nu-1}\,\cos(\pi\nu)\,P_{\nu}(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,d\tau\right]d\nu$$
$$+\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} r^{\nu}\,\cos(\pi\nu)\,P_{\nu}(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,d\tau\right]d\nu, \qquad (4.17)$$

in the case where Dirichlet boundary values are prescribed, or

$$q(r,\zeta) = \frac{a}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{2\nu+1}{\nu}\,\Big(\mathfrak{P}_{\nu}(\zeta)\,g_N(t)\Big)\,d\nu$$
$$-\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,a^{-\nu-1}\left[\int_0^a q(\tau,\zeta)\,\tau^{\nu}\,d\tau\right]d\nu$$
$$-\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\left(\frac{r}{a}\right)^{\nu}\frac{\nu+1}{\nu}\,a^{-\nu-1}\,\cos(\pi\nu)\,P_{\nu}(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{\nu}\,d\tau\right]d\nu$$
$$+\frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} r^{\nu}\,\cos(\pi\nu)\,P_{\nu}(\zeta)\left[\int_0^a q(\tau,-1)\,\tau^{-\nu-1}\,d\tau\right]d\nu, \qquad (4.18)$$

if Neumann data are given.

Following similar considerations as in section 4.1 the last three integrals on the rhs

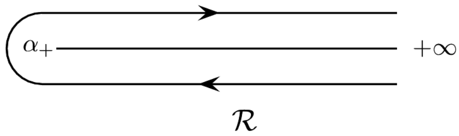

Figure 4: The contour $\mathcal{R}$.

of (4.17) and (4.18) yielding a zero contribution and we find

$$q(r,\zeta) = \frac{1}{2\pi i}\int_{\mathcal{R}} \left(\frac{r}{a}\right)^{\nu} (2\nu+1)\left(\mathfrak{P}_{\nu}(\zeta)\, g_D(t)\right) d\nu, \tag{4.19}$$

if Dirichlet boundary values are described, or

$$q(r,\zeta) = \frac{a}{2\pi i}\int_{\mathcal{R}} \left(\frac{r}{a}\right)^{\nu} \frac{2\nu+1}{\nu}\left(\mathfrak{P}_{\nu}(\zeta)\, g_N(t)\right) d\nu, \tag{4.20}$$

for Neumann data, respectively.

References

1. A.S. Fokas. *A Unified Approach To Boundary Value Problems.* SIAM, Philadelphia, Pennsylvania, 1st edition, 2008.
2. P.M. Morse and H. Feshbach. *Methods of Theoretical Physics*, Vol. 1. McGraw-Hill, 1953.