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Linear Rectangular Descriptor Matrix Differential Equations and Kronecker Products

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Abstract

In the control and system science theory, there are applications where the linear rectangular descriptor matrix differential systems of the following type are appeared

$$FX'(t) = AX(t) + X(t)B$$

whose coefficients are square time-invariant matrices.

Thus, in this paper, we provide analytical formulas for this class of differential equations when consistent and non-consistent initial conditions are given. The method used relies on Kronecker matrix products, basic tools of matrix pencil theory, and some related results and extensions are also given.

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1. Introduction

In the control and system science theory, see for instance [1], there are applications where the linear rectangular descriptor matrix differential systems of the following type are appeared

$$FX'(t) = AX(t) + X(t)B \tag{1}$$

whose coefficients are square time-invariant matrices. Here, the application of the Kronecker matrix products in transforming linear descriptor matrix equation into corresponding matrix-vector equations is a well-established technique; see [3], [5], [11], [16] and has received fairly widespread attention; see [2]. The main purpose of this brief paper is to provide explicit solutions for this class of differential equations (1) when consistent and non-consistent initial conditions are given. The method used relies on Kronecker matrix products, basic tools of matrix pencil theory, and some related results and extensions are also given.

1.1. **Properties of the Kronecker product**

Definition 1.1 Let $A = [a_{ij}] \in \mathcal{M}(\mathbb{F}, m \times n)$ and $B \in \mathcal{M}(\mathbb{F}, k \times l)$. The **kronecker tensor** product is defined as follows

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathcal{M}(\mathbb{F}, m \cdot k \times n \cdot l). \tag{2}$$

Associate with the matrix $X = [x_{ij}] \in \mathcal{M}(\mathbb{F}, m \times n)$, we can consider the useful transformation U, such as the following column vector is derived

$$U(X) = \left[x_{11}\ x_{12} \ldots\ x_{1n} \vdots\ x_{21}\ x_{22} \ldots\ x_{2n} \vdots \ldots \vdots x_{m1}\ x_{m2} \ldots\ x_{mn} \right] \in \mathcal{M}(\mathbb{F}, m \cdot n \times 1). \tag{3}$$

Moreover, we denote the "inverse" transformation

$$\mathcal{U}(U(X)) = \mathcal{U}\left(\left[x_{11}\ x_{12} \ldots\ x_{1n} \vdots\ x_{21}\ x_{22} \ldots\ x_{2n} \vdots \ldots \vdots x_{m1}\ x_{m2} \ldots\ x_{mn} \right] \right) = X \in \mathcal{M}(\mathbb{F}, m \times n) \tag{4}$$

This definition follows [14], [15] which is also slightly different from the adopted elsewhere [2] but is a somewhat more natural one since when applied to product the ordering is maintained, i.e.

$$U(CXD) = \left(C \otimes D^T \right) U(X), \tag{5}$$

where $C \in \mathcal{M}(\mathbb{F}, p \times m)$ and $D \in \mathcal{M}(\mathbb{F}, n \times q)$.

Lemma 1.1 *Let $A, B \in \mathcal{M}(\mathbb{F}, n \times n)$ and $C, D \in \mathcal{M}(\mathbb{F}, m \times m)$, then*

$$(A \otimes C) \cdot (B \otimes D) = AB \otimes CD. \tag{6}$$

Lemma 1.2 *Let $A, B \in \mathcal{M}(\mathbb{F}, n \times n)$ and $A \otimes B \in \mathcal{M}\left(\mathbb{F}, n^2 \times n^2\right)$ are non-singular matrices, then*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{7}$$

Lemma 1.3 *Let $A \in \mathcal{M}(\mathbb{F}, n \times n)$ and $B \in \mathcal{M}(\mathbb{F}, m \times m)$, then*

$$\det(A \otimes B) = (\det(A))^n (\det(B))^m. \tag{8}$$

Remark 1.1 Let $A \in \mathcal{M}(\mathbb{F}, n \times n)$ with $\det A = 0$, then $\det(A \otimes I_m) = (\det(A))^n = 0$.

1.2. Mathematical background for computational solutions

In this brief section, we introduce some preliminary concepts and definitions from matrix pencil theory, see [6], [7] etc.

Definition 1.2 Given $F, G \in \mathcal{M}(\mathbb{F}, m \times n)$ and an indeterminate $s \in \mathbb{F}$, the matrix pencil $sF - G$ is called *regular* when $m = n$ and $\det(sF - G) \neq 0$. In any other case, the pencil will be called *singular*.

Definition 1.3 The pencil $sF - G$is said to be *strictly equivalent* to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist nonsingular $P \in \mathcal{M}(\mathbb{F}, m \times m)$ and $Q \in \mathcal{M}(\mathbb{F}, n \times n)$ such as

$$P(sF - G)Q = s\tilde{F} - \tilde{G}.$$

In this paper, we consider the case that pencil is *regular*.

This is the set of elementary divisors (e.d.) obtained by factorizing the invariant polynomials $f_i(s, \hat{s})$ into powers of homogeneous polynomials irreducible over field $\mathbb{F}$.

In the case where $sF - G$ is a regular, we have e.d. of the following type:

— e.d. of the type s^p are called zero finite elementary divisors (**z. f.e.d.**)

— e.d. of the type $(s - a)^\pi$, $a \neq 0$ are called nonzero finite elementary divisors (**nz. f.e.d.**)

— e.d. of the type $\hat{s}^q$ are called infinite elementary divisors (**i.e.d.**).

Let $B_1, B_2, ..., B_n$ be elements of $\mathcal{M}(\mathbb{F}, n \times n)$. The direct sum of them denoted by $B_1 \oplus B_2 \oplus ... \oplus B_n$ is the *block diag* $\{B_1, B_2, ..., B_n\}$.

Then, the complex Weierstrass form $sF_w - Q_w$ of the regular pencil $sF - G$ is defined by $sF_w - Q_w \triangleq sI_p - J_p \oplus sH_q - I_q$, where the first normal Jordan type element is uniquely defined by the set of f.e.d.

$$(s - a_1)^{p_1}, \ldots, (s - a_\nu)^{p_\nu}, \quad \sum_{j=1}^{\nu} p_j = p \tag{9}$$

of $sF - G$ and has the form

$$sI_p - J_p \triangleq sI_{p_1} - J_{p_1}(a_1) \oplus \ldots \oplus sI_{p_\nu} - J_{p_\nu}(a_\nu) \tag{10}$$

And also the q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.

$$\hat{s}^{q_1}, \ldots, \hat{s}^{q_\sigma}, \quad \sum_{j=1}^{\sigma} q_j = q \tag{11}$$

of $sF - G$ and has the form

$$sH_q - I_q \triangleq sH_{q_1} - I_{q_1} \oplus \ldots \oplus sH_{q_\sigma} - I_{q_\sigma}. \tag{12}$$

Thus the H_q is a nilpotent element of $\mathcal{M}(\mathbb{F}, n \times n)$ with index $\tilde{q} = \max\{q_j : j = 1, 2, \ldots, \sigma\}$, where

$$H_{\tilde{q}} = \mathbb{O}, \tag{13}$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathcal{M}(\mathbb{F}, p_j \times p_j),$$

$$J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & 0 & \cdots & 0 \\ 0 & a_j & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_j & 1 \\ 0 & 0 & 0 & 0 & a_j \end{bmatrix} \in \mathcal{M}(\mathbb{F}, p_j \times p_j)$$

and

$$H_{q_j} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}(\mathbb{F}, q_j \times q_j). \tag{14}$$

Here, some elements for the analytic computation of $e^{A(t-t_o)}$, $t \in [t_o, \infty)$ are provided. To perform this computation, many theoretical and numerical methods have been developed. Thus, the interesting readers might consult papers [4], [5], [10], [12], [17] and the references therein. In order to obtain more analytic formulas, the following known results should be mentioned.

Lemma 1.4 *(see [5])*

$$e^{J_{p_j}(a_j)(t-t_o)} = (d_{kl})_{p_j}, \tag{15}$$

where

$$d_{kl} = \begin{cases} e^{a_j(t-t_o)} \dfrac{(t-t_o)^{l-1}}{(l-1)!}, 1 \leqslant k \leqslant l \leqslant p_j \\ 0 \text{ , } otherwise \end{cases}$$

Another expression for the exponential matrix of Jordan block is provided by the following Lemma.

Lemma 1.5 *(see [5])*

$$e^{J_{p_j}(a_j)(t-t_o)} = \sum_{i=0}^{p_j-1} f_i(t-t_o) \left[J_{p_j}(a_j)\right]^i \tag{16}$$

where the $f_k(t-t_o)$*' s satisfy the following system of* p_j *equations:*

$$\sum_{i=0}^{p_j-1}\begin{pmatrix} i \\ k \end{pmatrix} a_j^{i-k} f_i(t-t_o) = \frac{(t-t_o)^k}{k!} e^{a_j t}, k = 1, 2, ..., p_j$$

and

$$\left[J_{p_j}(a_j)\right]^i = \left(c_{kl}^{(i)}\right)_{p_j}, for 1 \leqslant k, l \leqslant p_j \tag{17}$$

where

$$c_{kl}^{(i)} = \begin{pmatrix} i \\ l-k \end{pmatrix} a_j^{i-(l-k)}.$$

2. Main results for (non-) consistent initial conditions

In this section the main results for (non-) consistent initial conditions for the regular case of system (18) are analytically presented. The whole discussion extends the existing literature; see for instance [4] and [8]. Moreover, it should be stressed out that these results offer the necessary mathematical framework for extending interesting known practical applications, see also introduction.

Lemma 2.1 *System (1) transposes to*

$$(F \otimes I_m) U'(X)(t) = \left(A \otimes I_m + I_n \otimes B^T\right) U(X)(t), \tag{18}$$

where $F \otimes I_m \in \mathcal{M}(\mathbb{F}, nm \times nm)$*, and* $U(X)(t) \in \mathcal{M}(\mathbb{F}, nm \times 1)$*.*

Proof. System (1) can be written as follows

$$F \cdot X'(t) \cdot I_m = A \cdot X(t) \cdot I_m + I_n \cdot X(t) \cdot B.$$

Considering the transformation U, we obtain

$$U(F \cdot X'(t) \cdot I_m) = U(A \cdot X(t) \cdot I_m + I_n \cdot X(t) \cdot B).$$

It can be easily shown that U is a linear transformation, so we take

$$U(A \cdot X(t) \cdot I_m + I_n \cdot X(t) \cdot B) = U(A \cdot X(t) \cdot I_m) + U(I_n \cdot X(t) \cdot B).$$

Consequently, we have

$$U(F \cdot X'(t) \cdot I_m) = U(A \cdot X(t) \cdot I_m) + U(I_n \cdot X(t) \cdot B) \Leftrightarrow$$
$$(F \otimes I_m) U(X'(t)) = (A \otimes I_m) U(X(t)) + \left(I_n \otimes B^T\right) U(X(t)).$$

And the eq. (18) derives. □

Remark 2.1 It is clear that system (18) is also descriptor, since $\det(F \otimes I_m) = 0$.

Now, from the regularity of the pencil $s\left(F\otimes I_m\right)-\left(A\otimes I_m+I_n\otimes B^T\right)$, there exist non-singular matrices P and Q on $\mathcal{M}\left(\mathbb{F},nm\times nm\right)$ such that (see also subsection 1.2):

$$P\left(F\otimes I_m\right)Q=\left(F\otimes I_m\right)_w=I_p\oplus H_q, \tag{19}$$

$$P\left(A\otimes I_m+I_n\otimes B^T\right)Q=\left(A\otimes I_m+I_n\otimes B^T\right)_w=J_p\oplus I_q, \tag{20}$$

where $I_p,\ J_p,\ H_q$ and I_q are given by (14) where

$$I_p=I_{p_1}\oplus\ldots\oplus I_{p_\nu},$$

$$J_p=J_{p_1}\left(a_1\right)\oplus\ldots\oplus J_{p_\nu}\left(a_\nu\right),$$

$$H_q\triangleq H_{q_1}\oplus\ldots\oplus H_{q_\sigma},$$

and

$$I_q=I_{q_1}\oplus\ldots\oplus I_{q_\sigma}.$$

Note that $\sum\limits_{j=1}^{\nu}p_j=p$ and $\sum\limits_{j=1}^{\sigma}q_j=q$, where $p+q=nm$.

Remark 2.2 Mathematically speaking, it would be very interested if we could connect the eigenstructure of the pencil $s\left(F\otimes I_m\right)-\left(A\otimes I_m+I_n\otimes B^T\right)$ with those of the pencil $sF-A$ (which is derived by the homogeneous system $FX'\left(t\right)=AX\left(t\right)$(*)).

Thus, we can consider the derived pencil of system (18) as a permutated pencil of system (*). Under this point of view, the **PAPS** formula should be transformed and generalized more, see [7] and [9]. For this direction, some research is in preparation.

Lemma 2.2 *System (18) may be written into two sub-systems. The so-called slow sub-system*

$$\tilde{U}'\left(X\right)_p\left(t\right)=J_p\tilde{U}\left(X\right)_p\left(t\right), \tag{21}$$

and the relative fast sub-system

$$H_q\tilde{U}'\left(X\right)_q=\tilde{U}\left(X\right)_q\left(t\right). \tag{22}$$

Proof. Consider the transformation

$$U\left(X\right)\left(t\right)=Q\tilde{U}\left(X\right)\left(t\right). \tag{23}$$

Substituting the previous expression into (18) we obtain

$$\left(F\otimes I_m\right)Q\tilde{U}'\left(X\right)\left(t\right)=\left(A\otimes I_m+I_n\otimes B^T\right)Q\tilde{U}\left(X\right)\left(t\right).$$

Whereby, multiplying by P, we arrive at

$$\left(F\otimes I_m\right)_w\tilde{U}'\left(X\right)\left(t\right)=\left(A\otimes I_m+I_n\otimes B^T\right)_w\tilde{U}\left(X\right)\left(t\right). \tag{24}$$

Moreover, we can write $\tilde{U}\left(X\right)\left(t\right)$ as $\tilde{U}\left(X\right)\left(t\right)=\begin{bmatrix}\tilde{U}\left(X\right)_p\left(t\right)\\ \tilde{U}\left(X\right)_q\left(t\right)\end{bmatrix}\in\mathcal{M}\left(nm\times 1,\mathbb{F}\right)$. And taking into account the above expressions, we arrive easily at (21) and (22). □

System (21) is an ordinary linear differential system and has a unique solution for any initial condition

$$\tilde{U}\left(X\right)_{p}\left(t_{o}\right)=\tilde{Q}_{p}U\left(X\right)_{p}\left(t_{o}\right)\in\mathcal{M}\left(p\times 1,\mathbb{F}\right), \tag{25}$$

where $Q^{-1}=\begin{bmatrix}\tilde{Q}_{p,nm}\\ \tilde{Q}_{q,nm}\end{bmatrix}$. It is well known, the solution has the form

$$\tilde{U}\left(X\right)_{p}\left(t\right)=e^{J_{p}\left(t-t_{o}\right)}\tilde{U}\left(X\right)_{p}\left(t_{o}\right). \tag{26}$$

However, considering [7], [8] etc it can be easily verified that the fast system (22) has only the zero solution (when we have consistent initial conditions), i.e.

$$\tilde{U}\left(X\right)_{q}\left(t\right)=\underset{\sim}{0}\in\mathcal{M}\left(\mathbb{F},q\times 1\right). \tag{27}$$

Remark 2.3 The characteristic polynomial is $\varphi\left(\lambda\right)=\prod_{j=1}^{v}\left(\lambda-a_{j}\right)^{p_{j}}$, with $a_i\neq a_j$ for $i\neq j$ and $\sum_{j=1}^{\nu}p_j=p$. Without loss of generality, we define that

$$d_1=\tau_1,\ d_2=\tau_2,\ ...,\ d_l=\tau_l\ \textit{and}\ d_{l+1}<\tau_{l+1},\ ...,\ d_v<\tau_v$$

where $d_j,\ \tau_j,\ j=1,2,\dots,v$ is the geometric and algebraic multiplicity of the given eigenvalues a_j, respectively.

— Consequently, when $d_j=\tau_j$, then

$$J_k\left(a_k\right)=\begin{bmatrix}a_k&&&\\&a_k&&\\&&\ddots&\\&&&a_k\end{bmatrix}\in\mathcal{M}\left(\mathbb{F},k\times k\right),$$

is also a diagonal matrix with diagonal elements the eigenvalue a_k, for $k=1,\dots,l$.

— When $d_j<\tau_j$, then

$$J_{k,z_k}=\begin{bmatrix}a_k&1&&&\\&a_k&1&&\\&&a_k&\ddots&\\&&&\ddots&1\\&&&&a_k\end{bmatrix}\in\mathcal{M}\left(\mathbb{F},z_k\times z_k\right)$$

for $k=l+1,l+2,\dots,v$, and $z_k=1,2,\dots,d_k$.

Thus, the set of consistent initial conditions for system (25) has the following form:

$$\left\{ \tilde{U}\left(X\right)\left(t_o\right) = \left[\begin{array}{c} \tilde{U}\left(X\right)_p\left(t_o\right) \\ \underline{0}_q \end{array} \right]; t_o \in (0, \infty) \right\}. \tag{28}$$

Proposition 2.1 *The analytic solution of (18) for consistent initial conditions is given by*

$$U\left(X\right)\left(t\right) = Q_{mn,p} \overset{v}{\underset{k=1}{\oplus}} e^{J_k(a_k)(t-t_o)} \tilde{Q}_{p,nm} U\left(X\right)\left(t_o\right). \tag{29}$$

Proof. Combine the eq. (26) and the above discussion, the solution is given by

$$\tilde{U}\left(X\right)_p\left(t\right) = \overset{v}{\underset{k=1}{\oplus}} e^{J_k(a_k)(t-t_o)} \tilde{U}\left(X\right)_p\left(t_o\right).$$

Now, consider eq. (23), then we obtain

$$\begin{aligned} U\left(X\right)\left(t\right) = Q\tilde{U}\left(X\right)\left(t\right) &= \left[\begin{array}{cc} Q_{mn,p} & Q_{mn,q} \end{array} \right] \left[\begin{array}{c} \tilde{U}\left(X\right)_p\left(t\right) \\ \tilde{U}\left(X\right)_q\left(t\right) \end{array} \right] \\ &= Q_{mn,p}\tilde{U}\left(X\right)_p\left(t\right) + Q_{mn,q}\tilde{U}\left(X\right)_q\left(t\right) \end{aligned}$$

and by using (27), we have

$$U\left(X\right)\left(t\right) = Q_{mn,p} \overset{v}{\underset{k=1}{\oplus}} e^{J_k(a_k)(t-t_o)} \tilde{U}\left(X\right)_p\left(t_o\right).$$

Moreover, we know that

$$\tilde{U}\left(X\right)\left(t\right) = \left[\begin{array}{c} \tilde{U}\left(X\right)_p\left(t\right) \\ \tilde{U}\left(X\right)_q\left(t\right) \end{array} \right] = Q^{-1} U\left(X\right)\left(t\right) = \left[\begin{array}{c} \tilde{Q}_{p,nm} \\ \tilde{Q}_{q,nm} \end{array} \right] U\left(X\right)\left(t\right).$$

Finally, the eq. (29) is derived. □

The following two expressions, i.e. (30) and (31) are based on Lemma 1.4 and 1.5 and the "inverse" transformation . Thus, two new analytical formulas are derived which are practically very interesting. Their proofs are straightforward exercise of Lemma 1.4, 1.5 and eq. (29).

Lemma 2.3 *Considering the results of Lemma 1.4, we obtain the expression*

$$X\left(t\right) = \mathcal{U}\left\{ Q_{mn,p} \left[\left(\overset{l}{\underset{k=0}{\oplus}} e^{a_k(t-t_o)} I_{\tau_k} \right) \oplus \left(\overset{v}{\underset{k=l+1}{\oplus}} \overset{d_j}{\underset{z_j=1}{\oplus}} \left(d_{k_1k_2}\right)_{z_j} \right) \right] \tilde{Q}_{p,nm} U\left(X\right)\left(t_o\right) \right\} \in \mathcal{M}\left(\mathbb{F}, m \times n\right). \tag{30}$$

where

$$d_{k_1k_2} = \begin{cases} e^{a_k(t-t_o)} \dfrac{\left(t-t_o\right)^{k_2-k_1}}{\left(k_2-k_1\right)!}, 1 \leqslant k_1 \leqslant k_2 \leqslant z_j \\ 0 \text{ , } otherwise \end{cases}$$

for $k = l+1, l+2, \ldots, v,$ *and* $z_j = 1, 2, \ldots, d_j$.

Lemma 2.4 *Considering the results of Lemma 1.5, we obtain the expression*

$$X(t) = \mathcal{U}\left\{Q_{mn,p}\left[(\underset{k=0}{\overset{l}{\oplus}} e^{a_k(t-t_o)} I_{\tau_k}) \oplus (\underset{k=l+1}{\overset{v}{\oplus}} \underset{z_j=1}{\overset{d_j}{\oplus}} \sum_{i=0}^{z_j-1} f_i(t-t_o)\left[J_{z_j}(\lambda_{jk})\right]^i)\right] \tilde{Q}_{p,nm} U(X)(t_o)\right\} \in \mathcal{M}(\mathbb{F}, m\times n), \tag{31}$$

where the $f_k(t-t_o)$' s satisfy the following system of z_j (for $z_j = 1, 2, \ldots, d_j$) equations:

$$\sum_{i=k}^{z_j-1} \binom{i}{k} a_j^{i-k} f_i(t-t_o) = \frac{(t-t_o)^k}{k!} e^{a_k t}, k = 1, 2, ..., z_j$$

and

$$\left[J_{z_j}(a_k)\right]^i = \left(c_{k_1 k_2}^{(i)}\right)_{z_j}, \ for\ 1 \leqslant k_1, k_2 \leqslant z_j.$$

where

$$c_{k_1 k_2}^{(i)} = \binom{i}{k_2 - k_1} a_j^{i-(k_2-k_1)}.$$

In the end of this section, we describe briefly the impulse behavior of the original system (18) at time t_o. In that case, we merely reformulate Proposition 2.1, so the impulse solution is finally obtained. First, we need to prove the Lemma 2.5. Its proof is based on Laplace transformation, and it is similar to [7], [8] etc. Thus, it is omitted.

Lemma 2.5 *The system (22) has the following solution*

$$\tilde{U}(X)_q(t) = -\sum_{k=0}^{q^*-2} \delta^{(k)}(t) H_q^{k+1} \tilde{U}(X)_q(t_o). \tag{32}$$

Proposition 2.2 *The analytic solution of (18) for non-consistent initial conditions is given by*

$$\begin{aligned} U(X)(t) &= Q_{mn,p} \underset{k=1}{\overset{v}{\oplus}} e^{J_k(a_k)(t-t_o)} \tilde{Q}_{p,nm} U(X)(t_o) \\ &\quad - Q_{mn,q} \sum_{k=0}^{q^*-2} \delta^{(k)}(t) H_q^{k+1} \tilde{Q}_{q,nm} U(X)(t_o) \end{aligned} \tag{33}$$

Proof. Combining the results of Proposition 2.1 and the above discussion, the solution is provided by eq. (33). □

Remark 2.4 For $t > t_o$, it is obvious that (29) is satisfied. Thus, we should stress out that the system (18) has the above impulse behaviour at time instant where a non-consistent initial value is assumed, while it returns to smooth behaviour at any subsequent time instant.

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