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On the Ellipsoidal Kernel Space for the BI-Laplacian Operator

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Abstract

The theory of ellipsoidal harmonics was introduced by Lamé as early as 1837 and was further developed mainly in the nineteenth and much less in the twentieth century. The actual structure of the ellipsoidal system is quiet involved and its analysis creates many problems both at the mathematical as well as at the computational level. Perhaps this is the reason why biharmonic functions in ellipsoidal coordinates are not present in the literature. In this work, we utilize the Almansi representation theorem to introduce ellipsoidal biharmonic functions in a natural way. Completeness of the these eigensolutions are justified in a straightforward manner. Nevertheless, the main difficulty comes with the orthogonality properties over the surface of any confocal ellipsoidal boundary. A cumbersome analysis is included, that results in an algebraic system, the solution of which expresses any intrinsically biharmonic eigensolution in terms of the complete set of orthogonal surface ellipsoidal harmonics. Explicit calculations for harmonics up to the fourth degree are also given.

1. Introduction

A function is called biharmonic if it is annihilated by the two successive applications of the Laplacian. Perhaps the most important contribution in the theory of biharmonic functions is the Almansi representation theorem [1], which states that any biharmonic function can be written as the sum of a harmonic function plus the square of the Euclidean distance multiplied by another harmonic function. This theorem is tailor made for the spherical coordinate system, since the Euclidean distance is a spherical coordinate. Nevertheless, it is still the best approach for any other coordinate system. Therefore, we consider here the case of the ellipsoidal coordinate system, which is the system with the appropriate fine structure that fits the needs of anisotropic behavior. As it will be demonstrated, the main difficulty of such a program is associated with the construction of an effective algorithm to express the second term of the Almansi representation in terms of surface ellipsoidal harmonics. This is necessary for the development of a useful spectral method to solve actual boundary value problems with the biharmonic equation. This is a complicated procedure which is accomplished through the following steps. We choose a particular ellipsoidal harmonic of degree n. When this harmonic is multiplied by the ellipsoidal expression of the square of the Euclidean distance we obtain an eigensolution of the biharmonic equation of degree

$n+2$, which has to be represented in terms of surface harmonics of degree less or equal to $n+2$. We assume such an expansion and we develop the algebraic system that determines the coefficients of the expansion. During this analysis we prove that, if an ellipsoidal harmonic is generated by a Lamé function that belongs to a particular class, then its product with the square of the distance is representable only with harmonics that came from Lamé functions of the same class. Finally, we obtain exact representation formulae for biharmonic functions of degree zero to four.

2. Eigensolutions of the Ellipsoidal Biharmonic Equation

The Almansi representation states that, if the function u is such that

$$\Delta^2 u(\boldsymbol{r}) = \Delta\Delta u(\boldsymbol{r}) = 0, \boldsymbol{r} \in \Omega \tag{1}$$

where Δ is the Laplacian and Ω domain in $\mathbb{R}^3$, then there exist two functions u_1 and u_2 such that

$$\Delta u_1(\boldsymbol{r}) = \Delta u_2(\boldsymbol{r}) = 0, \boldsymbol{r} \in \Omega \tag{2}$$

for which the following representation holds

$$u(\boldsymbol{r}) = u_1(\boldsymbol{r}) + r^2 u_2(\boldsymbol{r}), \boldsymbol{r} \in \Omega. \tag{3}$$

Using the definition of the ellipsoidal system (ρ, μ, ν) [3]

$$x_1^2 = \frac{\rho^2\mu^2\nu^2}{h_2^2 h_3^2} \tag{4}$$

$$x_2^2 = \frac{(\rho^2 - h_3^2)(\mu^2 - h_3^2)(h_3^2 - \nu^2)}{h_1^2 h_3^2} \tag{5}$$

$$x_3^2 = \frac{(\rho^2 - h_2^2)(h_2^2 - \mu^2)(h_2^2 - \nu^2)}{h_1^2 h_2^2}. \tag{6}$$

where

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1, 0 < a_3 < a_2 < a_1 < \infty \tag{7}$$

is the reference ellipsoid and

$$h_1^2 = a_2^2 - a_3^2, \quad h_2^2 = a_1^2 - a_3^2, h_3^2 = a_1^2 - a_2^2 \tag{8}$$

are the semi-focal distances of the system, we obtain the ellipsoidal representation of the Euclidean distance

$$r^2 = \rho^2 + \mu^2 + \nu^2 - h_3^2 - h_2^2. \tag{9}$$

The interior harmonic eigensolutions in ellipsoidal coordinates [3], [5], [6], [2] are given by

$$\mathbb{E}_n^m(\rho, \mu, \nu) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu) = E_n^m(\rho)S_n^m(\mu, \nu) \tag{10}$$

where $S_n^m(\mu,\nu)$ are the orthogonal surface ellipsoidal harmonics, introduced by Lamé in 1837 [4]. The functions E_n^m are the well known Lamé functions, where $n = 0, 1, 2, \ldots$ denotes the degree and $m = 1, 2, \ldots, 2n+1$ denotes the order of E_n^m. Since the set of harmonics $\mathbb{E}_n^m$ forms a complete set for the space $\ker\Delta$ in a bounded domain Ω, it follows that the two harmonics u_1, u_2 of the Almansi representation (3) can be expanded as

$$u_1(\boldsymbol{r}) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} A_n^m \mathbb{E}_n^m(\rho,\mu,\nu) \tag{11}$$

$$u_2(\boldsymbol{r}) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} B_n^m \mathbb{E}_n^m(\rho,\mu,\nu) \tag{12}$$

where

$$A_n^m = \frac{1}{E_n^m(\rho)\gamma_n^m}\oint_{S_{a_1}} u_1(\boldsymbol{r}) S_n^m(\mu,\nu) d\Omega(\mu,\nu) \tag{13}$$

$$B_n^m = \frac{1}{E_n^m(\rho)\gamma_n^m}\oint_{S_{a_1}} u_2(\boldsymbol{r}) S_n^m(\mu,\nu) d\Omega(\mu,\nu) \tag{14}$$

S_{a_1} denotes the boundary of the reference ellipsoid and $d\Omega(\mu,\nu)$ is the ellipsoidal form of the differential solid angle element. Inserting formulae (11) and (12) in the representation (3) we obtain

$$\begin{aligned} u(\boldsymbol{r}) &= \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}[A_n^m + (\rho^2+\mu^2+\nu^2-h_3^2-h_2^2)B_n^m]\mathbb{E}_n^m(\rho,\mu,\nu) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}[A_n^m E_n^m(\rho) + (\rho^2+\mu^2+\nu^2-h_3^2-h_2^2)B_n^m E_n^m(\rho)]S_n^m(\mu,\nu) \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}[A_n^m + B_n^m(\rho^2-h_3^2-h_2^2)]E_n^m(\rho)S_n^m(\mu,\nu) \\ &+ \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} B_n^m E_n^m(\rho)(\mu^2+\nu^2)S_n^m(\mu,\nu). \end{aligned} \tag{15}$$

The last sum on the right hand side of (15) involves the surface functions $(\mu^2 + \nu^2)S_n^m(\mu,\nu)$ which are not orthogonal. Therefore, expression (15), as it stands, can not be used to apply boundary conditions and calculate the coefficients. In order to be able to do that we need to express each one of the functions $(\mu^2+\nu^2)S_n^m(\mu,\nu)$ in terms of surface ellipsoidal harmonics of degree less or equal to $n+1$. This program will be followed in the next section.

3. Re-Orthogonalization of Surface Harmonics

If S_n^m is a surface ellipsoidal harmonic generated by the Lamé function E_n^m, that belongs in a certain class, then the function $(\mu^2+\nu^2)S_n^m(\mu,\nu)$ is expandable in surface harmonics $S_{n'}^{m}{}'$, with $n' \leqslant n+2$, that are generated by Lamé function of the same class. Indeed, this is obvious for Lamé functions of class K. For the other classes L, M and N, the same square root has to be a factor of every term of the expansion, which then is canceled out, and we actually work identically as with the class K.

Let us assume that E_n^m belongs to class K and that the degree n is even. We know [3] that for n even, there are $(n/2)+1$ Lamé functions of class K. Therefore, all we need to do is to calculate the coefficients $C_{n,2k}^{m,l}$ of the expansion

$$(\mu^2+\nu^2)S_n^m(\mu,\nu) = \sum_{k=0}^{\frac{n+2}{2}}\sum_{l=1}^{k+1} C_{n,2k}^{m,l} S_{2k}^l(\mu,\nu). \tag{16}$$

In order to keep the symbols as simple as possible we introduce the number $r = n/2$, and we eliminate the indices n and m from the coefficients. Hence, equation (16) is written in the following expanded form

$$\begin{aligned}(\mu^2+\nu^2)S_n^m(\mu,\nu) &= \sum_{l=1}^{r+2} C_{n+2}^l S_{n+2}^l(\mu,\nu)\\ &+ \sum_{l=1}^{r+1} C_n^l S_n^l(\mu,\nu)\\ &+ \sum_{l=1}^{r} C_{n-2}^l S_{n-2}^l(\mu,\nu)\\ &+ \ldots\ldots\ldots\ldots\\ &+ \sum_{l=1}^{3} C_4^l S_4^l(\mu,\nu)\\ &+ \sum_{l=1}^{2} C_2^l S_2^l(\mu,\nu)\\ &+ C_0^1 S_0^1(\mu,\nu).\end{aligned} \tag{17}$$

The plan we are going to follow next is to rewrite both sides of equation (17) as polynomials in the variables μ and ν and then to equate the coefficients of the same powers of μ and ν. That will give us a system for the determination of the unknown coefficients C_n^l. To this end, we write for every $k = 0, 1, 2, \ldots, r+1$ and $l = 1, 2, \ldots, k+1$

$$\begin{aligned}E_{2k}^l(x) &= x^{2k} + A_{2k,1}^l x^{2k-2} + A_{2k,2}^l x^{2k-4} + \ldots + A_{2k,k-1}^l x^2 + A_{2k,k}^l\\ &= \sum_{j=0}^{k} A_{2k,j}^l x^{2k-2j}\end{aligned} \tag{18}$$

with the understanding that we always have $A_{2k,0}^l = 1$. In view of (18) the left hand side of (17) gives

$$\begin{aligned}(\mu^2+\nu^2)S_n^m(\mu,\nu) &= (\sum_{i=0}^{r} A_{n,i}^m \mu^{n+2-2i})(\sum_{j=1}^{r+1} A_{n,j-1}^m \nu^{n+2-2j}) \\ &+ (\sum_{i=1}^{r+1} A_{n,i-1}^m \mu^{n+2-2i})(\sum_{j=0}^{r} A_{n,j}^m \nu^{n+2-2j}) \\ &= \sum_{i=0}^{r}\sum_{j=0}^{r} A_{n,i}^m A_{n,j}^m (\mu^{n+2-2i}\nu^{n-2j} + \mu^{n-2i}\nu^{n+2-2j}) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} (A_{n,i}^m A_{n,j-1}^m + A_{n,i-1}^m A_{n,j}^m)\mu^{n+2-2i}\nu^{n+2-2j} \\ &+ \sum_{i=0}^{r} A_{n,i}^m A_{n,r}^m (\mu^{n+2-2i} + \nu^{n+2-2i}) \\ &+ \sum_{i=1}^{r} A_{n,0}^m A_{n,i-1}^m (\mu^{n+2-2i}\nu^{n+2} + \mu^{n+2}\nu^{n+2-2i}) \qquad (19)\end{aligned}$$

and finally

$$\begin{aligned}(\mu^2+\nu^2)S_n^m(\mu,\nu) &= 2\sum_{i=1}^{r} A_{n,i}^m A_{n,i-1}^m \mu^{n+2-2i}\nu^{n+2-2i} \\ &+ \sum_{i=0}^{r-1}\sum_{j=i+1}^{r} (A_{n,i}^m A_{n,j-1}^m + A_{n,i-1}^m A_{n,j}^m) \cdot \\ &\cdot(\mu^{n+2-2i}\nu^{n+2-2j} + \mu^{n+2-2j}\nu^{n+2-2i}) \\ &+ \sum_{i=0}^{r} A_{n,i}^m A_{n,r}^m (\mu^{n+2-2i} + \nu^{n+2-2i}). \qquad (20)\end{aligned}$$

Next we focus on the right hand side of equation (17), which in view of the notation (18) can be written as

$$\begin{aligned}\sum_{k=0}^{r+1}\sum_{l=1}^{k+1} C_{2k}^l S_{2k}^l(\mu,\nu) &= \sum_{l=1}^{r+2} C_{n+2}^l [\sum_{i=0}^{r+1} A_{n+2,i}^l \mu^{n+2-2i}][\sum_{j=0}^{r+1} A_{n+2,j}^l \nu^{n+2-2j}\Bigg] \\ &+ \sum_{l=1}^{r+1} C_n^l [\sum_{i=0}^{r} A_{n,i}^l \mu^{n-2i}][\sum_{j=0}^{r} A_{n,j}^l \nu^{n-2j}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l=1}^{r} C_{n-2}^{l}[\sum_{i=0}^{r-1} A_{n-2,i}^{l}\mu^{n-2-2i}][\sum_{j=0}^{r-1} A_{n-2,j}^{l}\nu^{n-2-2j}\Bigg] \\
&+ \ldots\ldots\ldots\ldots \\
&+ \sum_{l=1}^{3} C_{4}^{l}[\sum_{i=0}^{2} A_{4,i}^{l}\mu^{4-2i}][\sum_{j=0}^{2} A_{4,j}^{l}\nu^{4-2j}\Bigg] \\
&+ \sum_{l=1}^{2} C_{2}^{l}[\sum_{i=0}^{1} A_{2,i}^{l}\mu^{2-2i}][\sum_{j=0}^{1} A_{2,j}^{l}\nu^{2-2j}\Bigg] \\
&+ C_0^1. \qquad (21)
\end{aligned}
$$

Through some algebraic manipulations, guided by the form (20), we can rewrite formula (21) in the following form

$$
\begin{aligned}
\sum_{k=0}^{r+1}\sum_{l=1}^{k+1} C_{2k}^{l} S_{2k}^{l}(\mu,\nu) &= [\sum_{l=1}^{r+2} C_{n+2}^{l}(A_{n+2,0}^{l})^2]\mu^{n+2}\nu^{n+2} \\
&+ \sum_{i=1}^{r+1}\Bigg[\sum_{l=1}^{r+2} C_{n+2}^{l}A_{n+2,0}^{l}A_{n+2,i}^{l}](\mu^{n+2}\nu^{n+2-2i}+\mu^{n+2-2i}\nu^{n+2}) \\
&+ \sum_{i=1}^{r+1}\sum_{j=1}^{r+1}[\sum_{l=1}^{r+2} C_{n+2}^{l}A_{n+2,i}^{l}A_{n+2,j}^{l} \\
&+ \sum_{l=1}^{r+1} C_{n}^{l}A_{n,i-1}^{l}A_{n,j-1}^{l}]\mu^{n+2-2i}\nu^{n+2-2j} \\
&+ \sum_{i=2}^{r+1}\sum_{j=2}^{r+1}[\sum_{l=1}^{r} C_{n-2}^{l}A_{n-2,i-2}^{l}A_{n-2,j-2}^{l}]\mu^{n+2-2i}\nu^{n+2-2j} \\
&+ \ldots\ldots\ldots\ldots \\
&+ \sum_{i=r}^{r+1}\sum_{j=r}^{r+1}[\sum_{l=1}^{2} C_{2}^{l}A_{2,i-r}^{l}A_{2,j-r}^{l}]\mu^{n+2-2i}\nu^{n+2-2j} \\
&+ C_0^1 \qquad (22)
\end{aligned}
$$

and finally in the form

$$\sum_{k=0}^{r+1}\sum_{l=1}^{k+1} C_{2k}^{l} S_{2k}^{l}(\mu,\nu)$$

$$= \sum_{i=0}^{r-1}\sum_{j=i+1}^{r}[\sum_{k=0}^{i}(\sum_{l=1}^{r+2-k} C_{n+2-2k}^{l} A_{n+2-2k,i-k}^{l} A_{n+2-2k,j-k}^{l})]\cdot$$

$$\cdot(\mu^{n+2-2i}\nu^{n+2-2j}+\mu^{n+2-2j}\nu^{n+2-2i})$$

$$+ \sum_{i=0}^{r}[\sum_{j=0}^{i}(\sum_{l=1}^{r+2-j} C_{n+2-2j}^{l} A_{n+2-2j,i-j}^{l} A_{n+2-2j,r+1-j}^{l})](\mu^{n+2-2i}+\nu^{n+2-2i})$$

$$+ \sum_{i=0}^{r+1}\sum_{j=0}^{i}[\sum_{l=1}^{r+2-j} C_{n+2-2j}^{l}(A_{n+2-2j,i-j}^{l})^{2}](\mu^{n+2-2i}\nu^{n+2-2i}). \tag{23}$$

By equation (16) the left-hand sides of equations (20) and (23) are equal. Therefore, the right-hand sides of these equations are also equal. Then, equating the coefficients of the same powers of μ and ν we arrive at the following system for the calculation of the unknown constants C_n^l. For $n = 0, 2, 4, 6, \ldots$ we obtain the equations

$$\sum_{l=1}^{r+2} C_{n+2}^{l}(A_{n+2,0}^{l})^{2} = 0 \tag{24}$$

$$\sum_{i=0}^{r+1}[\sum_{l=1}^{r+2-i} C_{n+2-2i}^{l}(A_{n+2-2i,r+1-i}^{l})^{2}] = 0 \tag{25}$$

$$\sum_{i=0}^{j}[\sum_{l=1}^{r+2-i} C_{n+2-2i}^{l} A_{n+2-2i,j-i}^{l} A_{n+2-2i,r+1-i}^{l}] = A_{n+2-2i,r}^{l} A_{n+2-2i,j}^{l} \quad \text{for}\, j=0,1,2,\ldots,r \tag{26}$$

while for $n = 2, 4, 6, \ldots$ we obtain the additional equations

$$\sum_{i=0}^{j}[\sum_{l=1}^{r+2-i} C_{n+2-2i}^{l}(A_{n+2-2i,j-i}^{l})^{2}] = 2A_{n+2-2i,j}^{l} A_{n+2-2i,j-1}^{l} \quad \text{for}\, j=1,2,\ldots,r \tag{27}$$

$$\sum_{i=0}^{j}[\sum_{l=1}^{r+2-i} C^l_{n+2-2i} A^l_{n+2-2i,j-i} A^l_{n+2-2i,k-i}] = A^l_{n+2-2i,k} A^l_{n+2-2i,j-1} + A^l_{n+2-2i,k-1} A^l_{n+2-2i,j} \tag{28}$$

$$\text{for}\, j = 0,1,2,\ldots,r-1 \quad \text{and}\, k = j+1, j+2, \ldots, r.$$

In the above equations we accept that $A^l_{-1} = 0$. For a fixed even degree n, we have to solve the system (24)-(28) to determine the constants C^l_n. Then expression (17) will provide the purely biharmonic part of the solution (15) expressed in terms of surface ellipsoidal harmonics.

A similar system for surface ellipsoidal harmonics of odd degree can be obtained the same way. Ellipsoidal harmonics generated by Lamé functions of the other classes are also handled identically.

4. The Leading Biharmonics

In order to illustrate the above analysis, as well as for the purpose of obtaining ready to use ellipsoidal biharmonic eigensolutions, we implement the algorithm in this section and the biharmonic eigenfunctions of degree less or equal to four. In order to do that we need the explicit form of the Lamé functions of degree 0,1,2,3 and 4, which are given in the Appendix. The notation has been simplified as much as it can be.

For $n = 0$, we have the expansion

$$(\mu^2+\nu^2)S^1_0(\mu,\nu) = C_0 S^1_0(\mu,\nu) + C_1 S^1_2(\mu,\nu) + C_2 S^2_2(\mu,\nu) \tag{29}$$

or

$$\begin{aligned}\mu^2+\nu^2 &= C_0 + C_1(\mu^2+\Lambda-a_1^2)(\nu^2+\Lambda-a_1^2) + C_2(\mu^2+\Lambda'-a_1^2)(\nu^2+\Lambda'-a_1^2)\\ &= C_0 + (C_1+C_2)\mu^2\nu^2 + [C_1(\Lambda-a_1^2)+C_2(\Lambda'-a_1^2)](\mu^2+\nu^2)\\ &+ [C_1(\Lambda-a_1^2)+C_2(\Lambda'-a_1^2)].\end{aligned} \tag{30}$$

Equating the coefficients of identical monomials we obtain the system

$$C_1 + C_2 = 0 \tag{31}$$

$$C_1(\Lambda-a_1^2) + C_2(\Lambda'-a_1^2) = 1 \tag{32}$$

$$C_0 + C_1(\Lambda-a_1^2) + C_2(\Lambda'-a_1^2) = 0 \tag{33}$$

which has the solution

$$C_0 = -(\Lambda+\Lambda') + 2a_1^2 \tag{34}$$

$$C_1 = \frac{1}{\Lambda-\Lambda'} \tag{35}$$

$$C_2 = -\frac{1}{\Lambda-\Lambda'}. \tag{36}$$

Therefore,

$$\begin{aligned}(\mu^2+\nu^2)S^1_0(\mu,\nu) = &\ \tfrac{1}{\Lambda-\Lambda'}[S^1_2(\mu,\nu) - S^2_2(\mu,\nu)]\\ &+[2a_1^2 - (\Lambda+\Lambda')]S^1_0(\mu,\nu)\end{aligned} \tag{37}$$

For $n = 1$, $m = 1$, using only Lamé functions of the same class, we have

$$(\mu^2 + \nu^2)S_1^1(\mu,\nu) = C_0S_1^1(\mu,\nu) + C_1S_3^1(\mu,\nu) + C_2S_3^2(\mu,\nu) \quad (38)$$

and working as before we obtain the expression

$$\begin{aligned}(\mu^2 + \nu^2)S_1^1(\mu,\nu) &= \frac{1}{\Lambda_1 - \Lambda_1'}[S_3^1(\mu,\nu) - S_3^2(\mu,\nu)] \\ &+[2a_1^2 - (\Lambda_1 + \Lambda_1')]S_1^1(\mu,\nu). \end{aligned} \quad (39)$$

Similarly, for $n = 1$, $m = 2$, we obtain

$$\begin{aligned}(\mu^2 + \nu^2)S_1^2(\mu,\nu) &= \frac{1}{\Lambda_2 - \Lambda_2'}[S_3^3(\mu,\nu) - S_3^4(\mu,\nu)] \\ &+[2a_1^2 - (\Lambda_2 + \Lambda_2')]S_1^2(\mu,\nu) \end{aligned} \quad (40)$$

and for $n = 1$, $m = 3$, we obtain

$$\begin{aligned}(\mu^2 + \nu^2)S_1^3(\mu,\nu) &= \frac{1}{\Lambda_3 - \Lambda_3'}[S_3^5(\mu,\nu) - S_3^6(\mu,\nu)] \\ &+[2a_1^2 - (\Lambda_3 + \Lambda_3')]S_1^3(\mu,\nu). \end{aligned} \quad (41)$$

Note that, since we use only functions of the same class, which imply that all functions have a common square root factor that cancels out, the relative system for all cases above is of the same form. Only the parameters are changed. That explains the similarity of the expansions (39)-(41) with the expansion (37).

For $n = 2$, $m = 1$, we have

$$\begin{aligned}(\mu^2 + \nu^2)S_2^1(\mu,\nu) &= C_0S_0^1(\mu,\nu) + C_1S_2^1(\mu,\nu) + C_2S_2^2(\mu,\nu) \\ &+ C_3S_4^1(\mu,\nu) + C_4S_4^2(\mu,\nu) + C_5S_4^3(\mu,\nu) \end{aligned} \quad (42)$$

or

$$\begin{aligned}(\mu^2 + \nu^2)(\mu^2 + \Lambda - a_1^2)(\nu^2 + \Lambda - a_1^2) =& C_0 \\ &+ C_1(\mu^2 + \Lambda - a_1^2)(\nu^2 + \Lambda - a_1^2) \\ &+ C_2(\mu^2 + \Lambda' - a_1^2)(\nu^2 + \Lambda' - a_1^2) \\ &+ C_3(\mu^4 + T\mu^2 + R)(\nu^4 + T\nu^2 + R) \\ &+ C_4(\mu^4 + T'\mu^2 + R')(\nu^4 + T'\nu^2 + R') \\ &+ C_5(\mu^4 + T''\mu^2 + R'')(\nu^4 + T''\nu^2 + R'') \end{aligned} \quad (43)$$

where the constants $\Lambda, \Lambda', T, T', T'', R, R'$ and R'' are specific parameters explained in the Appendix.

Rearranging both sides of equation (43) in monomials of the same degree in μ and ν, and equating the corresponding coefficients, we arrive at the following system for the unknown constants $C_i, i = 0, 1, \ldots, 5$

$$C_3 + C_4 + C_5 = 0 \quad (44)$$
$$TC_3 + T'C_4 + T''C_5 = 1 \quad (45)$$
$$RC_3 + R'C_4 + R''C_5 = \Lambda - a_1^2 \quad (46)$$

$$C_1 + C_2 + T^2C_3 + T'^2C_4 + T''^2C_5 = 2(\Lambda - a_1^2) \quad (47)$$
$$(\Lambda - a_1^2)C_1 + (\Lambda' - a_1^2)C_2 + TRC_3 + T'R'C_4 + T''R''C_5 = (\Lambda - a_1^2)^2 \quad (48)$$

$$C_0 + (\Lambda - a_1^2)^2C_1 + (\Lambda' - a_1^2)^2C_2 + R^2C_3 + R'^2C_4 + R''^2C_5 = 0. \quad (49)$$

The above system is decomposable in three subsystems. Equations (44)-(46) can be solved to determine the constants C_3, C_4, C_5. Then we substitute the values of C_3, C_4, C_5 in equations (47) and (48) and we solve the resulting system with respect to C_1, C_2. Finally, substituting C_1 to C_5 in equation (49) we obtain the constant C_0. This way we obtain the following values of the constants

$$C_0 = -(\Lambda - a_1^2)^2C_1 - (\Lambda' - a_1^2)^2C_2 - R^2C_3 - R'^2C_4 - R''^2C_5 \quad (50)$$
$$C_1 = \frac{-(\Lambda' - a_1^2)K_1 + K_2}{\Lambda - \Lambda'} \quad (51)$$
$$C_2 = \frac{(\Lambda - a_1^2)K_1 - K_2}{\Lambda - \Lambda'} \quad (52)$$

where

$$K_1 = 2(\Lambda - a_1^2) - T^2C_3 - T'^2C_4 - T''^2C_5 \quad (53)$$
$$K_2 = (\Lambda - a_1^2)^2 - TRC_3 - T'R'C_4 - T''R''C_5 \quad (54)$$

and

$$C_i = \frac{D_i}{D}, i = 3, 4, 5 \quad (55)$$

with

$$D = \begin{vmatrix} 1 & 1 & 1 \\ T & T' & T'' \\ R & R' & R'' \end{vmatrix} \quad (56)$$

$$D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & T' & T'' \\ \Lambda - a_1^2 & R' & R'' \end{vmatrix} \quad (57)$$

$$D_4 = \begin{vmatrix} 1 & 0 & 1 \\ T & 1 & T'' \\ R & \Lambda - a_1^2 & R'' \end{vmatrix} \quad (58)$$

$$D_5 = \begin{vmatrix} 1 & 1 & 0 \\ T & T' & 1 \\ R & R' & \Lambda - a_1^2 \end{vmatrix}. \quad (59)$$

For $n = 2$, $m = 2$, we have the same representation as in (42), that is

$$\begin{aligned}(\mu^2+\nu^2)S_2^2(\mu,\nu) &= C_0S_0^1(\mu,\nu)+C_1S_2^1(\mu,\nu)+C_2S_2^2(\mu,\nu)\\ &+ C_3S_4^1(\mu,\nu)+C_4S_4^2(\mu,\nu)+C_5S_4^3(\mu,\nu) \qquad (60)\end{aligned}$$

with the only difference that in the expressions of the constants $C_i, i = 0, 1, \ldots, 5$, given by the formulae (50)-(59), the values of Λ and Λ' are interchanged.

For the cases $n = 2$ and $m = 3, 4, 5$, the situation is simpler, since there are cancellations of factors that are second degree in the variables μ and ν, and the process is similar to the case $n = 0$. Hence, it is straightforward to deduce the representations

$$(\mu^2+\nu^2)S_2^3(\mu,\nu) = -(V+V')S_2^3(\mu,\nu)+\frac{1}{V-V'}[S_4^4(\mu,\nu)-S_4^5(\mu,\nu)] \qquad (61)$$

$$(\mu^2+\nu^2)S_2^4(\mu,\nu) = -(U+U')S_2^4(\mu,\nu)+\frac{1}{U-U'}[S_4^6(\mu,\nu)-S_4^7(\mu,\nu)] \qquad (62)$$

$$(\mu^2+\nu^2)S_2^3(\mu,\nu) = -(W+W')S_2^5(\mu,\nu)+\frac{1}{W-W'}[S_4^8(\mu,\nu)-S_4^9(\mu,\nu)]. \qquad (63)$$

It is obvious that we can keep developing expansions of this form, but the calculations become eventually very cumbersome.

5. Appendix

The variable x represents one of the ellipsoidal coordinates, $\rho \in [h_2, \infty)$, $\mu \in [h_3, h_2]$, $\nu \in [0, h_3]$. There are $2n + 1$ Lamé functions of degree n. For $n = 0, 1, 2, 3, 4$, these functions are :

$$E_0^1(x) = 1.$$

$$\begin{aligned}E_1^1(x) &= x\\ E_1^2(x) &= \sqrt{|x^2-h_3^2|}\\ E_1^3(x) &= \sqrt{|x^2-h_2^2|}\end{aligned}$$

$$\begin{aligned}E_2^1(x) &= x^2+\Lambda-a_1^2\\ E_2^2(x) &= x^2+\Lambda'-a_1^2\\ E_2^3(x) &= x\sqrt{|x^2-h_3^2|}\\ E_2^4(x) &= x\sqrt{|x^2-h_2^2|}\\ E_2^5(x) &= \sqrt{|x^2-h_3^2|}\sqrt{|x^2-h_2^2|}\end{aligned}$$

where the constants Λ and Λ' are the roots of the equation

$$\sum_{n=1}^{3}\frac{1}{\Lambda-a_n^2} = 0$$

$$
\begin{aligned}
E_3^1(x) &= x(x^2+\Lambda_1-a_1^2)\\
E_2^2(x) &= x(x^2+\Lambda_1'-a_1^2)\\
E_3^3(x) &= \sqrt{|x^2-h_3^2|}(x^2+\Lambda_2-a_1^2)\\
E_3^4(x) &= \sqrt{|x^2-h_3^2|}(x^2+\Lambda_2'-a_1^2)\\
E_3^5(x) &= \sqrt{|x^2-h_2^2|}(x^2+\Lambda_3-a_1^2)\\
E_3^6(x) &= \sqrt{|x^2-h_2^2|}(x^2+\Lambda_3'-a_1^2)\\
E_3^7(x) &= x\sqrt{|x^2-h_3^2|}\sqrt{|x^2-h_2^2|}
\end{aligned}
$$

where the constants Λ_k and Λ_k' are the roots of the equation

$$
\sum_{n=1}^{3}\frac{1+2\delta_{kn}}{\Lambda_k-a_n^2} = 0, k=1,2,3
$$

$$
\begin{aligned}
E_4^1(x) &= x^4+\frac{(h_3^2+h_2^2)(M_1-16)}{14}x^2-\frac{h_3^2h_2^2(M_1-16)}{7M_1}\\
E_4^2(x) &= x^4+\frac{(h_3^2+h_2^2)(M_1'-16)}{14}x^2-\frac{h_3^2h_2^2(M_1'-16)}{7M_1'}\\
E_4^3(x) &= x^4+\frac{(h_3^2+h_2^2)(M_1''-16)}{14}x^2-\frac{h_3^2h_2^2(M_1''-16)}{7M_1''}\\
E_4^4(x) &= x\sqrt{|x^2-h_3^2|}[x^2+\frac{(h_3^2+h_2^2)(M_2-9)-7h_2^2}{14}]\\
E_4^5(x) &= x\sqrt{|x^2-h_3^2|}[x^2+\frac{(h_3^2+h_2^2)(M_2'-9)-7h_2^2}{14}]\\
E_4^6(x) &= x\sqrt{|x^2-h_2^2|}[x^2+\frac{(h_3^2+h_2^2)(M_3-9)-7h_3^2}{14}]\\
E_4^7(x) &= x\sqrt{|x^2-h_2^2|}[x^2+\frac{(h_3^2+h_2^2)(M_3'-9)-7h_3^2}{14}]\\
E_4^8(x) &= \sqrt{|x^2-h_3^2|}\sqrt{|x^2-h_2^2|}[x^2+\frac{(h_3^2+h_2^2)(M_4-9)}{14}]\\
E_4^9(x) &= \sqrt{|x^2-h_3^2|}\sqrt{|x^2-h_2^2|}[x^2+\frac{(h_3^2+h_2^2)(M_4'-9)}{14}]
\end{aligned}
$$

where the constants M_1, M_1', M_1'' are the roots of the equation

$$
\frac{10}{M_1}+\frac{4}{M_1-4}-\frac{14}{M_1-16} = \frac{(h_3^2+h_2^2)^2}{h_3^2h_2^2}
$$

the constants M_2, M_2' are the roots of the equation

$$9\frac{5h_2^2+12h_3^2}{M_2-1}+7\frac{5h_2^2-4h_3^2}{M_2-9} = 8\frac{(h_3^2+h_2^2)^2}{h_2^2}$$

the constants M_3, M_3' are the roots of the equation

$$9\frac{5h_3^2+12h_2^2}{M_3-1}+7\frac{5h_3^2-4h_2^2}{M_3-9} = 8\frac{(h_3^2+h_2^2)^2}{h_3^2}$$

and the constants M_4, M_4' are the roots of the equation

$$\frac{1}{M_4-1}-\frac{1}{M_4-9} = \frac{2}{7}\frac{(h_3^2+h_2^2)^2}{h_3^2h_2^2}.$$

In order to facilitate the notation we rewrite the functions of degree 4 in the following form, where the constants $T, T', T'', R, R', R'', V, V', U, U', W, W'$ are obtained from the above formulae by inspection

$$\begin{aligned}
E_4^1(x) &= x^4+Tx^2+R\\
E_4^2(x) &= x^4+T'x^2+R'\\
E_4^3(x) &= x^4+T''x^2+R''\\
E_4^4(x) &= x\sqrt{|x^2-h_3^2|}[x^2+V]\\
E_4^5(x) &= x\sqrt{|x^2-h_3^2|}[x^2+V']\\
E_4^6(x) &= x\sqrt{|x^2-h_2^2|}[x^2+U]\\
E_4^7(x) &= x\sqrt{|x^2-h_2^2|}[x^2+U']\\
E_4^8(x) &= \sqrt{|x^2-h_3^2|}\sqrt{|x^2-h_2^2|}[x^2+W]\\
E_4^9(x) &= \sqrt{|x^2-h_3^2|}\sqrt{|x^2-h_2^2|}[x^2+W'].
\end{aligned}$$

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