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Blow-up of solutions for a non-local degenerate Filtration and Porous Medium problem

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Abstract

We consider a non-local Filtration equation of the form: $u_t = \Delta K(u) + \lambda f(u)/\big(\int_\Omega f(u)\big)^p$ and a Porous Medium equation in the case where $K(u) = u^m,\ m > 1$, with some boundary and initial data $u_0 \geq 0$, where $0 < p < 1$ and $f(s),\ f'(s),\ f''(s) > 0,\ s > 0,\ f(0) = 0$, both in degenerate form. We prove blow-up of L_1-norm of solutions, for large enough values of the parameter $\lambda > 0$ and for any $u_0 \geq 0$.

Keywords: Degenerate filtration, Blow-up, Non-local non-linear diffusion.

1. Introduction

In this work we prove the blow-up of solutions, actually the blow-up of L_1-norm of solutions, in finite time, of the following non-local degenerate initial boundary value problem:

$$u_t = \Delta K(u) + \frac{\lambda f(u)}{\big(\int_\Omega f(u)dx\big)^p}, \quad x \in \Omega, \quad t > 0, \tag{1a}$$

$$B(u) \equiv \frac{\partial K(u)}{\partial \hat{n}} + \beta(x)K(u) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1b}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{1c}$$

where $0 < p < 1$, $\hat{n}$ is the outward pointing unit normal vector field and Ω is a bounded domain of $\mathbb{R}^N,\ N \geq 1$ with smooth enough boundary $\partial\Omega$. We impose initial data $u_0(x) \in C(\overline{\Omega}),\ u_0(x) \geq 0$ and $u_0(x)$ is not identically zero. Moreover for the Dirichlet and the Robin problem we require $u_0(x)$ to decrease near $\partial\Omega$ in suitable way [15, p.119]. u is a classical solution to (1), if it satisfies problem (1) and $u \in C^{2,1}_{x,t}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T))$, for some $T > 0$, see [13, 17]. It is also possible for the parabolic equation to become degenerate, depending on the assumptions of the nonlinear diffusion $K(u)$ and nonlinear reaction term $f(u)$.

We introduce boundary conditions of the form $B(u)$. This type of conditions are a consequence of Fourier's law for diffusion and conservation of mass, or heat conduction and conservation of energy. For classical solutions, β $(0 \leq \beta = \beta(x) \leq \infty)$ is $C^{1+\alpha}(\partial\Omega)$, $\alpha > 0$, whenever it is bounded. ($\beta \equiv 0,\ \beta \equiv \infty,\ 0 < \beta < \infty$ means Neumann, Dirichlet and Robin boundary condition respectively; we note that we may have mixed boundary conditions on $\partial\Omega$ i.e. Dirichlet on $\partial\Omega_1$, Neumann on $\partial\Omega_2$ with $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$). The parameters λ, p are positive with $p \in (0,1)$.

We consider the following assumptions for the data of problem (1):

$$K(s),\ K'(s),\ K''(s) > 0, \text{ for } s > 0 \quad \text{and} \quad K(0) \geq 0,\ K'(0) \geq 0,\ K''(0) > 0, \tag{2a}$$

$$f(s) > 0,\ f'(s) > 0,\ f''(s) > 0, \text{ for } s > 0 \text{ and } f(0) = 0 \text{ (unforced case).} \tag{2b}$$

$$\text{Also,} \quad \int_b^\infty \frac{ds}{f^{1-p}(s)} < \infty \text{ for some } b \geq 0 \quad \text{and} \quad f^{1-p}(s) \text{ is convex.} \tag{2c}$$

If $K(u) = u^m,\ m > 1,\ m \in \mathbb{R}$, then (1) becomes the non-local Porous Medium (PM) problem:

$$u_t = \Delta u^m + \frac{\lambda f(u)}{\left(\int_\Omega f(u)dx\right)^p}, \quad x \in \Omega,\ t > 0, \tag{3a}$$

$$B(u) = \frac{\partial u^m}{\partial \hat{n}} + \beta(x) u^m = 0, \quad x \in \partial\Omega,\ t > 0, \tag{3b}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega. \tag{3c}$$

If $u_0(x) \geq 0$ but not identically zero in Ω i.e. $u_0(x)$ has compact support, or $u(x,t) = 0$ on $\partial\Omega$, this implies a degenerate parabolic equation (this is due to the loose of uniform parabolicity of the problem, see [9]).

Our motivation to address (3a), concerning the conduction term Δu^m (or $\nabla \cdot u^{m-1}\nabla u$) comes from [16]. In [16], the plasma heating equation $u_t = (u^3 u_x)_x + \lambda f(u)/(\int_{-1}^1 f(u)dx)^2$, for $f(s)$ positive and decreasing, is studied. More precisely the conduction term $(u^4)_{xx}$ or $(u^3 u_x)_x$ is introduced, where the term u^3 accounts for heat transport dominated by thermal radiation by assuming the Stefan-Boltzman law for emission of thermal radiation. Actually, equation (3a) is a generalization of the plasma heating equation. Problem (1) can also be considered as a generalization of (3).

Concerning the non-local reaction term of problem (1), this comes from modelling Ohmic heating phenomena as well as in shear bands of metals which are being deformed at high strain rates (see [2, 3, 4]), in the theory of gravitational equilibrium of polytropic stars [11], in the investigation of the fully turbulent behavior of flows, using invariant measures for the Euler equation [5] and in modelling aggregation of cells via interaction with a chemical substance (chemotaxis), see [18]. It is worth mentioning that Galaktionov and Levine in [8] and later on Afanas'eva and Tedeev in [1] have treated, among other, a non-local Porous Medium problem with critical Fujita exponent and proved, for the Cauchy problem, global existence and blow-up of solutions. Moreover, some existence and nonexistence theorems for solutions of degenerate parabolic equations, see in [14].

Degeneracy. Justification of using very weak solutions.

If we examine the PM equation without reaction term, namely:

$$u_t = \Delta u^m = \nabla \cdot \left(m u^{m-1} \nabla u \right), \quad m > 1 \tag{4}$$

and set $v = cu^{m-1}$, we get the equation

$$v_t = av\Delta v + b\left|\nabla v\right|^2 \tag{5}$$

where $a = m/c,\ \ b = m/c(m-1)$.

For $v = 0$, we get $v_t = b\left|\nabla v\right|^2$, a first order equation where the solutions propagate along characteristic curves, formatting a free boundary which propagates with finite speed and separates the regions of $v > 0$ and $v = 0$, [17].

On the moving free boundary, the solution $u(x,t)$ is not classical. So we should work with generalized solutions, let say very weak solutions. The same occurs and for the non-local problems, under some assumptions on the function $f(u)$ that we discuss in remark (1) below.

In the case of the general filtration problem (1), we have:

$$u_t = K'(u)\Delta u + K''(u)\left|\nabla u\right|^2 + \frac{\lambda f(u)}{\left(\int_\Omega f(u)dx\right)^p}, \tag{6}$$

and the diffusivity coefficient now is $K'(u)$. If for $u = 0$, $K'(0) = 0$, the equation becomes degenerate and we have to work again with very weak solutions.

Remark 1 If $f(0) > 0$ we would have: $u_t - \nabla \cdot K'(u)\nabla u = \lambda f(u)/\left(\int_\Omega f(u)dx\right)^p$, and for $u = 0$, even though the left hand side becomes degenerate parabolic, because of $K'(0) = 0$, the positivity of the right hand side will make the solution immediately positive in all over domain Ω, hence all the solutions will be classical.

Finally, we conclude that we have degeneracy if $K'(0) = 0$ and $f(0) = 0$, and hence we have to work with generalized solutions (very weak solutions).

This work has been organized as follows: In Section 2 we refer to the existence and give some preliminaries. In the Section 3 we prove blow-up of solutions of a non-local filtration problem.

We point out that for the proof of blow-up for the Dirichlet and Robin boundary problem, we require **the domain Ω to be convex**.

2. Existence and preliminaries

- **Formulation of filtration equation for very weak solutions**

Definition 2 A function $u \in C(t_1, t_2; L_1(\Omega))$ defined on $\Omega_{t_1 t_2} := \Omega \times (t_1, t_2)$ is a very weak solution of the problem (1) if u, $K(u)$, $f(u) \in L_1(\Omega_{t_1 t_2})$ and satisfies the identity:

$$\int_\Omega [u\eta]_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_\Omega u\eta_t dxds - \int_{t_1}^{t_2} \int_\Omega K(u)\Delta\eta\, dxds - \lambda \int_{t_1}^{t_2} \frac{\int_\Omega f(u)\eta\, dx}{\left(\int_\Omega f(u)dx\right)^p}\, ds = 0, \tag{7a}$$

$$\forall\ t_1,\ t_2 :\ 0 \le t_1 < t_2 \le T, \quad \forall\ \eta = \eta(x,t) \ge 0 \quad \text{and}$$

$$\eta \in C^\infty(t_1, t_2; C^\infty(\Omega)), \quad \frac{\partial \eta}{\partial \hat{n}} + \beta\eta = 0 \text{ on } \partial\Omega, \ \ 0 \le \beta = \beta(x) < \infty \ \text{ or} \tag{7b}$$

$$\eta \in C^\infty(t_1, t_2; C_c^\infty(\Omega)), \ \text{ for the Dirichlet problem,} \tag{7c}$$

where for the Robin problem ($\beta(x) > 0$), while for the Neumann ($\beta(x) = 0$); $C_c^\infty(\Omega)$ is the space of all $C^\infty-$functions with compact support in Ω, see also [7, 17].

We also can simply say that u is a distribution solution of (1) if it satisfies (7). For problem (7), the integral on the boundary becomes zero, i.e.

$$\int_{t_1}^{t_2} \left\{ \int_{\partial\Omega} \left(\eta \frac{\partial K(u)}{\partial \hat{n}} - K(u) \frac{\partial \eta}{\partial \hat{n}} \right) d\sigma(x) \right\} ds = 0,$$

and therefore does not appear in (7a); this is a trick of weak formulation of problem (7). Actually for (7a), we only need $\eta \in C_{x,t}^{2,1}$ or the same smoothness but with compact support in Ω, ($\eta \in C_{c x,t}^{2,1}$). Moreover, if we take $t_1 = 0,\ t_2 = T$ and assume test functions $\eta = \eta(x,t)$ which vanishes at $t = T$ then the first integral of (7a) becomes $\int_\Omega u_0(x)\eta_0(x)dx$, where $\eta_0(x)=\eta(x,0)$. Also the initial data, if $t_1=0$, can also be considered in the sense:

$$\lim_{t\to 0+}\int_\Omega \eta(x,t)u(x,t)dx=\int_\Omega \eta_0(x)u_0(x)dx.$$

- **Comparison principle via very weak lower-upper solution pair**

In contrast to the decreasing case, where a direct weak comparison principle holds, when $f(s)$ is an increasing function, the existence of an upper and a lower solution in the usual sense does not guarantee the existence of a solution of problem (1), laying between them. In order to use proper comparison arguments, which helps us to obtain existence, we introduce the concept of **a very weak lower-upper solution pair**.

Definition 3 For two functions $z=z(x,t),\ v=v(x,t)\in C(t_1,t_2;L_1(\Omega))$, let the operator V be defined by:

$$\begin{aligned} V(z,v;\eta):=&\int_\Omega [z\eta]_{t_1}^{t_2}dx-\int_{t_1}^{t_2}\int_\Omega z\eta_t dxds \\ &-\int_{t_1}^{t_2}\int_\Omega K(z)\Delta\eta\, dxds-\lambda\int_{t_1}^{t_2}\frac{\int_\Omega f(z)\eta dx}{\left(\int_\Omega f(v)dx\right)^p}ds \end{aligned} \tag{8}$$

$$\forall\, t_1,\ t_2:\ 0\le t_1<t_2\le T,\quad \forall\,\eta\in C^\infty(t_1,t_2;C_c^\infty(\Omega))\ \text{ and }\ \eta\ge 0.$$

Then (z,v) is called a very weak lower-upper solution pair to problem (1), if the following hold:

$$V(z,v;\eta)\le 0\le V(v,z;\eta),\quad x\in\Omega,\quad t_1\le t\le t_2, \tag{9a}$$

$$0\le\int_\Omega \eta_0(x)z(x,0)dx\le\int_\Omega \eta_0(x)u(x,0)dx\le\int_\Omega \eta_0(x)v(x,0)dx, \tag{9b}$$
$$\forall\ t_1,\ t_2:\ 0\le t_1<t_2\le T\quad\text{and}\quad\forall\,\eta\quad\text{satisfying (7b), (7c).}$$

The integral on the boundary does not appear in (9a) since

$$\int_{t_1}^{t_2}\int_{\partial\Omega}\{\eta[\frac{\partial K(z)}{\partial\hat n}-\frac{\partial K(v)}{\partial\hat n}]-[K(z)-K(v)]\frac{\partial\eta}{\partial\hat n}\}d\sigma(x)ds\le 0,$$

(for the Dirichlet problem $\eta=0$ and $\frac{\partial\eta}{\partial\hat n}<0$, for the Neumann $\frac{\partial\eta}{\partial\hat n}=0$ and for the Robin $\frac{\partial\eta}{\partial\hat n}+\beta(x)\eta=0$ on $\partial\Omega$).

Now we have the following comparison, existence and uniqueness results.

Lemma 4 Let (z, v) be a very weak lower-upper solution pair to (1), then $z \leq u \leq v$. This result holds for strict inequalities as well.

Proposition 5 (**Local existence and uniqueness**). If f satisfies (2), is Lipschitz continuous, u is defined in $\Omega_T = \Omega \times (0, T)$, with u, $K(u)$, $f(u) \in L_1(\Omega_T)$ and $u_0(x) \geq 0$, with u_0 is not identically zero and having compact support, then there exists a unique very weak solution $u(x, t; \lambda)$ to problem (1) in $L_1(\Omega_T)$ to each $\lambda > 0$ and for some $T > 0$.

Proof. The proof is based on comparison arguments and on using lower-upper solution pair, see for the proof [7]. Also for similar arguments see [13] as well as [6, 17]. □

Lemma 6 Let (z, v) be a very weak lower-upper solution pair and u a solution both of (1), then $z \leq u \leq v$.

- **Steklov averages**

We define now the Steklov averages of a function $u(x, t)$:

$$u_h(x,t) = \frac{1}{h}\int_{t-h}^{t} u(x,s)ds \quad \text{or} \quad u_h(x,t) = \frac{1}{h}\int_{t}^{t+h} u(x,s)ds, \quad 0 < h \leq T - t.$$

For this function the first time derivative exists and is equal:

$$\frac{d}{dt}u_h(x,t) = u_{h,t} = \frac{1}{h}\left(u(x,t) - u(x,t-h)\right).$$

The Steklov averages has the following main properties, which we use widely in the proof of blow-up (next section) of very weak solutions.

($\mathbf{S_1}$) $\lim\limits_{h \to 0} u_h = u$, in $L_1(\Omega)$ [6, p.11],

($\mathbf{S_2}$) $u_{t,h} = \left(\dfrac{d}{dt}u(x,t)\right)_h = \dfrac{d}{dt}u_h(x,t) = u_{h,t}$,

($\mathbf{S_3}$) $(f(u))_h \geq f(u_h)$ for $f(u)$ convex, $\quad (f(u))_h \leq f(u_h)$ for $f(u)$ concave,

($\mathbf{S_4}$) $\left(\displaystyle\int_\Omega u dx\right)_h = \displaystyle\int_\Omega u_h dx$,

($\mathbf{S_5}$) If $v \leq (\geq) u$, then $v_h \leq (\geq) u_h$.

The property (S_1) is mentioned in [6].

We give the proof of the property (S_3), in the case $f(u)$ is convex. Analogously when $f(u)$ is concave.

$$\begin{aligned}(f(u(x,t)))_h &= \frac{1}{h}\int_{t-h}^{t} f(u(x,s))ds \geq \frac{1}{h}(t-t+h) f\left(\frac{1}{(t-t+h)}\int_{t-h}^{t}(u(x,s))ds\right)\\ &= f\left(\frac{1}{h}\int_{t-h}^{t}(u(x,s))ds\right) = f(u_h(x,t)).\end{aligned}$$

The properties $S_2,\ S_4$ and S_5 are obvious.

The very weak solution of problem (7) also can be written in terms of Steklov averages, namely setting $t_1 = t-h,\ t_2 = t$ (or $t_1 = t,\ t_2 = t+h$), multiply by $h > 0$ and taking as test functions $0 \le \eta = \eta(x) \in C^\infty(\Omega)$, we obtain the equivalent Steklov form (see below relation (10)).

3. BLOW-UP

In this section we discuss the blow-up of solutions of (1). Actually we prove the blow-up of the very weak solution for the non-local degenerate Filtration problem and for the non-local degenerate Porous Medium problem. The main tools we use in this section are Kaplan's method, Jensen's inequality, Steklov averages, as well as the very weak form of solutions.

We shall prove that the blow-up occurs in the following cases:

- **For the Neumann problem.**

For all $\lambda > 0$ and for all $u_0(x) \ge 0$.

- **For the Dirichlet and Robin problem and for convex $\boldsymbol{\Omega}$.**

For sufficiently large λ and for all $u_0(x) \ge 0$.
In both cases it is required the functions $K(s)$ and $f(s)$ to satisfy a certain condition, see below (13).

3.1. The Neumann problem

We have $\beta(x) \equiv 0$ and we consider the following problem in terms of Steklov averages:

$$\frac{d}{dt}\int_\Omega \eta u_h dx - \int_\Omega K_h(u)\Delta\eta dx - \lambda\frac{1}{h}\int_t^{t+h}\frac{\int_\Omega f(u)\eta dx}{\left(\int_\Omega f(u)dx\right)^p}ds$$
$$-\frac{1}{h}\int_t^{t+h}\left\{\int_{\partial\Omega}\left(\eta\frac{\partial K(u)}{\partial \hat{n}} - K(u)\frac{\partial \eta}{\partial \hat{n}}\right)d\sigma(x)\right\}ds = 0, \tag{10a}$$

$$\frac{1}{h}\int_t^{t+h}\left(\int_{\partial\Omega}\eta\frac{\partial K(u)}{\partial \hat{n}}d\sigma(x)\right)ds = 0, \quad \int_\Omega \eta(x)u(x,0)dx \ge 0, \tag{10b}$$

$$0 \le t \le T-h \quad \text{and for every} \quad \eta(x) \in C^\infty(\Omega),\ \eta(x) \ge 0.$$

Here we choose as test functions the constant function $\eta = \eta(x) \equiv 1,\ x \in \overline{\Omega}$. Hence the integral on the boundary becomes zero and is omitted.

Proposition 7 Let Ω be a bounded domain of $\mathbb{R}^N$ and $f(s)$ be taken to satisfy (2). Then, the solution $u(x,t)$ of (10a) blows-up in L_1-norm sense in finite time for all values of the parameter $\lambda > 0$ and for every $u_0(x) \geq 0$ ($u_0(x)$ is not identically zero).

Proof. We have for problem (10a):

$$\frac{d}{dt}\int_\Omega u_h dx = \lambda \frac{1}{h}\int_t^{t+h} \frac{\int_\Omega f(u)dx}{\left(\int_\Omega f(u)dx\right)^p} ds = \lambda \frac{1}{h}\int_t^{t+h} \left(\int_\Omega f(u)dx\right)^{1-p} ds \text{ (Jensen on} f)$$

$$\geq \lambda \frac{1}{h}\int_t^{t+h} \left(|\Omega|\, f\left(\frac{1}{|\Omega|}\int_\Omega udx\right)\right)^{1-p} ds = \quad \text{(by definition of Steklov averages)}$$

$$= \lambda\,|\Omega| \left[\left(f\left(\frac{1}{|\Omega|}\int_\Omega udx\right)\right)^{1-p}\right]_h = \lambda\,|\Omega| \left[f^{1-p}\left(\frac{1}{|\Omega|}\int_\Omega udx\right)\right]_h$$

$$\geq \lambda\,|\Omega|\, f^{1-p}\left(\frac{1}{|\Omega|}\int_\Omega u_h dx\right),$$

(on using that f^{1-p} is a convex function and from S_3 - S_4 properties of Steklov averages).

Now we put $A_h(t) = \frac{1}{|\Omega|}\int_\Omega u_h(x,t)dx$, with $A_h(t) \to A(t) = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx$ as $h \to 0+$, for all $t \geq 0$, and the above inequality becomes:

$$\frac{d}{dt}A_h(t) \geq \lambda f^{1-p}\left(A_h(t)\right) \quad \text{or} \quad \int_{A_h(0)}^{A_h(t)} \frac{ds}{f^{1-p}(s)} \geq \lambda t, \ \text{ this implies}$$

$$t \leq \frac{1}{\lambda}\int_{A_h(0)}^{A_h(t)} \frac{ds}{f^{1-p}(s)} \leq \frac{1}{\lambda}\int_{A_h(0)}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty,$$

by condition (2), $\displaystyle\int_{b>0}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty$ and $\displaystyle A_h(0) = \frac{1}{|\Omega|}\int_\Omega u_h(x,0)dx > 0.$

Thus we conclude that:

$$A_h(t) = \frac{1}{|\Omega|}\int_\Omega u_h dx = \frac{1}{|\Omega|}\left\|u_h(\cdot,t)\right\|_{L_1} \to \infty, \quad t \to t_h^* \leq \frac{1}{\lambda}\int_{A_h(0)}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty.$$

The parameter h is an arbitrary positive number. So we can pass to the limit as $h \to 0+$ and by S_1 Steklov property we get:

$$A_h(t) \to A(t) = \frac{1}{|\Omega|}\int_\Omega u\, dx = \frac{1}{|\Omega|}\left\|u(\cdot,t)\right\|_{L_1} \ \text{ and } \ t_h^* \to t^* \leq \int_{A(0)}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty.$$

Finally, we obtain:

$$\|u(\cdot,t)\|_{L_1} \to \infty \quad \text{as} \quad t \to t^* \leq \frac{1}{\lambda}\int_{A(0)}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty,$$

so we prove the blow-up of L_1-norm of $u(\cdot,t)$ in finite time t^*.

3.2. **The Dirichlet and Robin problem**

(i). Blow-up for sufficiently large $\boldsymbol{\lambda}$ ($\boldsymbol{\lambda \gg 1}$).

The equation in very weak (Steklov) form is written:

$$\frac{d}{dt}\int_{\Omega} \eta u_h dx - \frac{1}{h}\int_{t-h}^{t}\int_{\Omega} K(u)\Delta\eta dx ds - \lambda\frac{1}{h}\int_{t-h}^{t} \frac{\int_{\Omega} f(u)\eta dx}{\left(\int_{\Omega} f(u)dx\right)^p} ds - \\ \frac{1}{h}\int_{t-h}^{t}\left\{\int_{\partial\Omega}\left(\eta\frac{\partial K(u)}{\partial \hat{n}} - K(u)\frac{\partial \eta}{\partial \hat{n}}\right)d\sigma(x)\right\}ds = 0, \tag{11a}$$

$$\frac{1}{h}\int_{t-h}^{t}\left\{\int_{\partial\Omega}\left(\eta\left[\frac{\partial K(u)}{\partial \hat{n}} + \beta(x)K(u)\right]\right)d\sigma(x)\right\}ds = 0, \quad 0 < \beta(x) \leq \infty, \tag{11b}$$

$$\int_{\Omega} \eta(x,0)u(x,0)dx \geq 0, \tag{11c}$$

where $\eta(x,t) \in C_c^{\infty}(\Omega_T)$ or $C^{\infty}(\Omega_T)$, for the Dirichlet or for the Robin problem, respectively, and $\eta(x) \geq 0$; hence in both cases the integral on the boundary becomes zero.

Now we consider the eigenvalue problem:

$$-\Delta\phi = \mu\phi, \quad x \in \Omega, \tag{12a}$$

$$\frac{\partial \phi}{\partial \hat{n}} + \beta(x)\phi = 0, \quad x \in \partial\Omega, \tag{12b}$$

with $\mu > 0$, $\phi = \phi(x) > 0,\ x \in \Omega$ and $0 \leq \inf_{\Omega} \phi(x) \leq \phi(x) \leq \overline{k} = \sup_{\Omega} \phi(x)$.

Proposition 8 Let the following condition hold:

$$\int_{\Omega}\left[f^{1-p}(u(x,t)) - K(u(x,t))\right]\phi(x)dx > 0. \tag{13}$$

Then, for sufficiently large λ ($\lambda > \lambda_0 = \mu\left[(\gamma+1)/k\right]^p > 0$), there exists a $t^* < \infty$, such that the very weak solution of (11) blows-up in finite time in the sense of L_1-norm.

Proof. We take $\int_\Omega \phi(x)dx = 1$, for Jensen's inequality to hold in its simple form. On taking as test function, the function $\phi(x)$ of problem (12) ($\eta = \phi$ in (11)) then the boundary term in (11a) is zero and we obtain:

$$\frac{d}{dt}\int_\Omega \phi u_h dx - \frac{1}{h}\int_{t-h}^t \int_\Omega K(u)\Delta\phi dx ds = \lambda \frac{1}{h}\int_{t-h}^t \frac{\int_\Omega f(u)\phi dx}{\left(\int_\Omega f(u)dx\right)^p} ds, \quad \text{or}$$

$$\frac{d}{dt}\int_\Omega \phi u_h dx + \mu \frac{1}{h}\int_{t-h}^t \int_\Omega \phi K(u) dx ds = \lambda \frac{1}{h}\int_{t-h}^t \frac{\int_\Omega f(u)\phi dx}{\left(\int_\Omega f(u)dx\right)^p} ds.$$

By assumption (13) we have:

$$\int_\Omega \left[f^{1-p}(u(x,t)) - K(u(x,t))\right]\phi(x)dx > 0, \quad \text{or}$$

$$\int_{t-h}^t \int_\Omega \phi f^{1-p}(u) dx ds > \int_{t-h}^t \int_\Omega \phi K(u) dx ds,$$

so we get the inequality:

$$\frac{d}{dt}\int_\Omega \phi u_h dx + \mu \frac{1}{h}\int_{t-h}^t \int_\Omega \phi f^{1-p}(u) dx ds > \lambda \frac{1}{h}\int_{t-h}^t \frac{\int_\Omega f(u)\phi dx}{\left(\int_\Omega f(u)dx\right)^p} ds.$$

At this point we notice that the method of parallel moving planes, which is known to hold for classical solutions, can also be extended in the degenerate parabolic case, by using approximation arguments, provided that the initial data decrease near $\partial\Omega$ in suitable way, see for details in [15, p.119] and the references therein. So by this method (see also [10]), since Ω is a convex domain and $f(s)$ is increasing, there is a relative compact set $\Omega_0 \subset \Omega$, $(\overline{\Omega}_0 \subset \Omega)$ such that:

$$\int_\Omega f(u)dx \le (\gamma+1)\int_{\Omega_0} f(u)dx$$

for some $\gamma = \gamma(\Omega) \in N^*$. Let $k = \inf_{x\in\Omega_0} \phi(x)$. Because of $\overline{\Omega}_0 \subset \Omega$ and on using maximum principle arguments, as mentioned in [15], we have $k > 0$, so:

$$\int_\Omega f(u)dx \le \frac{\gamma+1}{k}\int_{\Omega_0} f(u)\phi(x)dx \le \frac{\gamma+1}{k}\int_\Omega f(u)\phi(x)dx. \tag{14}$$

Now by using (14), our main inequality becomes:

$$\frac{d}{dt}\int_\Omega \phi u_h dx + \mu \frac{1}{h}\int_{t-h}^t \int_\Omega \phi f^{1-p}(u) dx ds > \frac{\lambda k^p}{(\gamma+1)^p}\frac{1}{h}\int_{t-h}^t \left(\int_\Omega f(u)\phi dx\right)^{1-p} ds.$$

Applying Jensen inequality on the term $\int_\Omega \phi f^{1-p}(u)dx$, (the inverse version because the function s^{1-p} is concave), we get:

$$\int_\Omega \phi f^{1-p}(u)dx \leq \left(\int_\Omega \phi f(u)dx\right)^{1-p}.$$

Hence we obtain the inequality:

$$\frac{d}{dt}\int_\Omega \phi u_h dx + \mu\frac{1}{h}\int_{t-h}^{t}\left(\int_\Omega \phi f(u)dx\right)^{1-p}ds > \frac{\lambda k^p}{(\gamma+1)^p}\frac{1}{h}\int_{t-h}^{t}\left(\int_\Omega \phi f(u)dx\right)^{1-p}ds,$$

$$\text{or }\ \frac{d}{dt}\int_\Omega \phi u_h dx > \left(\frac{\lambda k^p}{(\gamma+1)^p} - \mu\right)\frac{1}{h}\int_{t-h}^{t}\left(\int_\Omega \phi f(u)dx\right)^{1-p}ds.$$

Now if we have $c = \lambda k^p (\gamma+1)^{-p} - \mu > 0$, or $\lambda > \mu(\gamma+1)^p k^{-p}$, we apply again Jensen inequality on f, and take:

$$\frac{d}{dt}\int_\Omega \phi u_h dx > c\frac{1}{h}\int_{t-h}^{t}\left(f\left(\int_\Omega \phi u dx\right)\right)^{1-p}ds = c\frac{1}{h}\int_{t-h}^{t} f^{1-p}\left(\int_\Omega \phi u dx\right)ds =$$

$$\text{(definition of Steklov average)} = c\left[f^{1-p}\left(\int_\Omega \phi u dx\right)\right]_h \geq$$

$$(S_3 \text{ Steklov for } f^{1-p} \text{ convex and } S_4 \text{ Steklov}) \geq c f^{1-p}\left(\int_\Omega \phi u_h dx\right).$$

We again set $A(t) = \int_\Omega \phi(x)u(x,t)dx$ then $A_h(t) = \int_\Omega \phi u_h dx$, and obtain:

$$\frac{d}{dt}\left(A_h(t)\right) > c f^{1-p}\left(A_h(t)\right) \text{ or } \int_{A_h(0)}^{A_h(t)}\frac{ds}{f^{1-p}(s)} > ct \text{ or}$$

$$t < \frac{1}{c}\int_{A_h(0)}^{A_h(t)}\frac{ds}{f^{1-p}(s)} < \frac{1}{c}\int_{A_h(0)}^{\infty}\frac{ds}{f^{1-p}(s)} = t_h^* < \infty,$$

on using (2) and $A_h(0) = \int_\Omega \phi(x)u_h(x,0)dx > 0$.

So, as $t \to t_h^*-$ this implies that $A_h(t) \to \infty$ and because this convergence holds for every $h > 0$, we can pass to the limit as $h \to 0$ and obtain:

$$A(t) \to \infty \quad \text{as} \quad t \to t^* \leq \frac{1}{c}\int_{A(0)}^{\infty}\frac{ds}{f^{1-p}(s)} < \infty.$$

Now we have:

$A(t) = \int_\Omega \phi\, u dx \leq \overline{k}\int_\Omega u dx = \overline{k}\left\|u(\cdot,t)\right\|_{L_1}$, where $\overline{k} = \max_{x\in\Omega}\phi(x)$.

Finally we conclude:

$$\|u(\cdot,t)\|_{L_1} \to \infty \quad \text{as} \quad t \to T^*- \leq t^* \leq \frac{1}{c}\int_{A(0)}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty,$$

where T^* is the blow-up time for the L_1-norm, $c = \lambda k^p (\gamma+1)^{-p} - \mu > 0$ and for all $\lambda > \frac{\mu(\gamma+1)^p}{k^p}$.

So we prove the blow-up of L_1-norm of $u(\cdot,t)$ in finite time.

4. Discussion

In this work, we consider the local existence and uniqueness of very weak solutions for a degenerate non-local initial boundary value problem for both the Filtration (see (1)) and the Porous Medium (see (3)) equations, (for these results see [7, 13]). We assume nonnegative initial data ($u_0(x) \geq 0$ in Ω having compact support in Ω), which decrease near $\partial\Omega$ in suitable way [15]; this is required for applying parallel moving plane method for degenerate parabolic case and to show that (14) holds. For both problems, we prove blow-up of very weak solutions for sufficiently large λ and for any $u_0(x) \geq 0$. The method we use is Kaplan's method on very weak solutions as well as on Steklov averages. In the case of Dirichlet and Robin problems, we restrict Ω to be a convex domain. For the Neumann problem the blow-up occurs for any $\lambda > 0$ and for any $u_0(x) \geq 0$.

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