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On an Approximation Result for Piecewise Polynomial Functions

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Abstract

We provide a new approach for proving approximation results for piecewise polynomials used in finite element methods. In particular, we improve on the classical results that have been used since the 70's.

Keywords: Piecewise polynomial approximation, discontinuous Galerkin method.

1. Introduction

In proving error estimates for finite element (Galerkin) approximations to solutions of PDE's, one typically tries to reduce the problem into one in approximation theory e.g. Céa's Lemma [4]. Hence, the basic approximation properties of piecewise polynomial functions play a pivotal role in the proof of such error estimates. Specifically, given a smooth function u defined on some domain Ω, and a partition $\mathcal{T}_h = \{K_1, \dots, K_N\}$ of Ω, a piecewise polynomial function χ defined on $\mathcal{T}_h$ is sought that is optimally close to u as measured in some norm. A standard tool for establishing such estimates has been the Bramble-Hilbert Lemma [4, 3]. However, there are some limitations associated with this approach. Specifically, the Lemma cannot be applied in a straighforward manner to domains with curved boundaries. A more serious shortcoming is that the estimates are tied to the degree q in an essential way. To illustrate this point, consider the following estimate

$$\|u - \chi\|_{L^2(\Omega)} \le ch^{q+1}|u|_{H^{q+1}(\Omega)}$$

derived via the Bamble-Hilbert Lemma and where q is the degree of the piecewise polyomials used. In many instances, such as in proving a posteriori error estimates, it is desirable to also have the estimate

$$\|u - \chi\|_{L^2(\Omega)} \le ch|u|_{H^1(\Omega)}$$

which cannot be derived from the Bramble-Hilbert Lemma.

In this paper we will show that estimates such as this, and others, can be obtained using a technique discovered recently in the context of a posteriori error estimation of elliptic problems [7]. Essentially the result states that a discontinuous piecewise polynomial function of degree q over a mesh $\mathcal{T}_h$ can be approximated by a continuous piecewise polynomial function of the same degree defined over the same mesh and that the difference between the two can be bounded by the jumps of the discontinuous function on the interfaces of the mesh. Complementing this result will be the fact that a smooth function u can be appoximated on each cell by a Taylor polynomial type approximation [5], [2]. Indeed, the combination of these two approaches yields a powerful approximation strategy with a wide range of applicability. However, to simplify the exposition, we shall restrict demonstration of this new technique to the case of Lagrangian elements defined on a triangulation of a domain in R^2.

2. Preliminaries

2.1. Notation

For a domain $D \subseteq R^d$ and integer $m \geq 0$, $H^m(D)$ will denote the (Hilbertian) Sobolev space with inner product $(u, v)_{m,D} = \sum_{|\alpha| \leq m} \int_D D^\alpha u \, D^\alpha v \, dx$ and norm $\|u\|_{m,D} = (u, u)_{m,D}^{1/2}$ (cf. [1]). Also, $|\cdot|_{m,D}$ will the denote the seminorm in $H^m(D)$. To simplify the notation, we shall drop m when its value is zero.

Extensive use will be made of edge/surface integrals. So for a $(d-1)$-dimensional subset e of R^d, we set $< u, v >_e = \int_e u \, v \, ds$ and $|u|_e = < u, u >_e^{1/2}$.

2.2. Partitions of Ω

Let $\mathcal{T}_h = \{K_i : \ i = 1, 2, \cdots, m_h\}$ be a family of partitions of the domain Ω parametrized by $0 < h \leq 1$. Such partitions consist of *cells* e.g. triangles or quadrilaterals or other simple geometric shapes. We should note that if Ω is not polygonal, i.e. if $\partial\Omega$ is *curved*, then an *outlying* cell will have one or more curved sides, these being the intersection of the boundary of the cell and $\partial\Omega$. On the other hand, we shall assume that the intersection of the boundaries of two cells is an affine function.

We further assume that

(i) Each cell in $\mathcal{T}_h$ is starlike,

(ii) Every cell in $\mathcal{T}_h$ is *shape-regular*, i.e. the diameters of its inscribed and circumscribed spheres are commensurate. This is referred to often as satisfying the minimum angle condition.

(iii) $\mathcal{T}_h$ is locally quasi-uniform, that is if two cells K_j and K_ℓ are *adjacent* in the sense that $\mu_{d-1}(\partial K_j \cap \partial K_\ell) > 0$ then $\mathrm{diam(K_j)} \approx \mathrm{diam}(K_\ell)$.

We shall find it convenient to use the so-called *broken* Sobolev spaces $H^m(\mathcal{T}_h) = \Pi_{K \in \mathcal{T}_h} H^m(K)$. In this context we consider cells to be open so that elements of $H^m(\mathcal{T}_h)$ are single-valued. We shall also use the discontinuous finite element spaces $V_h^q = \Pi_{K \in \mathcal{T}_h} \mathcal{P}_q(K)$, $q \geq 0$ where $\mathcal{P}_q(K)$ is the space of polynomials of total degree less than or equal to q defined on K.

We define $\mathcal{E}_h^I$ and $\mathcal{E}_h^B$ to be the set of all interior and boundary edges (faces when $d = 3$), respectively:

$$
\begin{aligned}
\mathcal{E}_h^I &= \{e = \partial K_j \cap \partial K_\ell, \quad \mu_{d-1}(\partial K_j \cap \partial K_\ell) > 0\} \\
\mathcal{E}_h^B &= \{e = \partial K \cap \partial\Omega, \quad \mu_{d-1}(\partial K \cap \partial\Omega) > 0\}, \\
\mathcal{E}_h &= \mathcal{E}_h^I \cup \mathcal{E}_h^B.
\end{aligned}
$$

For each $e \in \mathcal{E}_h^I$, we denote the two cells that "share" it by K^+ and K^- respectively. As to which of the two is K^+ is completely arbitrary but not irrelevant! If $e \in \mathcal{E}_h^B$, then $e = \partial K^+ \cap \partial\Omega \equiv \partial K \cap \partial\Omega$.

It is essential to be able to define values of functions in $H^m(\mathcal{T}_h)$ and V_h^q on the edges e. So for $v \in H^m(\mathcal{T}_h)$, $m \geq 1$, and $e \in \mathcal{E}_h$, v_e^+ will denote the trace on e of the restriction of v to K^+. Similarly we define v_e^- for $e \in \mathcal{E}_h^I$.

We also define *jumps* of such traces

$$[v]_e = v_e^+ - v_e^-,\ e \in \mathcal{E}_h^I, \qquad [v]_e = v_e^+,\ e \in \mathcal{E}_h^B$$

Essential use will be made of the following well known *trace* inequality for H^1 functions: Let D be starlike and shape-regular domain with boundary ∂D and let h denote its diameter. Then

$$|v|_{\partial D}^2 \leq c_{tr}(h^{-1}\|v\|_D^2 + h\|\nabla v\|_D^2), \quad \forall v \in H^1(D). \tag{1}$$

Here c_{tr} is independent of h. The explicit dependence of the right-hand-side on h is important for the derivation of error estimates and can be found in [6]. We shall also make use of the following inverse inequality [4]

$$\|v\|_{L^\infty(D)} \leq ch^{-\frac{d}{2}}\|v\|_{L^2(D)} \tag{2}$$

3. The approximation results

We now state and prove our main results. For the sake of brevity, we restrict the exposition to a very specific case: The domain is two-dimensional and is polygonal, the partition consists of a conforming triangulation (no hanging nodes), the finite element space is a subspace of $C^0(\Omega)$ with Lagrangian type basis elements corresponding to Lagrangian nodes (see e.g. Figure 1). At the end of this section we will indicate as to how the proof can be generalized to handle curved boundaries. Further generalizations of this approach in all the above directions will be the subject of a subsequent work.

We begin by quoting the following local approximation result due to Dupont and Scott [5]. See also [2] for an extension of this result to solenoidal vector fields.

Theorem 3.1 *Let integers q, m be given satisfying $0 \leq q \leq m-1$. Let D be a shape-regular and starlike domain in R^d, $d \geq 1$ and let $u \in H^m(D)$. Then, there exists $v_h \in \mathcal{P}_q(D)$ satisfying*

$$|u - v_h|_{s,D} \leq ch^{\ell+1-s}|u|_{\ell+1,D}, \quad 0 \leq s \leq \ell \leq q. \tag{3}$$

In view of the above result and given a triangulation $\mathcal{T}_h$ of Ω, we obtain a global approximation $v_h \in V_h^q$ by patching together the approximations on the individual triangles. Next, we construct an approximation $\chi \in V_h^q \cap C^0(\Omega)$ by an averaging process. As a first step in that direction, we need the following lemma

Lemma 3.1 Given N real numbers $\{\alpha_1, \ldots, \alpha_N\}$ let $\beta = \frac{1}{N}\sum_{j=1}^N \alpha_j$. Then,

$$\sum_{j=1}^{N} |\alpha_j - \beta|^2 \leq C \sum_{j=1}^{N-1} |\alpha_{j+1} - \alpha_j|^2, \tag{4}$$

where C depends only on N.

Proof. For any j, the Cauchy-Schwarz inequality gives

$$|\alpha_j - \beta|^2 = \frac{1}{N^2}\left|\sum_{i=1}^{N}(\alpha_j - \alpha_i)\right|^2 \leq \frac{N-1}{N^2}\sum_{i=1}^{N}|\alpha_j - \alpha_i|^2.$$

Summing over j, we obtain $\sum_{j=1}^{N}|\alpha_j - \beta|^2 \leq \frac{2(N-1)}{N}\sum_{j>i}|\alpha_j - \alpha_i|^2$. The required result now follows upon writing $\alpha_j - \alpha_i = \sum_{k=i}^{j-1}(\alpha_{k+1} - \alpha_k)$ and using the arithmetic-geometric mean inequality. □

The actual construction of χ is done next. The following result, the proof of which we provide next, is a special case of a similar result that appeared in [7]. However, it also extends it in different directions.

Theorem 3.2 *Let $\mathcal{T}_h$ be a conforming triangulation of a domain Ω in R^2. Then for any $v_h \in V_h^q$, $q \geq 1$, there exists $\chi \in V_h^q \cap C^0(\Omega)$ satisfying*

$$\sum_{K\in\mathcal{T}_h} |v_h - \chi|_{s,K}^2 \leq c \sum_{e\in\mathcal{E}_h^I} h_e^{1-2s}\|[v_h]\|_e^2, \quad s \geq 0, \tag{5}$$

where h_e is the length of e.

Proof.

For each $K \in \mathcal{T}_h$ let $\mathcal{N}_K = \{x_K^{(j)},\ j = 1, \cdots, m\}$ be the Lagrange nodes (points) of K and $\{\phi_K^{(j)},\ j = 1, \cdots, m\}$ the corresponding (local) basis functions satisfying $\phi_K^{(j)}(x_K^{(i)}) = \delta_{ij}$. Set $\mathcal{N} = \cup_{K\in\mathcal{T}_h}\mathcal{N}_K$. These are the *global* nodes.

For each $\nu \in \mathcal{N}$, let $\omega_\nu = \{K \in \mathcal{T}_h|\ \nu \in \overline{K}\}$ and denote its cardinality by $|\omega_\nu|$. Note that $|\omega_\nu|$ depends only on the mesh $\mathcal{T}_h$ in that it is bounded by a constant depending only on the minimum angle θ_0 of the triangles in $\mathcal{T}_h$. On the other hand, it is important to distinguish the nodes $\nu \in \mathcal{N}$ for which $|\omega_\nu| = 1$ from the rest. Indeed, if a node ν is such that $|\omega_\nu| = 1$, then ν could be in the interior of a triangle and such nodes exist when $q \geq 3$. Another possibility is that ν belongs to a boundary edge but to only one triangle.

To each node $\nu \in \mathcal{N}$ we associate the global basis function $\phi^{(\nu)}$ given by

$$\operatorname{supp}\phi^{(\nu)} = \bigcup_{K\in\omega_\nu} K, \quad \phi^{(\nu)}\big|_K = \phi_K^{(j)},\ x_K^{(j)} = \nu.$$

Now given $v_h \in V_h^q$, written $v_h = \sum_{K\in\mathcal{T}_h}\sum_{j=1}^{m}\alpha_K^{(j)}\phi_K^{(j)}$, we define the function $\chi \in V_h^q \cap C^0(\Omega)$ by

$$\chi = \sum_{\nu\in\mathcal{N}} \beta^{(\nu)}\phi^{(\nu)}, \text{ where } \beta^{(\nu)} = \frac{1}{|\omega_\nu|}\sum_{x_K^{(j)}=\nu}\alpha_K^{(j)}, \tag{6}$$

and introduce the cell coordinates $\beta_K^{(j)}$, $K \in \mathcal{T}_h$, $j = 1, \ldots, m$ by $\beta_K^{(j)} = \beta^{(\nu)}$ whenever $x_K^{(j)} = \nu$.

A simple scaling argument, say using an affine transformation between a triangle K and the so-called master triangle, shows that

$$\left|\phi_K^{(j)}\right|_{s,K}^2 \leq c h_K^{d-2s}, \quad s \geq 0. \tag{7}$$

Note that we have kept the dimension d in the formula for the sake of possible generalization. Hence

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} |v_h-\chi|_{s,K}^2 &\leq c\,m \sum_{K\in\mathcal{T}_h} h_K^{d-2s} \sum_{j=1}^m \left|\alpha_K^{(j)} - \beta_K^{(j)}\right|^2 \\ &\leq c \sum_{\nu\in\mathcal{N}} h_\nu^{d-2s} \sum_{x_K^{(j)}=\nu} \left|\alpha_K^{(j)} - \beta^{(\nu)}\right|^2, \quad (h_\nu = \max_{K\in\omega_\nu} h_K), \\ &= c \sum_{\nu\in\mathcal{N},\,|\omega_\nu|>1} h_\nu^{d-2s} \sum_{x_K^{(j)}=\nu} \left|\alpha_K^{(j)} - \beta^{(\nu)}\right|^2. \end{aligned} \tag{8}$$

Note that there are no contributions from nodes ν for which $|\omega_\nu| = 1$. Another important fact is that when $|\omega_\nu| > 1$, then ν must be shared by at least two triangles and therefore must belong to some interior edge $e \in \mathcal{E}_h^I$ and this may be the case even if ν belongs to $\partial\Omega$.

For $\nu \in \mathcal{N}$, with $|\omega_\nu| > 1$, we enumerate the elements of ω_ν as $\{K_1, \ldots, K_{|\omega_\nu|}\}$ so that any consecutive pair K_i, K_{i+1} in that list share an edge. Then from Lemma 3.1, with some constant c depending only on $|\omega_\nu|$ and thus on θ_0, we have

$$\sum_{x_K^{(j)}=\nu} \left|\alpha_K^{(j)} - \beta^{(\nu)}\right|^2 \leq c \sum_{i=1}^{|\omega_\nu|-1} \left|\alpha_{K_i}^{(j_i)} - \alpha_{K_{i+1}}^{(j_{i+1})}\right|^2. \tag{9}$$

Using (9) in (8) we have

$$\sum_{K\in\mathcal{T}_h} |v_h-\chi|_{s,K}^2 \leq c \sum_{e\in\mathcal{E}_h^I} \sum_{\nu\in e} h_\nu^{d-2s} \left|\alpha_{K^+}^{(j_\nu^+)} - \alpha_{K^-}^{(j_\nu^-)}\right|^2, \tag{10}$$

with $x_{K^+}^{(j_\nu^+)} = x_{K^-}^{(j_\nu^-)} = \nu$. Note that $\alpha_{K^+}^{(j_\nu^+)} - \alpha_{K^-}^{(j_\nu^-)}$ is the jump in the values of v_h at ν across e. Also, since the mesh $\mathcal{T}_h$ is locally quasiuniform, we have $h_e \approx h_\nu$. Hence, noting that e is $(d-1)$-dimensional, it follows from the inverse inequality (2) that

$$\sum_{\nu\in e} h_\nu^{d-2s} \left|\alpha_{K^+}^{(j_\nu^+)} - \alpha_{K^-}^{(j_\nu^-)}\right|^2 \leq c h_e^{d-2s} |[v_h]|_{L^\infty(e)}^2 \leq c h_e^{1-2s} |[v_h]|_e^2. \tag{11}$$

where the constant c depends on the number of nodes in e.

Using (11) in (10) concludes the proof. □

Remark 3.1 The proof of theorem 3.2 that is contained in [7] is a special case of the present one in several respects. There we had $s = 0$, Ω was assumed to be polyhedral and χ vanished on $\partial\Omega$. On the other hand, it encompassed more general two- and three-dimensional partitions. Furthermore, the same paper contained an extension to nonconforming meshes.

We now briefly indicate how theorem 3.2 and consequently our main result below can be extended to cover cells more general than triangles including those having curved sides; see Figure 1. Indeed, an examination of the proof of theorem 3.2 shows that the only part that requires modification is (7) and this can be easily shown to hold for a vast array of shapes provided they are starlike and shape-regular.

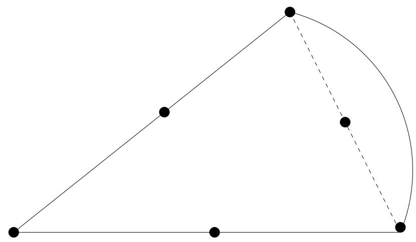

Figure 1: Lagrangian nodes for quadratics

Combining theorems 3.1 and 3.2 we arrive at the main result of this paper

Theorem 3.3 *Under the conditions of theorems 3.1 and 3.2, there exists* $\chi \in V_h^q \cap C^0(\Omega)$ *satisfying*

$$\sum_{K\in\mathcal{T}_h} |u-\chi|_{s,K}^2 \le c \sum_{K\in\mathcal{T}_h} h_K^{2(\ell+1-s)} |u|_{\ell+1,K}^2, \quad 0 \le s \le \ell \le q. \tag{12}$$

Proof. Let v_h and χ be as constructed along the lines of theorems theorems 3.1 and 3.2 respectively. Using the triangle inequality, from (3) and (5) we obtain

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} |u-\chi|_{s,K}^2 &\le 2\sum_{K\in\mathcal{T}_h} |u-v_h|_{s,K}^2 + 2\sum_{K\in\mathcal{T}_h} |v_h-\chi|_{s,K}^2 \\ &\le c\sum_{K\in\mathcal{T}_h} h_K^{2(\ell+1-s)} |u|_{\ell+1,K}^2 + c\sum_{e\in\mathcal{E}_h^I} h_e^{1-2s} |[v_h]|_e^2. \end{aligned} \tag{13}$$

Since u is smooth (continuity is sufficient), its jumps of on interior edges are zero. Hence,

$$|[v_h]|_e^2 = |[v_h-u]|_e^2 \le c \sum_{K=K^+,K^-} \Big(h_K^{-1} \|u-v_h\|_K^2 + h_K \|\nabla(u-v_h\|_K^2 \Big), \tag{14}$$

by virtue of the trace inequality (1). Also, since the mesh is locally quasi uniform, we have

$h_e \approx h_{K^+} \approx h_{K^-}$. Finally, using the local approximation result (3) to bound the terms in (14), we see that (12) follows, thus concluding the proof of the theorem. □

It is interesting to note that the same χ is optimal in a scale of seminorms. Other potentially useful extensions of theorem 3.2 that will be worked out are

— The estimate (12) is global in character. An estimate of the form

$$|u-\chi|_{s,K}^2 \le c \sum_{T\in\mathcal{T}_h^K} h_T^{2(\ell+1-s)} |u|_{\ell+1,T}^2,$$

where $\mathcal{T}_h^K$ is a small patch of cells surrounding K would constitute an obvious improvement.

— Often it is useful to impose specific values of χ on $\partial\Omega$.

— Extension to other types of finite elements e.g. Crouzeix-Raviart, Raviart-Thomas etc. would be obviously of great interest. For elements whose local bases are specified by a set of linear functionals, this should be straightforward. As above, the extension hinges on verifying (7).

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