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On the Character of Neuronal Currents as Sources for EEG and MEG Fields

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Abstract

The basic electric and magnetic differences between the spherical and the non-spherical model of the brain and its response under neuronal excitation is investigated. It is shown that, for the spherical model, no part of the neuronal current contributes both to the exterior electric potential and to the magnetic field that are observed outside the head. Therefore, in this case, Magnetoencephalography provides information that is missed by Electroencephalography, and vice versa. On the other hand, for the non-spherical models, there is a component of the current that contributes both to the electric potential and to the magnetic field. Nevertheless, a part of the current is 'visible' only by the magnetic field. Consequently, for any geometrical model of the brain, Magnetoencephalography carries information which can not be carried by Electroencephalography. This conclusion resolves the question of whether there exists information encoded in the magnetoencephalographic recordings that are independent of the synchronously obtained electroencephalographic data.

1. Introduction

In a celebrated paper, published in 1853, Helmholtz [21] demonstrated that it is impossible to identify uniquely a primary current within a conductor from the knowledge of the magnetic field generated outside the conductor. This result implies that the inverse problem of Magnetoencephalography (MEG) is not uniquely solvable. Similarly, the primary current inside a conductor is not uniquely specified if the electric potential is given on the boundary of the conductor. Hence, the inverse problem of Electroencephalography (EEG) is not uniquely solvable either. A complete mathematical analysis of the inverse MEG problem for the sphere was given in [16] and [17], where it was proved that, among the three scalar functions that are used to represent the current via the Hansen [20] decomposition, only one is needed to express the exterior magnetic field in terms of the current. It turns out that the corresponding analysis for the inverse EEG problem is harder, since it involves a more complicated system of singularities [9]. It is of interest though to see that if the same Hansen decomposition, as in the MEG case, is assumed for the current, then the electric potential is expressed in terms of the two scalar functions that the magnetic potential misses. This result, which proves that, for the case of spherical geometry, no part of the current contributes both to the EEG and MEG, was proved recently by Hadjiloizi and the author [9]. The analysis in [17] and [9] is based on differential representations of the current, integral representations of the fields, distribution theory and the theory of moments. Equivalently, one can use spectral methods and appropriate eigen-function expansions to arrive at similar results [6]. Indeed, if we replace the Hansen decomposition of the current with an expansion in terms of vector spherical harmonics of the $\boldsymbol{P}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ type [25], then we can prove that, the component of the current that lives in the subspace spanned by the $\boldsymbol{C}$-functions contributes only to the magnetic field, and the component of the current that lives in the subspace spanned by the $\boldsymbol{P}$ and $\boldsymbol{B}$-functions contributes only to the electric potential.

This, complementary behaviour between EEG and MEG is not preserved if the geometry of the conductor is not spherical [3]. This can be demonstrated by considering the smallest sphere that circumscribes the conductor, and work as in the case of a sphere for the region outside this sphere. In this way, it can be shown that there is a part of the current, actually the one that lives in the subspace spanned by the $\boldsymbol{P}$ and $\boldsymbol{B}$-functions, that influences both the electric potential and the magnetic field. Nevertheless, the component of the current that lives in the subspace spanned by the $\boldsymbol{C}$-functions, even in this non-symmetric case, influences only to the magnetic field. Therefore, outside the spherical symmetry, EEG and MEG do not behave in a complimentary way. Whether this behaviour is an intrinsic one, or it is a consequence of the lack of appropriate vector decomposition for the current that fits the actual geometry of the conductor, remains to be investigated. Interesting results along this direction, have been reported in [15], where the Hansen decomposition of the current has been replaced by the Helmholtz decomposition [7].

From the realistic point of view, the best geometrical approximation of the brain-head system is provided by a triaxial ellipsoid. Nevertheless, solving boundary value problems in ellipsoidal coordinates is a highly technical and complicated process, which also has definite limitations. On the other hand, such solutions reveal some structural properties of the EEG and MEG methods which are invisible in the presence of spherical symmetry. We mention for example the fact that, in the spherical case, shells of different conductivity are not recognizable by the MEG recordings, while they are clearly identified in the case of ellipsoidal conductors [14], [8]. Explicit results on EEG and MEG in ellipsoidal geometry can be found in [2], [4], [5], [8], [10], [11], [12], [13], [14], [22] and [23].

The present work is organized as follows. Section 2 contains a brief introduction to the mathematics of the electric and magnetic brain activity. Then the spherical model is analyzed in detail in Section 3. The case of non-spherical geometries is investigated in Section 4. Finally, in Section 5, we summarize and discuss all the results mentioned in this presentation.

2. The Mathematics of EEG and MEG

Let Ω be a bounded and connected domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. The domain Ω provides a simplified geometrical model of the brain-head system as an isotropic and homogeneous conductor with conductivity σ. A neuronal current $\boldsymbol{J}^p$ is supported within Ω. The electric and magnetic activity that is generated from $\boldsymbol{J}^p$ is governed by the quasi-static theory of Electromagnetism [26], [24]. Hence, the electric field $\boldsymbol{E}$ is irrotational and therefore is represented by an electric potential u as

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla u(\boldsymbol{r}) \tag{1}$$

and the magnetic field $\boldsymbol{B}$ is the solution of the equation

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \mu_0 \boldsymbol{J}^p(\boldsymbol{r}) - \mu_0 \sigma \nabla u(\boldsymbol{r}) \tag{2}$$

which is also solenoidal, where μ_0 denotes the magnetic permeability in the regions inside and outside the brain. Taking the divergence of equation (2) and using the fact that Ω is a conductor it follows that the electric potential u^- in Ω has to satisfy the interior Neumann problem

$$\sigma \Delta u^-(\boldsymbol{r}) = \nabla \cdot \boldsymbol{J}^p(\boldsymbol{r}), \quad \boldsymbol{r} \in \Omega \tag{3}$$

$$\frac{\partial}{\partial n} u^-(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \partial\Omega. \tag{4}$$

The electric potential u^+ in the exterior of Ω satisfies the exterior Dirichlet problem

$$\Delta u^+(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \Omega^c \tag{5}$$

$$u^+(\boldsymbol{r}) = u^-(\boldsymbol{r}), \quad \boldsymbol{r} \in \partial\Omega. \tag{6}$$

Once the boundary value problem (3)–(4) is solved, the trace of the electric potential on $\partial\Omega$ is obtained which can be used to express the electric potential as

$$4\pi\sigma u^-(\boldsymbol{r}) = \int_\Omega \boldsymbol{J}^p(\boldsymbol{\tau}) \cdot \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} dv(\boldsymbol{\tau}) - \sigma \oint_{\partial\Omega} u^-(\boldsymbol{\tau})\hat{\boldsymbol{n}}(\boldsymbol{\tau}) \cdot \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} ds(\boldsymbol{\tau}) \tag{7}$$

and the magnetic field as

$$\frac{4\pi}{\mu_0}\boldsymbol{B}(\boldsymbol{r}) = \int_\Omega \boldsymbol{J}^p(\boldsymbol{\tau}) \times \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} dv(\boldsymbol{\tau}) - \sigma \oint_{\partial\Omega} u^-(\boldsymbol{\tau})\hat{\boldsymbol{n}}(\boldsymbol{\tau}) \times \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} ds(\boldsymbol{\tau}). \tag{8}$$

Formulae (7) and (8) were obtained by Geselowitz in [18] and [19], respectively. They form the backbone of the mathematical theory of EEG and MEG.

Since, in the exterior of the head there is no neuronal current and the medium is not conductive it follows from (3) that the magnetic field is also irrotational there. Therefore, there exists a scalar magnetic potential U such that

$$\frac{4\pi}{\mu_0}\boldsymbol{B}(\boldsymbol{r}) = \nabla_{\boldsymbol{r}} U(\boldsymbol{r}), \quad \boldsymbol{r} \notin \Omega. \tag{9}$$

It was shown in [7] that the scalar magnetic potential U satisfies the integral representation

$$U(\boldsymbol{r}) = \int_\Omega \frac{\boldsymbol{J}^p(\boldsymbol{\tau}) \times \boldsymbol{\tau} \cdot \boldsymbol{r}}{F(\boldsymbol{r};\boldsymbol{\tau})} dv(\boldsymbol{r}) - \sigma \oint_{\partial\Omega} u^-(\boldsymbol{\tau}) \frac{\hat{\boldsymbol{n}}(\boldsymbol{\tau}) \times \boldsymbol{\tau} \cdot \boldsymbol{r}}{F(\boldsymbol{r};\boldsymbol{\tau})} ds(\boldsymbol{\tau}), \quad \boldsymbol{r} \notin \Omega \tag{10}$$

where the function F [27], which is given by

$$F(\boldsymbol{r};\boldsymbol{\tau}) = |\boldsymbol{r}-\boldsymbol{\tau}|^2|\boldsymbol{r}| + |\boldsymbol{r}-\boldsymbol{\tau}|\boldsymbol{r} \cdot (\boldsymbol{r}-\boldsymbol{\tau}) \tag{11}$$

satisfies the differential equation

$$\frac{\partial}{\partial r} \frac{r}{F(\boldsymbol{r};\boldsymbol{\tau})} = -\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|^3}. \tag{12}$$

In view of (7), the representation (6) can be rewritten as

$$\frac{4\pi}{\mu_0}\boldsymbol{B}(\boldsymbol{r}) = \nabla_{\boldsymbol{r}} \times \left[\int_\Omega \frac{\boldsymbol{J}^p(\boldsymbol{\tau})}{|\boldsymbol{r}-\boldsymbol{\tau}|} dv(\boldsymbol{\tau}) - \sigma \oint_{\partial\Omega} \frac{u^-(\boldsymbol{\tau})\hat{\boldsymbol{n}}(\boldsymbol{\tau})}{|\boldsymbol{r}-\boldsymbol{\tau}|} ds(\boldsymbol{\tau}) \right] \tag{13}$$

which shows that the function

$$\boldsymbol{A}(\boldsymbol{r}) = \int_{\Omega} \frac{\boldsymbol{J}^p(\boldsymbol{\tau})}{|\boldsymbol{r}-\boldsymbol{\tau}|} dv(\boldsymbol{\tau}) - \sigma \oint_{\partial\Omega} \frac{u^-(\boldsymbol{\tau})\hat{\boldsymbol{n}}(\boldsymbol{\tau})}{|\boldsymbol{r}-\boldsymbol{\tau}|} ds(\boldsymbol{\tau}) \tag{14}$$

is a vector potential for the solenoidal field $\boldsymbol{B}$. Comparing the gradient representation (7)–(8) with the rotational representation (10), we see that the function 1/F plays, for the scalar potential U, a role similar to the one that the fundamental solution $1/|\boldsymbol{r}-\boldsymbol{\tau}|$ plays for the vector potential $\boldsymbol{A}$. The fundamental role of the function $1/F$ in the expression of the scalar magnetic potential was originally recognized by Bronzan [1] in connection to nuclear potentials, and by Sarvas [27] in connection to magnetoencephalography.

For a dipolar current with moment $\boldsymbol{Q}$, which is located at the point $\boldsymbol{\tau}$, the current reads

$$\boldsymbol{J}^p(\boldsymbol{r}) = \boldsymbol{Q}\delta(\boldsymbol{r}-\boldsymbol{\tau}). \tag{15}$$

In this case, formulae (5), (6) and (10) become

$$4\pi\sigma u^-(\boldsymbol{r}) = \boldsymbol{Q} \cdot \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} - \sigma \oint_{\partial\Omega} u^-(\boldsymbol{\tau})\hat{\boldsymbol{n}}(\boldsymbol{\tau}) \cdot \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} ds(\boldsymbol{\tau}) \tag{16}$$

$$\frac{4\pi}{\mu_0}\boldsymbol{B}(\boldsymbol{r}) = \boldsymbol{Q} \times \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} - \sigma \oint_{\partial\Omega} u^-(\boldsymbol{\tau})\hat{\boldsymbol{n}}(\boldsymbol{\tau}) \times \frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} ds(\boldsymbol{\tau}) \tag{17}$$

and

$$U(\boldsymbol{r}) = \frac{\boldsymbol{Q} \times \boldsymbol{\tau} \cdot \boldsymbol{r}}{F(\boldsymbol{r};\boldsymbol{\tau})} - \sigma \oint_{\partial\Omega} u^-(\boldsymbol{\tau}) \frac{\hat{\boldsymbol{n}}(\boldsymbol{\tau}) \times \boldsymbol{\tau} \cdot \boldsymbol{r}}{F(\boldsymbol{r};\boldsymbol{\tau})} ds(\boldsymbol{\tau}) \tag{18}$$

respectively.

3. The Spherical Model

Here we assume that the domain Ω is a sphere of radius α centered at the origin. Then, for the current (15), the solution of the Neumann problem (3)–(4) is given by [6]

$$4\pi\sigma u^-(\boldsymbol{r}) = \boldsymbol{Q} \cdot \nabla_{\boldsymbol{\tau}} \frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|} + \sum_{n=1}^{\infty} \frac{n+1}{n} \frac{\tau^n r^n}{\alpha^{2n+1}} P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\tau}}), \quad r < \alpha \tag{19}$$

where P_n denotes the Legendre polynomial of degree n. Utilizing the fundamental expansion

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|} = \sum_{n=0}^{\infty} \frac{\tau^n}{r^{n+1}} P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\tau}}), \quad r > \tau \tag{20}$$

we can show that the electric potential on the boundary of the sphere is given by

$$4\pi\sigma u^-(\alpha\hat{\boldsymbol{r}}) = \boldsymbol{Q} \cdot \nabla_{\boldsymbol{\tau}} \sum_{n=1}^{\infty} \frac{2n+1}{n} \frac{\tau^n}{\alpha^{n+1}} P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\tau}}). \tag{21}$$

Similarly, the exterior electric potential can be calculated as

$$4\pi\sigma u^+(\boldsymbol{r}) = \boldsymbol{Q} \cdot \nabla_{\boldsymbol{\tau}} \sum_{n=1}^{\infty} \frac{2n+1}{n} \frac{\tau^n}{r^{n+1}} P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\tau}}), \quad r > \alpha \tag{22}$$

For the scalar magnetic potential, the integral term on the right hand side of (16) vanishes [27]. Then from (10) and (16) we obtain

$$\partial_r U(\boldsymbol{r}) = -\frac{\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\hat{\boldsymbol{r}}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} = \frac{1}{r}\boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}}\frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|} \tag{23}$$

and finally, using the expansion (20) and a trivial integration, we obtain

$$U(\boldsymbol{r}) = \boldsymbol{Q}\times\boldsymbol{\tau}\cdot\nabla_{\boldsymbol{\tau}}\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{\tau^n}{r^{n+1}}P_n(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\tau}}), \quad r>\alpha. \tag{24}$$

Comparing (22) and (24) we observe that the two exterior potentials are expressed in terms of two expansions that differ only in the coefficients, and that the critical difference is that for each expansion we need to take the directional derivative in different direction. These directional derivatives are dependent on the characteristics of the dipolar current. Indeed, for the electric potential we need to differentiate in the direction of the moment $\boldsymbol{Q}$, and for the magnetic potential we need to differentiate in a direction perpendicular to the plane formed by $\boldsymbol{Q}$ and the position vector $\boldsymbol{\tau}$ where the dipole is located.

We remind here the definition of the vector spherical harmonics [25]

$$\boldsymbol{P}_n^m(\hat{\boldsymbol{\tau}}) = Y_n^m(\hat{\boldsymbol{\tau}})\widehat{\boldsymbol{\tau}} \tag{25}$$

$$\boldsymbol{B}_n^m(\hat{\boldsymbol{\tau}}) = \frac{1}{\sqrt{n(n+1)}}\tau\nabla Y_n^m(\hat{\boldsymbol{\tau}}) \tag{26}$$

$$\boldsymbol{C}_n^m(\hat{\boldsymbol{\tau}}) = -\frac{1}{\sqrt{n(n+1)}}\boldsymbol{\tau}\times\nabla Y_n^m(\hat{\boldsymbol{\tau}}) \tag{27}$$

which form a complete set of orthogonal eigenvectors over the unit sphere.

Then, by using the addition theorem

$$P_n(\widehat{\boldsymbol{r}}\cdot\widehat{\boldsymbol{\tau}}) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n}Y_n^m(\widehat{\boldsymbol{r}})Y_n^m(\widehat{\boldsymbol{\tau}})^* \tag{28}$$

where Y_n^m are the normalized complex form of spherical harmonics [25], we obtain the expansion

$$4\pi\sigma u^+(\boldsymbol{r}) = \boldsymbol{Q}\cdot\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{\tau^{n-1}}{r^{n+1}}\left[\boldsymbol{P}_n^m(\hat{\boldsymbol{\tau}})^* + \sqrt{\frac{n+1}{n}}\boldsymbol{B}_n^m(\hat{\boldsymbol{\tau}})^*\right]Y_n^m(\hat{\boldsymbol{r}}) \tag{29}$$

for the electric potential and the expansion

$$U(\boldsymbol{r}) = -\boldsymbol{Q}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{4\pi}{2n+1}\sqrt{\frac{n}{n+1}}\frac{\tau^n}{r^{n+1}}\boldsymbol{C}_n^m(\hat{\boldsymbol{\tau}})^*Y_n^m(\hat{\boldsymbol{r}}) \tag{30}$$

for the magnetic potential. Consequently, the magnetic potential, and therefore magnetoencephalography, depends only on the component of the neuronal current that lives in the subspace spanned by the eigen-functions $\boldsymbol{C}_n^m$, while the electric potential, and therefore electroencephalography, depends only on the complementary component of the current that lives in the subspace spanned by the eigen-functions $\boldsymbol{P}_n^m$ and $\boldsymbol{B}_n^m$. This means that the information carried by anyone of the two potentials comes from the part of the information that the other potential is missing. This is a characteristic of the spherical geometry and it reflects its highly symmetric properties.

If we express the $\boldsymbol{r}$-dependence in terms of vector harmonics as well we arrive at the form

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{\boldsymbol{Q}}{\sigma}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(n+1)\frac{\tau^{n-1}}{r^{n+2}}\cdot\left[\boldsymbol{P}_n^m(\hat{\boldsymbol{\tau}})^* + \sqrt{\frac{n+1}{n}}\boldsymbol{B}_n^m(\hat{\boldsymbol{\tau}})^*\right]\left[\boldsymbol{P}_n^m(\hat{\boldsymbol{r}}) - \sqrt{\frac{n}{n+1}}\boldsymbol{B}_n^m(\hat{\boldsymbol{r}})\right] \quad (31)$$

for the electric field and at the form

$$\boldsymbol{B}(\boldsymbol{r}) = \mu_0\boldsymbol{Q}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{\sqrt{n(n+1)}}{2n+1}\frac{\tau^n}{r^{n+2}}\cdot\boldsymbol{C}_n^m(\hat{\boldsymbol{\tau}})^*\left[\boldsymbol{P}_n^m(\hat{\boldsymbol{r}}) - \sqrt{\frac{n}{n+1}}\boldsymbol{B}_n^m(\hat{\boldsymbol{r}})\right] \quad (32)$$

for the magnetic field. Note that the dependence of the electric and the magnetic fields on the position is expressed in terms of the eigen-functions $\boldsymbol{P}_n^m$ and $\boldsymbol{B}_n^m$ alone.

In the case of distributed current we expand the current as

$$\boldsymbol{J}^p(\boldsymbol{\tau}) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}p_n^m(\tau)\boldsymbol{P}_n^m(\hat{\boldsymbol{\tau}}) + \sum_{n=1}^{\infty}\sum_{m=-n}^{n}b_n^m(\tau)\boldsymbol{B}_n^m(\hat{\boldsymbol{\tau}}) + \sum_{n=1}^{\infty}\sum_{m=-n}^{n}c_n^m(\tau)\boldsymbol{C}_n^m(\hat{\boldsymbol{\tau}}) \quad (33)$$

we replace, in (29) and (30), the dipolar current $\boldsymbol{Q}$ with the above expansion of the current, and we integrate over the sphere to obtain

$$\sigma u^+(\boldsymbol{r}) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{1}{r^{n+1}}Y_n^m(\hat{\boldsymbol{r}})\int_{\tau<\alpha}\tau^{n-1}\boldsymbol{J}^p(\boldsymbol{\tau})\cdot\left[\boldsymbol{P}_n^m(\hat{\boldsymbol{\tau}})^* + \sqrt{\frac{n+1}{n}}\boldsymbol{B}_n^m(\hat{\boldsymbol{\tau}})^*\right]dv(\boldsymbol{\tau}) \quad (34)$$

and

$$U(\boldsymbol{r}) = -\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{4\pi}{2n+1}\sqrt{\frac{n}{n+1}}\frac{1}{r^{n+1}}Y_n^m(\hat{\boldsymbol{r}})\int_{\tau<\alpha}\tau^n\boldsymbol{J}^p(\boldsymbol{\tau})\cdot\boldsymbol{C}_n^m(\hat{\boldsymbol{\tau}})^*dv(\boldsymbol{\tau}). \quad (35)$$

Utilizing the orthogonality relations of the vector spherical harmonics [25] the expressions (34) and (35) are also written as

$$\sigma u^+(\boldsymbol{r}) = 4\pi\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{1}{2n+1}\frac{(n+m)!}{(n-m)!}\times \left[\int_0^{\alpha}\tau^{n+1}p_n^m(\tau)d\tau + \sqrt{\frac{n+1}{n}}\int_0^{\alpha}\tau^{n+1}b_n^m(\tau)d\tau\right]\frac{1}{r^{n+1}}Y_n^m(\hat{\boldsymbol{r}}) \quad (36)$$

and

$$U(\boldsymbol{r}) = -(4\pi)^2 \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{1}{(2n+1)^2} \frac{(n+m)!}{(n-m)!} \sqrt{\frac{n}{n+1}} \times \\ \times \left[\int_0^{\alpha} \tau^{n+1} c_n^m(\tau) d\tau \right] \frac{1}{r^{n+1}} Y_n^m(\hat{\boldsymbol{r}}) \quad (37)$$

where the actual current enters through the moments of the radially dependent coefficients.

The inverse EEG and MEG problems, where one has to identify the neuronal current from measurements of the exterior electric potential and magnetic field, are reduced to the solution of particular moment problems. Indeed, if we assume that the electric and magnetic potentials are known outside the sphere S, then we can expand these potentials in exterior spherical harmonics as

$$4\pi\sigma u^{+}(\boldsymbol{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} E_n^m \frac{(4\pi)^2}{r^{n+1}} Y_n^m(\hat{\boldsymbol{r}}) \quad (38)$$

and

$$U(\boldsymbol{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} M_n^m \frac{(4\pi)^2}{r^{n+1}} Y_n^m(\hat{\boldsymbol{r}}) \quad (39)$$

where the coefficients E_n^m and M_n^m are known. We remind here that u^{+} is harmonic since there is no current outside Ω, and U is harmonic since $\boldsymbol{B}$ is irrotational and solenoidal in Ω. Comparing (36) with (38) and (37) with (39), and using orthogonality arguments, we arrive at the following moment problems

$$\int_0^{\alpha} \left(p_n^m(\tau) + \sqrt{\frac{n+1}{n}} b_n^m(\tau) \right) \tau^{n+1} d\tau = (2n+1) \frac{(n-m)!}{(n+m)!} E_n^m, \quad n \geqslant 1,\ |m| \leqslant n \quad (40)$$

and

$$\int_0^{\alpha} c_n^m(\tau) \tau^{n+2} d\tau = -(2n+1)^2 \sqrt{\frac{n+1}{n}} \frac{(n-m)!}{(n+m)!} M_n^m, \quad n \geqslant 1,\ |m| \leqslant n. \quad (41)$$

We observe that the solutions of the relative inverse problems are given in global form.

4. The Non-Spherical Model

In the case where the conductor Ω is not spherical we take the smallest circumscribing sphere S and we consider the fields outside this sphere. In order to find out how the neuronal current enters the surface integrals on the right hand side of (7) and (10), we need to know how the electric potential depends on this current. From equation (3) we see that the electric potential depends only on the divergence of the current. Since

$$\nabla \cdot \boldsymbol{P}_n^m(\hat{\boldsymbol{\tau}}) = \frac{2}{\tau} Y_n^m(\hat{\boldsymbol{\tau}}) \quad (42)$$

$$\nabla \cdot \boldsymbol{B}_n^m(\hat{\boldsymbol{\tau}}) = -\frac{\sqrt{n(n+1)}}{\tau} Y_n^m(\hat{\boldsymbol{\tau}}) \quad (43)$$

$$\nabla \cdot \boldsymbol{C}_n^m(\hat{\boldsymbol{\tau}}) = 0 \quad (44)$$

it follows that

$$\nabla \cdot \boldsymbol{J}^p(\boldsymbol{\tau}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[\frac{d}{d\tau} p_n^m(\tau) + \frac{2}{\tau} p_n^m(\tau) + \frac{\sqrt{n(n+1)}}{\tau} b_n^m(\tau) \right] Y_n^m(\hat{\boldsymbol{\tau}}) \tag{45}$$

which implies that the component of the current that lives in the space spanned by the eigen-functions $\boldsymbol{C}_n^m$ does not enter the surface integrals on the right hand side of the representations (7) and (10). Furthermore, for all points outside the sphere S we have

$$\begin{aligned}
\frac{\boldsymbol{r}-\boldsymbol{\tau}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} &= \nabla_{\boldsymbol{\tau}} \frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|} \\
&= \nabla_{\boldsymbol{\tau}} \sum_{n=0}^{\infty} \frac{\tau^n}{r^{n+1}} P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\tau}}) \\
&= 4\pi \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{1}{2n+1} \frac{\tau^{n-1}}{r^{n+1}} \left[n\hat{\boldsymbol{\tau}} Y_n^{m*}(\hat{\boldsymbol{\tau}}) + \tau \nabla_{\boldsymbol{\tau}} Y_n^{m*}(\hat{\boldsymbol{\tau}}) \right] Y_n^m(\hat{\boldsymbol{r}}) \\
&= 4\pi \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{\sqrt{n}}{2n+1} \frac{\tau^{n-1}}{r^{n+1}} \left[\sqrt{n} \boldsymbol{P}_n^{m*}(\hat{\boldsymbol{\tau}}) + \sqrt{n+1} \boldsymbol{B}_n^{m*}(\hat{\boldsymbol{\tau}}) \right] Y_n^m(\hat{\boldsymbol{r}}).
\end{aligned} \tag{46}$$

Inserting (33) and (46) in the volume integral on the right hand side of the representation (7) for the electric potential and using the fact that the eigen-functions $\boldsymbol{C}_n^m$ are orthogonal to $\boldsymbol{P}_n^m$ and $\boldsymbol{B}_n^m$, we see that this volume integral is also independent of the component of the current that lives in the space spanned by the $\boldsymbol{C}_n^m$.

For the volume integral on the right hand side of the representation (10) for the magnetic field we first note that

$$\frac{\partial}{\partial r} \frac{\boldsymbol{J}^p(\boldsymbol{\tau}) \times \boldsymbol{\tau} \cdot \boldsymbol{r}}{F(\boldsymbol{r};\boldsymbol{\tau})} = -\frac{\boldsymbol{J}^p(\boldsymbol{\tau}) \times \boldsymbol{\tau} \cdot \hat{\boldsymbol{r}}}{|\boldsymbol{r}-\boldsymbol{\tau}|^3} = -\frac{1}{r} \boldsymbol{J}^p(\boldsymbol{\tau}) \times \boldsymbol{\tau} \cdot \nabla_{\boldsymbol{\tau}} \frac{1}{|\boldsymbol{r}-\boldsymbol{\tau}|} \tag{47}$$

which, by using the addition theorem (28) and orthogonality properties, gives

$$\frac{\boldsymbol{J}^p(\boldsymbol{\tau}) \times \boldsymbol{\tau} \cdot \boldsymbol{r}}{F(\boldsymbol{r};\boldsymbol{\tau})} = -\sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{4\pi}{2n+1} \sqrt{\frac{n}{n+1}} \frac{\tau^n}{r^{n+1}} [\boldsymbol{J}^p(\boldsymbol{\tau}) \cdot \boldsymbol{C}_n^{m*}(\hat{\boldsymbol{\tau}})] Y_n^m(\hat{\boldsymbol{r}}). \tag{48}$$

Consequently, the volume integral in (10) depends only on the component of the current that belongs to the subspace spanned by the functions $\boldsymbol{C}_n^m$.

5. Conclusions

In this presentation we have shown that, if we decompose the neuronal current into three components, corresponding to the three subspaces spanned by the three types of vector spherical harmonics $\boldsymbol{P}$, $\boldsymbol{B}$, $\boldsymbol{C}$, then in the case of spherical geometry, the $\boldsymbol{P}$-component and the $\boldsymbol{B}$-component influence only the electric potential, while the $\boldsymbol{C}$-component of the current influences only the magnetic potential. In other words, there is some kind of complementarity between the EEG and the MEG methods, in the sense that no part of the current effects both of them. Obviously, that does not mean that what we miss from one method is recovered by the other, since it is possible that every one of the three components of the current could have an 'invisible' subcomponent. All it says is that, the information we obtain from one method is completely independent of the information carried by the other. Furthermore, we show that, if we move away from the spherical geometry, then there is overlapping information carried by the EEG and the MEG measurements. On the other hand, there is a component of the current, i.e. the $\boldsymbol{C}$-component, which is 'visible' solely by the MEG recordings. Hence, for the non spherical model of the brain, MEG brings in information that is not encoded by the EEG.

It seems that these results provide an answer to the long standing controversy of whether there is information in MEG that is independent of EEG. Most theoreticians were using the spherical model and therefore they could not detect any overlapping information between EEG and MEG. On the other hand, experimentalists, that were using real data which were coming from non spherical geometry, were observing overlapping information, as they should. Nevertheless, for any geometrical model of the brain, MEG does carry information about the current that is not observed in EEG.

There are many interesting questions that are left open though. Some of them are:

1 What are the recoverable components of the current in each one of the subspaces spanned by the $\boldsymbol{P}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ functions?

2 What kind of a-priori information about the current will eliminate its 'invisible' part?

3 How the 'invisible' subspaces of the current depend on simple geometries, such as spheroids or ellipsoids?

4 What is the situation if we use a model of the brain that includes shells of different conductivity?

5 Do multipoles of higher order offer a better inside to the EEG and MEG problems ?

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