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Dynamic Polynomial Combinants and Generalised Resultants

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Abstract

The theory of constant polynomial combinants has been well developed [2] and it is linked to the linear part of the constant Determinantal Assignment problem [1] that provides the unifying description of the pole and zero assignment problems in Linear Systems. Considering the case of dynamic pole, zero assignment problems leads to the emergence of dynamic polynomial combinants. This paper aims to demonstrate the origin of dynamic polynomial combinants from Linear Systems, and develop the fundamentals of the relevant theory by establishing their link to the theory of *Generalised Resultants* and examining issues of their parameterization according to the notions of order and degree. The paper provides a description of the key spectral assignment problems, derives the conditions for arbitrary assignability of spectrum and introduces a parameterization of combinants according to their order and degree.

Key Words: Linear Systems, Spectrum Assignment, Generalised Resultants, Diofantine Equations, Polynomial Combinants.

1. Introduction

For the study of problems of linear feedback synthesis which are of the determinantal type [1] (such as pole zero assignment, stabilisation) a specific school of thought has been developed which is specially suited to tackle such problems. This framework is referred to as algebro-geometric because it relies on tools from algebra and algebraic geometry. The essence of the problems faced in this set-up is that they are of a multi-linear nature. The main difficulty of the determinantal problems in the case of frequency assignment lies in that the problem is equivalent to finding real solutions to sets of nonlinear and linear equations; in the case of stabilisation, this is equivalent to determining solutions of nonlinear equations and nonlinear inequalities. The first of the two problems naturally belongs to the intersection theory of complex algebraic varieties, whereas, the latter belongs to the intersection theory of semi-algebraic sets. Determining real intersections is not an easy problem [9]; furthermore, it is also important to be able to compute solutions whenever they exist. The use of algebraic Geometry in the study of spectrum assignment problems was originally introduced in [5], [6], where an affine space approach has been used. The main emphasis in this approach has been the use of intersection theory for the development of necessary conditions and the deployment of special techniques for establishing generic sufficient conditions. Issues of dealing with non-generic cases as well as computation of solutions are hardly addressed.

The Determinantal Assignment Problem Approach (DAP) [1] has been formulated as a unifying approach for all problems of frequency assignment (dynamic and constant pole zero) and its basis lies on the fact that determinantal problems are of a multi-linear nature and thus may be naturally split into a linear and multi-linear problem (decomposability of multivectors). In this framework, the final solution is thus reduced to the solvability of a set of linear equations (characterising the linear problem) together with quadratics (characterising the multi-linear problem of decomposability). The approach heavily relies on exterior algebra and this has implications on the computability of solutions (reconstruction of solutions whenever they exist) and introduces new sets of invariants (of a projective character) which, in turn, characterise the solvability of the problem. This approach has been further developed in [7], [8] by the development of a "blow-up" methodology for linearization of multilinear maps that permit the development of computations, as well as techniques for establishing the development of real solutions [9]. The distinct advantages of the DAP approach, which is a projective space approach, are: it provides the means for computing the solutions; it can handle both generic and exact solvability investigations, and it introduces new criteria for the characterisation of solvability of different problems. Furthermore, it provides a set-up for exterior algebra computations by using the methodology of "Global Linearization" [7], [8]. Most of the work in the DAP framework has been on problems dealing with non-dynamic compensation, where the linear part of the problem is expressed as a constant polynomial combinant, and the study of its properties is well developed [2]. Dynamic compensation problems may also be studied within the DAP framework, but their linear sub-problem depends on dynamic polynomial combinants which have much richer properties. Of course, real intersection theory of varieties is once more the central issue, but the linear varieties (linear part of the problem), as well as the multi-linear part becomes more complex in the dynamic case.

This paper deals with the development of the fundamentals of the theory of dynamic combinants, which define the linear part of the dynamic DAP problem, by examining the origin of the dynamic combinants in Control Theory problems, introducing the basic problems related to spectrum assignment, examining their parameterization according to their order and degree, consider their representation in terms of *generalised resultants* and finally establishing the conditions spectral assignability, which are equivalent to the solvability of a Diofantine Equation over $\mathbb{R}[s]$. The work here provides the means for studying the properties of the linear varieties of the Dynamic DAP and set up the appropriate framework that allows the study of spectrum assignment properties of dynamic combinants.

Throughout the paper the following notation is adopted: If $\mathcal{F}$ is a field, or ring then $\mathcal{F}^{m\times n}$ denotes the set of $m \times n$ matrices over $\mathcal{F}$ If H is a map, then $\mathcal{R}(H)$, $\mathcal{N}_r(H)$, $\mathcal{N}_l(H)$ denote the range, right, left nullspaces respectively. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from the set $\tilde{n} \triangleq \{1, 2, \ldots, n\}$. If $\mathcal{V}$ is a vector space and $\{\underline{v}_{i_1}, \ldots, \underline{v}_{i_k}\}$ are vectors of $\mathcal{V}$ then $\underline{v}_{i1} \wedge \ldots \wedge \underline{v}_{ik} = \underline{v}_{\omega} \wedge,\ \omega = (i_1, \ldots, i_k)$ denotes their exterior product and $\wedge^r \mathcal{V}$ the $r-$th exterior power of $\mathcal{V}$. If $H \in \mathcal{F}^{m\times n}$ and $r \leq \min\{m, n\}$, then $C_r(H)$ denotes the $r-$th compound matrix of H [10]. We shall denote by $\mathbb{R}$[s], $\mathbb{R}$(s), $\mathbb{R}_{pr}$(s)the ring of polynomials, rational functions and proper rational functions over $\mathbb{R}$ respectively.

2. Linear Systems and Dynamic Polynomial Combinants

Consider the linear system [11] described by $S(\mathrm{A, B, C, D})$:

$$\underline{\dot{x}} = A\underline{x} + B\underline{u},\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times p},\ \underline{y} = C\underline{x} + D\underline{u},\ C \in \mathbb{R}^{m\times n},\ D \in \mathbb{R}^{m\times p} \quad (1)$$

where (A, B) is controllable, (A, C) is observable, or by the transfer function matrix $G(s) = C(sI - A)^{-1}B + D$, where . In terms of left, right coprime matrix fraction descriptions (LCMFD, RCMFD), $G(s)$ may be represented as

$$G(s) = D_l^{-1}(s)N_l(s) = N_r(s)D_r^{-1}(s) \quad (2)$$

where $N_\imath(s), N_r(s) \in \mathbb{R}^{m\times p}[s], D_\imath \in \mathbb{R}^{m\times m}[s]$ and $D_r(s) \in \mathbb{R}^{p\times p}$ The system will be called *square* if $m = p$ and *nonsquare* if $m \neq p$. Within the state space framework we may define a number of constant, frequency assignment problems such as the Pole assignment by state feedback, Design of an n-state observer, Pole assignment by constant output feedback and Zero assignment by squaring down, which are all reduced to a Constant Determinantal assignment problem [1]. A number of dynamic assignment problems may be defined on a linear system as shown below:

Dynamic Compensation Problems

Consider the standard feedback configuration [11]

If $G(s) \in \mathbb{R}_{pr}(s)^{m\times p},\ C(s) \in \mathbb{R}(s)^{p\times m}$, and assume coprime MFD"s as in (2) and

$$C(s) = A_\ell(s)^{-1}B_\ell(s) = B_r(s)A_r(s)^{-1} \quad (3)$$

The closed loop characteristic polynomial may be expressed as [10]:

$$\mathrm{f(s)} = \det\left\{[D_\ell(s),\ N_\ell(s)]\begin{bmatrix} A_r(s) \\ B_r(s) \end{bmatrix}\right\} = \det\left\{[A_\ell(s),\ B_\ell(s)]\begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \quad (4)$$

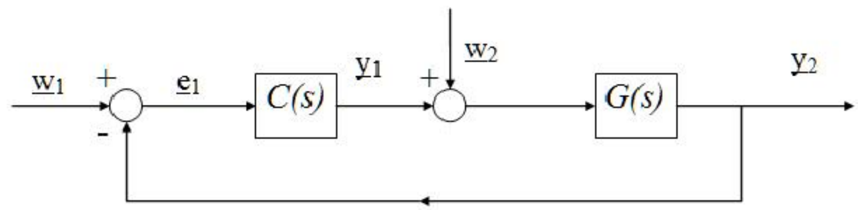

Figure 1: Feedback Configuration

1 if $p \leq m$, then $C(s)$ may be interpreted as *feedback compensator* and we will use the expression of the closed loop polynomial described by (4a)

2 if $p \geq m$, the $C(s)$ may be interpreted as *pre-compensator* and we will use the expression of the closed loop polynomial described by (4b).

The above general dynamic formulation covers a number of important families of $C(s)$ compensators [11] as : **(a)** Constant**, (b)** PI, **(c)** PD, **(d)** PID, **(e)** Bounded degree. In fact,

(a) Constant Controllers : If $p \leq m,\ A_\ell = I_p,\ B_\ell = K \in \mathbb{R}^{p\times m}$, then (4) expresses the constant output feedback case, whereas if $p \geq m$, $A_r = I_m, B_r = K \in \mathbb{R}^{p\times m}$ expresses the constant pre-compensation formulation of the problem.

(b) Proportional plus Integral Controllers: Such controllers are defined by

$$C(s) = K_0 + \frac{1}{s}K_1 = [sI_p]^{-1}[sK_0 + K_1] \tag{5}$$

where $K_0, K_1 \in \mathbb{R}^{p\times m}$ and the left MFD for C(s) is coprime, iff $rank(K) = p$. From the above the determinantal problem for the output feedback PI design is expressed as :

$$\begin{aligned} f(s) &= det\left\{[sI_p, sK_0 + K_1]\begin{bmatrix} D_r(s) \\ N_r(s)\end{bmatrix}\right\} \\ &= det\left\{[I_p, K_0, K_1]\begin{bmatrix} sD_r(s) \\ sN_r(s) \\ N_r(s)\end{bmatrix}\right\} \end{aligned} \tag{6}$$

(c) Proportional plus Derivative Controllers: Such controllers are expressed as

$$C(s) = sK_0 + K_1 = [I_p]^{-1}[sK_0 + K_1] \tag{7}$$

where $K_0,\ K_1\ \in \mathbb{R}^{p\times m}$ and the left MFD for C(s) is coprime for finite s and also for $s=\infty$ if $rank(K_0)=p$. From the above the determinantal output PD feedback is expressed as :

$$f(s)=det\{[I_p, sK_0+K_1]\begin{bmatrix} D_r(s)\\ N_r(s)\end{bmatrix}\}=det\{[I_p,K_1,K_0]\begin{bmatrix} D_r(s)\\ N_r(s)\\ sN_r(s)\end{bmatrix}\} \qquad (8)$$

(d) PID Controllers: These controllers are expressed as

$$C(s)=K_0+\frac{1}{s}K_1+sK_2=[sI_p]^{-1}[s^2K_2+sK_0+K_1] \qquad (9)$$

where $K_0,K_1\in\mathbb{R}^{p\times m}$ and the left MFD is coprime with the only exception possibly at $s=0, s=\infty$ (coprimeness at s=0 is guaranteed by $rank(K_1)=p$ and at $s=\infty$ by $rank(K_2)=p$. From the above, the determinantal output PID feedback is expressed as :

$$\begin{aligned} f(s)&=det\left\{[sI_p, s^2K_2+sK_0+K_1]\begin{bmatrix} D_r(s)\\ N_r(s)\end{bmatrix}\right\}\\ &=\det\left\{[I_p,K_2,K_0,K_1]\begin{bmatrix} sD_r(s)\\ s^sN_r(s)\\ sN_r(s)\\ N_r(s)\end{bmatrix}\right\} \end{aligned} \qquad (10)$$

(e) Observability Index Bounded Dynamics (OBD) Controllers: These are defined by the property that their McMillan degree is equal to pk, where k is the observability index [11] of the controller. Such controllers are expressed by the composite MFD representation as

$$[A_l(s),B_l(s)]=T_ks^k+\ldots+T_0 \qquad (11)$$

$T_k,T_{k-1},\ldots,T_0\in\mathbb{R}^{p\times(p\times m)}$ and $T_k=[I_p,X]$. Note that the above representation is not always coprime, and coprimeness has to be guaranteed first for McMillan degree to be pk; otherwise, the McMillan degree is less than pk. The dynamic determinantal OBD output feedback problem is expressed as

$$f(s)=det\left\{[T_ks^k+\ldots+T_0]\begin{bmatrix} D_r(s)\\ N_r(s)\end{bmatrix}\right\}=det\{(T_ks^k+\ldots+T_0)M(s)\}=$$

$$=det\left\{[T_k,T_{k-1},\ldots,T_0]\begin{bmatrix} s^kM(s)\\ s^{k-1}M(s)\\ \vdots\\ M(s)\end{bmatrix}\right\} \qquad (12)$$

Remark 2.1 The above formulation of the determinantal dynamic assignment problems is based on the assumption that $p \leq m$ and thus output feedback configuration is used. If $p \geq m$, we can similarly formulate the corresponding problems as determinantal dynamic pre-compensation problems and use right coprime MFDs for $C(s)$.

Abstract Determinantal Assignment Problem

All the problems introduced above, belong to the same problem family i.e. the determinantal assignment problem (DAP) [1]. This problem is to solve the following equation with respect to polynomial matrix $H(s)$:

$$det(H(s)M(s)) = f(s) \tag{13}$$

where $f(s)$ is a polynomial of an appropriate degree d. The difficulty for the solution of DAP is mainly due to the multi-linear nature of the problem, as this is described by its determinantal character. We should note, however, that in all cases mentioned previously, all dynamics can be shifted from $H(s)$ to $M(s)$, which, in turn, transforms the problem to a constant DAP. This problem may be described as follows:

Let $M(s) \in \mathbb{R}^{p\times m}[s]$, $r \leq p$, such that rank $M(s) = r$ and let $\mathcal{H}$ be a family of full rank $r \times p$ constant matrices having a certain structure. Solve with respect to $H \in \mathcal{H}$ the equation:

$$f_M(s,H) = det(HM(s)) = f(s) \tag{14}$$

where f(s) is a real polynomial of an appropriate degree d.

Remark 2.2 The degree of the polynomial $f(s)$ depends firstly upon the degree of $M(s)$ and secondly, upon the structure of H. Generically, the degree of $f(s)$ is equal to the degree of $M(s)$.

The determinantal assignment problem has two main aspects. The first has to do with the solvability conditions for the problem and the second, whenever this problem is solvable, to provide methods for constructing these solutions. If $\underline{h}_i(s)^t,\ \underline{m}_i(s),\ i \in \tilde{r}$, we denote the rows of $H(s)$, columns of $M(s)$ respectively, then

$$C_r(H) = \underline{h}_1^t \wedge \ldots \wedge \underline{h}_r^t = \underline{h}^t \wedge \quad \in \mathbb{R}^{l\times\sigma} \tag{15}$$

$$C_r(M(s)) = \underline{m}_1(s) \wedge \ldots \wedge \underline{m}_r(s) = \underline{m}(s)\wedge \in \mathbb{R}^{\sigma}[s],\ \sigma = \binom{p}{r}. \tag{16}$$

and by Binet-Cauchy theorem [10] we have that [1] :

$$f_M(s,H) = C_r(H)C_r(M(s)) = <\underline{\mathrm{h}}(s)\wedge, \underline{\mathrm{m}}(s)\wedge> = \sum_{\omega\in Q_{r,p}} h_\omega(s)m_\omega(s) \tag{17}$$

where $< \cdot, \cdot >$ denotes inner product, $\omega = (i_1, \ldots, i_r) \in Q_{r,p}$ and $h_{\omega(s)},\ m_{\omega(s)}$ are the coordinates of $\underline{\mathrm{h}}(s)\wedge, \underline{\mathrm{m}}(s)\wedge$ respectively. Note that $h_{\omega(s)}$ is the $r \times r$ minor of $H(s)$, which corresponds to the ω set of columns of $H(s)$ and thus $h_{\omega(s)}$, is a multilinear alternating function of the entries $h_{ij}(s)$ of $H(s)$. The multilinear, skew symmetric nature of DAP suggests that the natural framework for its study is that of exterior algebra. The essence of exterior algebra is that it reduces the study of multilinear skew-symmetric functions to the simpler study of linear functions. The study of the zero structure of the multilinear function $f_M(s,H)$ may thus be reduced to a linear subproblem and a standard multilinear algebra problem as it is shown below.

1 **Linear subproblem of DAP:** Set $\underline{m}(s)\wedge = \underline{p}(s) \in \mathbb{R}^{\sigma}$[s]. Determine whether there exists a $\underline{k}(s) \in \mathbb{R}^{\sigma}$[s], $\underline{k}(s) \neq \underline{0}$, such that

$$f_M(s,\underline{k}) = \underline{k}^t \underline{p}(s) = \sum k_i p_i(s) = f(s), i \in \underset{\sim}{\sigma},\ f(s) \in \mathbb{R}[s] \tag{18}$$

2 **Multilinear subproblem of DAP :** Assume that $\mathcal{K}$ is the family of solution vectors $\underline{k}(s)$ of (18). Determine whether there exists $H(s)^t = [\underline{\mathrm{h}}_1(s),...,\underline{\mathrm{h}}_r(s)]$, where $H(s)^t \in \mathbb{R}^{p\times r}[s]$, such that

$$\underline{h}_1(s) \wedge \ldots \wedge \underline{h}_k(s) = \underline{h}(s)\wedge = \underline{k}(s) \in \mathcal{K} \tag{19}$$

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The polynomials $f_M(s,\underline{\mathrm{k}}(s))$ are generated by $\underline{p}(s) = [\underline{p}_1(s),\ldots,p_i(s),\ldots,p_\sigma(s)]^t \in \mathbb{R}^{\sigma}[s]$, or as linear combinations of the set $\mathcal{P} = \{p_i(s) \in \mathbb{R}[s],\ i \in \widetilde{\sigma}\}$ and they will be referred to as *dynamic polynomial combinants.* The study of the spectral properties of such polynomials is the objective of this paper.

3. Basic Definitions and Representation of Dynamic Combinants

Given a set of polynomials $\mathcal{P} = \{p_i(s) \in \mathbb{R}[s],\ i \in \widetilde{m}\}$ and a family of polynomial sets $< \mathcal{K} >= \{\mathcal{K}_d, \forall d \in \mathbb{Z}^+ : \mathcal{K} = (k_1(s) : k_1(s) \in \mathbb{R}[s],\ i \in \widetilde{m}),\ d = max\{deg(k_i(s))\}\}$, we consider

$$f(s,\mathcal{K},\mathcal{P}) = \sum k_i(s)p_i(s),\ where\ k_i(s)\ \in \mathcal{K}_d \tag{20}$$

which are referred to as *d order dynamic-polynomial combinants* of $\mathcal{P}$ and are polynomials with some degree p. Dynamic compensation of linear systems always involves polynomial combinants generated by the corresponding system descriptions. Concepts such as those of multivariable zeros and decoupling zeros are related to the greatest common divisor [12], [4], [13] of certain sets $\mathcal{P}$ associated with the system they define fixed zeros of the associated combinants. The pole, zero assignment and stabilizability properties of linear systems are based on properties of corresponding combinants and thus on the structure of sets $\mathcal{P}$, which generate these combinants. The examination of those properties of a set $\mathcal{P}$ which affect the assignability, stabilizability and "nearly fixed" zero phenomena of the corresponding combinants $f(s,\mathcal{K},\mathcal{P})$ is the main drive for the research here. This paper develops the fundamentals of the theory of polynomial combinants. The representation problem of given order and degree dynamic polynomial combinants is considered here, which involves a parameterization of all sets $< \mathcal{K} >= \{\mathcal{K}_d, \forall d \in \mathbb{Z}^+ : \mathcal{K} = (k_1(s) : k_1(s) \in \mathbb{R}[s],\ i \in \widetilde{m}),\ d = max\{deg(k_i(s))\}\}$ which lead to a polynomial combinant of a given degree p.

Given the sets $\mathcal{P}$ with m elements and maximal degree n and the set $\mathcal{K}$ of m elements and maximal degree d of $\mathbb{R}[s]$, the generated combinant is denoted by

$$f_d(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s) = \phi(s) \tag{21}$$

This is a polynomial generated by the set $\mathcal{P}$ and characterised by the order d of $\mathcal{K}$ and the resulted degree $\partial[f_d(s,\mathcal{K},\mathcal{P})]$ of the combinant. We always assume that the maximal degree polynomial in $\mathcal{K}$, $k_1(s) \neq 0$ and such sets $\mathcal{K}$ are referred to as *proper*. If we explicitly define $\mathcal{P}$ as

$$\mathcal{P} = \{p_i(s) \in \mathbb{R}[s], i \in \widetilde{m},\ \ n = deg\{p_1(s)\} \geq deg\{p_i(s)\},\ i = 2,\dots,m,$$
$$q = max\{deg\{p_i(s)\}; i = 2,\dots,m\} \tag{22}$$

$$p_1(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0,\ \ p_i(s) = b_{i,q}s^q + \dots + b_{i,1}s + b_{1,0},$$
$$i = 2,\dots,m \tag{23}$$

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ p_2(s) \\ \vdots \\ p_m(s) \end{bmatrix} = [\underline{p}_0, \underline{p}_1, \dots, \underline{p}_n] \begin{bmatrix} 1 \\ s \\ \vdots \\ s^n \end{bmatrix} = P\underline{e}_n(s) \tag{24}$$

Then the set $\mathcal{P}$ will be referred to as an *(m;n(q))-ordered set* of $\mathbb{R}[s]$. Consider now a set of m polynomials of maximal degree d, $\mathcal{K} = \{k_i(s) \in \mathbb{R}[s],\ i \in \tilde{m},\ \ deg\{k_i(s)\} \leq d\}$, referred to in short as an $(m;d)$ set of $\mathbb{R}[s]$. The resulting polynomial combinant is

$$f_d(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m}, k_i(s)p_i(s) = \underline{k}^t(s)\underline{p}(s) \tag{25}$$

where

$$\underline{k}^t(s) = [k_1(s), k_2(s), \dots, k_m(s)]^t = \underline{k}_0^t + s\,\underline{k_1}^t + \dots + s^d\,\underline{k}_d^t \tag{26}$$

is defined as a *d-order polynomial combinant* of , or in short as $d-\ \ \mathbb{R}[s]\ \ -$ *combinant of* $\mathcal{P}$. The matrix $P\ \in \mathbb{R}^{m\times(n+1)}$ generates the representative $\underline{p}(s)\ \in \mathbb{R}^m[s]$ of and it is referred to as the *basis matrix* of. Clearly $f_d(s,\mathcal{K},) \in \mathbb{R}[s]$and some interesting problems related to its spectrum stem from the fact that the set$\mathcal{K}$may take arbitrary form in terms of its degree and selection of free parameters. The combinant $f_d(s,\mathcal{K},)$ as a polynomial of $\mathbb{R}[s]$ has degree $\partial[f_d(s,\mathcal{K},)]$ that clearly satisfies the inequality

$$-\infty \leq \partial[f_d(s,\mathcal{K},\mathcal{P})] \leq n + q \tag{27}$$

In the following we consider two different representations of $f_d(s,\mathcal{K},\mathcal{P})$ and the parametrisation of all combinants of different order and degree and show how these lead to standard linear algebra problem formulations. The order and degree parameterisations introduce some interesting links with the theory of generalised resultants.

Fixed Order Representations of Dynamic Combinants: Generalised Resultant Representations

For the general $(m;d)$ set $\mathcal{K}$ with a representative vector

$$\underline{k}(s)^t = \underline{k}_0^t + s\underline{k}_1^t + \ldots + s^d\underline{k}_d^t = [k_1(s), k_2(s), \ldots, k_m(s)] \tag{28}$$

where $k_i(s) = k_{i,0} + k_{i,1}s + \ldots + k_{i,d}s^d$, then $f_d(s, \mathcal{K}, \mathcal{P})$ may be expressed as

$$f_d(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^{m}[k_{i,d}, \ldots, k_{i,1}, k_{1,0}] \begin{bmatrix} s^d p_i(s) \\ \vdots \\ sp_i(s) \\ p_i(s) \end{bmatrix} = [\underline{k}_{1,d}^t, \ldots, \underline{k}_{m,d}^t] \begin{bmatrix} \underline{p}_{i,d}(s) \\ \vdots \\ \underline{p}_{m,d}(s) \end{bmatrix} \tag{29}$$

The above leads to the following representation of dynamic combinants:

Proposition 3.1 *Every dynamic combinant $f_d(s, \mathcal{K}, \mathcal{P})$ defined by an (m;d) set $\mathcal{K}$ is equivalent to a constant polynomial combinant defined by the (m(d+1);0) set $\mathcal{K}^0$ and generated by the (m(d+1);(n+d)(q+d)) the d-th power of the (m;n(q)) set $\mathcal{P}$, defined by*

$$\mathcal{P}^d = \{s^d p_1(s), \ldots, sp_1(s), p_1; \ldots; s^d p_m(s), \ldots, sp_m(s), p_m(s)\} \tag{30}$$

The above leads to the following representation of dynamic combinants as equivalent constant combinants. If $\mu = n+d$, $\underline{\widetilde{e}}_\mu(s)^t = [s^\mu, s^{\mu-1}, \ldots, s, 1]$, then $\partial\, [p_{1,d}(s)] = n+d$, $\partial\, [p_{i,d}(s)] \leq q + d$ for all *i=2,3,...,m* and

$$\underline{p}_{1,d}(s) = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_1 & a_0 \end{bmatrix} \underline{\widetilde{e}}_\mu(s) \tag{31}$$

or

$$\underline{p}_{1,d}(s) = \mathcal{S}_{n,d}(p_1)\underline{\widetilde{e}}_\mu(s), \quad \mathcal{S}_{n,d}(p_1) \in \mathbb{R}^{(d+1)\times(\mu+1)} \tag{32}$$

and for $i = 2, 3, \ldots, m$

$$\underline{p}_{i,d}(s) = \begin{bmatrix} 0 & \ldots & 0 & b_{i,q} & \ldots & b_{i,1} & b_{i,0} & 0 & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & 0 & b_{i,q} & \ldots & b_{i,1} & b_{i,0} & 0 & \ldots & 0 \\ \vdots & & 0 & \vdots & \ddots & \ddots & & & & & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & b_{i,q} & \ldots & \ldots & b_{i,1} & b_{i,0} \end{bmatrix} \underline{\widetilde{e}}_\mu(s) \tag{33}$$

(3.9b)

$$\underline{p}_{i,d}(s) = \mathcal{S}_{n,d}(p_i)\underline{\widetilde{e}}_\mu(s), \quad \mathcal{S}_{n,d}(p_i) \in \mathbb{R}^{(d+1)\times(\mu+1)} i = 2, 3, \ldots, m. \tag{34}$$

The set $\mathcal{P}^d$ has then a matrix representation as shown below

$$\underline{p}_d(s) = \begin{bmatrix} \underline{p}_{1,d}(s) \\ \underline{p}_{2,d}(s) \\ \vdots \\ \underline{p}_{m,d}(s) \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{n,d}(p_1) \\ \mathcal{S}_{n,d}(p_2) \\ \vdots \\ \mathcal{S}_{n,d}(p_m) \end{bmatrix} \underline{\widetilde{e}}_\mu(s) = \mathcal{S}_{\mathcal{P},d}\underline{\widetilde{e}}_\mu(s) \tag{35}$$

where $\mathcal{S}_{\mathcal{P},d} \in \mathbb{R}^{m(d+1)\times(\mu+1)}$ and is referred to as the *d-th Resultant representation of* the set $\mathcal{P}$. Clearly,$\mathcal{S}_{\mathcal{P},d}$ is the basis matrix $\mathcal{P}^d$ set.

Fixed Order Representations of Dynamic Combinants: Toeplitz Representation

An alternative expression for the dynamic combinant is obtained using the basis matrix description of the set $\mathcal{P}$. Thus , let us assume that

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ \vdots \\ p_m(s) \end{bmatrix} = P\underline{\widetilde{e}}_n(s), P = [\underline{p}_n, \dots, \underline{p}_1, \underline{p}_0] \in \mathbb{R}^{m\times(n+1)} \tag{36}$$

where P is the basis matrix of $\mathcal{P}$. Then,

$$f_d(s, \mathcal{K}, \mathcal{P}) = (\underline{k}_0^t + s\underline{k}_1^t + \dots + s^d\underline{k}_d^t)\underline{P\widetilde{e}}_n(s) =$$

$$= \underline{k}_0^t P\underline{\widetilde{e}}_n(s) + s\underline{k}_1^t P\underline{\widetilde{e}}_n(s) + \dots + s^d\underline{k}_d^t P\underline{\widetilde{e}}_n(s) =$$

$$= \underline{k}_d^t[P, 0, \dots, 0]\underline{\widetilde{e}}_\mu(s) + \underline{k}_{d-1}^t[0, P, 0, \dots, 0]\underline{\widetilde{e}}_\mu(s) + \dots + \underline{k}_0^t[0, \dots, 0, P]\underline{\widetilde{e}}_\mu(s) =$$

$$= [\underline{k}_d^t, \underline{k}_{d-1}^t, \dots, \underline{k}_0^t] \begin{bmatrix} \underline{p}_0, & \underline{p}_1, & \underline{p}_2, & \dots, & \underline{p}_n, & \underline{0}, & \cdots & \cdots & \underline{0} \\ \underline{0} & \underline{p}_0, & \underline{p}_1, & \underline{p}_2, & \dots, & \underline{p}_n, & \underline{0}, & \cdots & \underline{0} \\ \vdots & & \ddots & \ddots & & & \ddots & \underline{0} & \\ \underline{0}, & \cdots & \cdots & \underline{0} & \underline{p}_0, & \underline{p}_1, & \underline{p}_2, & \dots, & \underline{p}_n, \end{bmatrix} \underline{e}_\mu(s) \tag{37}$$

or equivalently

$$f_d(s, \mathcal{K}, \mathcal{P}) = \underline{k}_{d+1,m}^t \mathcal{Q}_{\mathcal{P},d}\underline{\widetilde{e}}_\mu(s), \ \mathcal{Q}_{\mathcal{P},d} \in \mathbb{R}^{m\times(d+1)\times(\mu+1)}. \tag{38}$$

The matrix $\mathcal{Q}_{\mathcal{P},d}$ generating the dynamic combinant as a constant combinant is referred to as the *d-th Toeplitz Toeplitz Representation* of the set $\mathcal{P}$. From the construction of the matrices $\mathcal{S}_{\mathcal{P},d}, \ \mathcal{Q}_{\mathcal{P},d}$ we have

Remark 3.1 The matrices $\mathcal{Q}_{\mathcal{P},d}$ and $\mathcal{S}_{\mathcal{P},d}$ associated with $\mathcal{P}$ have the same dimensions and are permutation equivalent, i.e. $\exists$ permutation matrices $P_L, \ P_R$ such that

$$\mathcal{Q}_{\mathcal{P},d} = P_L \mathcal{S}_{\mathcal{P},d} P_R. \tag{39}$$

The above implies that establishing the rank properties of $\mathcal{S}_{\mathcal{P},d}$ implies the same properties for $\mathcal{Q}_{\mathcal{P},d}$ and vice versa. Thus either of the two representations may be used. In the following we shall concentrate on the Generalised Resultant representation and the general properties may be referred back to the Toeplitz Representations as well.

4. Fixed degree and order Parametrisation of $(m;d)$ sets $\mathcal{K}$ and Combinants

The general unstructured representation of dynamic combinants considered before may lead to combinants of varying degree. An alternative characterisation based on the fixed degree of $f_d(s,\mathcal{K},\mathcal{P})$, but with varying order $\mathcal{K}$ provides an alternative parametrisation of the $\mathcal{K}$ sets. We always assume proper sets $\mathcal{K}$, i.e. $k_1(s) \neq 0$. The fixed degree parametrisation of combinants is summarised by the following result

Theorem 4.1 *Given the $(m;q(n))$ set $\mathcal{P}$ and a general proper $(m;d)$ set $\mathcal{K}$, then the following properties hold true*

(i) *For all proper $(m;d)$ sets $\mathcal{K}$*

$$n \leq \partial[f_d(s,\mathcal{K},\mathcal{P})] \leq n+d \tag{40}$$

(ii) *If $p \in \mathbb{N}_{>0},\ p \geq n$ then the family $\{\mathcal{K}_p\}$ for which $\partial[f_d(s,\mathcal{K},\mathcal{P})] = p$, satisfies the conditions*

$$\partial[k_1(s)] \leq p-n,\ \partial[k_i(s)] \leq p-q, \quad i=2,\ldots,m \tag{41}$$

where at least one of the first two conditions holds as an equality.

(iii) *The fixed degree p family $\{\mathcal{K}_p\}$ contains $n-q+1$ subfamilies parameterised by a fixed order d. The possible values for the order are:*

$$d_1 = p-q > \ d_2 = p-q-1 > \ \ldots > d_{n-q+1} = p-n \tag{42}$$

and the corresponding subfamilies are

$$\{\mathcal{K}_p^{d_1}\} = \{k_i(s) : \partial[k_1(s)] \leq p-n,\ \partial[k_2(s)] = d_1 = p-q,\ \partial[k_i(s)] \leq d_1, i=3,\ldots,m\}$$

$$\{\mathcal{K}_p^{d_2}\} = \{k_i(s) : \partial[k_1(s)] = p-n,\ \partial[k_2(s)] = d_2 = p-q-1,\ \partial[k_i(s)] \leq d_2, \\ i=3,\ldots,m\}$$

$$\vdots$$

$$\{\mathcal{K}_p^{d_{n-q+1}}\} = \{k_i(s) : \partial[k_1(s)] = \partial[k_2(s)] = d_{n-q+1} = p-n,\ \partial[k_i(s)] \leq p-n, \\ i=3,\ldots,m\} \tag{43}$$

Proof: Parts **(i)** and **(ii)** are rather straight forward and follow from the definition of the combinant. The parameterisation implied by part (iii) follows by the construction of the combinant as indicated by the following table

$$\begin{array}{llll} p_1(s): & \partial[p_1(s)] = n, & k_1(s) & \partial[k_1(s)] \leq p-n \\ p_2(s): & \partial[p_2(s)] = q, & k_2(s) & \partial[k_2(s)] \leq p-q \\ \vdots & \vdots & \vdots & \vdots \\ p_m(s): & \partial[p_m(s)] \leq q, & k_m(s) & \partial[k_m(s)] \leq p-q \end{array} \tag{44}$$

where amongst the first two relationships at least one is an equality. The above table follows from the need to guarantee degree p to the $f_d(s,\mathcal{K},\mathcal{P})$ combinant. The condition from the above implies:

— If $\partial[k_2(s)] = p-q > p-n$ then we have the maximal degree $d_1 = p-q$ subfamily of $\{\mathcal{K}_p^{d_1}\}$ with degrees

$$\partial[k_1(s)] \leq p-n,\ \partial[k_2(s)] = d_1 = p-q,\ \partial[k_i(s)] \leq d_1, i = 3,\ldots,m.$$

— If $\partial[k_2(s)] = p-q-1 > p-n$ then we have the next value of degree $d_2 = p-q-1$ and the $\{\mathcal{K}_p^{d_2}\}$ subfamily with degrees

$$\partial[k_1(s)] = p-n,\ \partial[k_2(s)] = d_2 = p-q-1,\ \partial[k_i(s)] \leq d_2, i = 3,\ldots,m$$

— the process finishes when $\partial[k_1(s)] = p-n = \partial[k_2(s)] = d_{n-q+1}$, when

$$\partial[k_1(s)] = \partial[k_2(s)] = d_{n-q+1} = p-n,\ \partial[k_i(s)] \leq p-n, i = 3,\ldots,m.$$

Clearly this is the last family in $\{\mathcal{K}_p\}$ for which the degree has minimal value $d_{n-q+1} = p-n$.

Remark 4.1 For the $(m, n(q))$ set $\mathcal{P}$ the degree of the proper combinants (corresponding to proper sets $\mathcal{K}$) takes values $p \geq n$.

The entire family of proper combinants of $\mathcal{P}$ may thus be parameterised by degree and orders and the entire set may be characterised by the sets of $\mathcal{K}$ vectors which will be denoted as $<\mathcal{K}>$ Clearly,

$$<\mathcal{K}> = \{\mathcal{K}_n\}\bigcup\{\mathcal{K}_{n+1}\}\bigcup\ldots\bigcup\ldots\{\mathcal{K}_{n+q-1}\} \tag{45}$$

whereas each subset $\{\mathcal{K}_p\}$ has the structure defined by the previous result.

Corollary 4.1 *Given an (m;q) set $\mathcal{P}$ and a general (m;q) set $\mathcal{K}$, then:*

(i) *The minimal degree family p=n, $\{\mathcal{K}_n\}$ is expressed as*

$$\begin{aligned}\{\mathcal{K}_n\} = \quad & \{\{\mathcal{K}_n^0\} : <\mathcal{K}_n^0> = (0,\ldots,0);\\ & \{\mathcal{K}_n^1\} : <\mathcal{K}_n^1> = (0,1,\ldots,1);\\ & \vdots \\ & \{\mathcal{K}_n^{n-q}\} : <\mathcal{K}_n^{n-q}> = (0, n-q,\ldots,n-q)\}\end{aligned} \tag{46}$$

(ii) *The general degree family $p = n + d$, $\{\mathcal{K}_p\}$ is then expressed as*

$$\begin{aligned}\{\mathcal{K}_p\} = \quad & \{\{\mathcal{K}_p^d\} : <\mathcal{K}_p^d> = (0,\ldots,0) + (d,d,\ldots,d)\\ & \{\mathcal{K}_p^{d+1}\} : <\mathcal{K}_p^{d+1}> = (0,1,\ldots,1) + (d,d,\ldots,d)\\ & \vdots \\ & \{\mathcal{K}_p^{d+n-q}\} : <\mathcal{K}_p^{d+n-q}> = (0,n-q,\ldots,n-q) + (d,d,\ldots,d)\}.\end{aligned} \tag{47}$$

(iii) *For the general degree p family, $p \geq n$, the values of possible orders, in decreasing order, are*

$$d_1 = p - q > d_2 = p - q - 1 > \cdots > d_{n-q} = p - n + 1 > d_{n-q+1} = p - n \quad (48)$$

and are given as $d_i = p - q + 1 - i$, $i = 1, 2, \ldots, n - q + 1$, or in increasing order

$$\widetilde{d}_1 = p - n < \widetilde{d}_2 = p - n + 1 < \cdots < \widetilde{d}_{n-q} = p - q - 1 < \widetilde{d}_{n-q+1} = p - q$$

and are given recursively as $\widetilde{d}_i = p - n - 1 + i$, $i = 1, 2, \ldots, n - q + 1$.

The proof of the above result follows by induction. Amongst all $(m; d)$ sets $\mathcal{K}$, the set which is defined by

$$\{\mathcal{K}_{n+q-1}^{n-1}\} = \{k_1(s) : \partial[k_1(s)] = q - 1, k_i(s) : \partial[k_i(s)] = n - 1, \; i = 2, \ldots, m\} \quad (49)$$

plays a particular role in our study and it is referred to as the *Sylvester set* of $\mathcal{P}$. The general p degree family may be expressed as

$$\{\mathcal{K}_p\} = \{\mathcal{K}_p^{\widetilde{d}_i}, \; \widetilde{d}_i = p - n - 1 + i, \; i = 1, 2, \ldots, n - q + 1\} =$$

$$= \{\mathcal{K}_p^{p-n}; \mathcal{K}_p^{p-n+1}; \ldots; \mathcal{K}_p^{p-q+1}; \mathcal{K}_p^{p-q}\}. \quad (50)$$

The element $\mathcal{K}_p^{p-q}$ that corresponds to the highest order $p - q$ will be referred to as the *generator* of the family and its degrees are

$$\{\mathcal{K}_p^{p-q}\} = (p - n, p - q, \ldots, p - q). \quad (51)$$

Similarly, the element $\mathcal{K}_p^{p-n}$ that corresponds to the lowest order $p - n$ is referred to as a *co-generator* of the family and its degrees are

$$\{\mathcal{K}_p^{p-n}\} = (p - n, p - n, \ldots, p - n). \quad (52)$$

The above suggests that the entire family of vector sets $\mathcal{K}$ may be expressed in a "direct sum" form as

$$< \mathcal{K} >= \{\mathcal{K}_n\} \bigcup \{\mathcal{K}_{n+1}\} \bigcup \ldots \{\mathcal{K}_{n+q-1}\} \bigcup \ldots$$

$$\{\mathcal{K}_p\} = \{\mathcal{K}_p^{p-n}\} \bigcup \{\mathcal{K}_p^{p-n+1}\} \bigcup \ldots \bigcup \{\mathcal{K}_p^{p-q}\} \quad (53)$$

for all $p \geq n$. This parametrisation of $\mathcal{K}$ sets leads to a corresponding parametrisation of generalised resultants that is considered next.

5. Generalised Resultants based on Fixed degree and order Parametrisations

The parameterisation of the sets $\mathcal{K}$ based on degree and order induces in a natural parameterisation of the corresponding Generalized Resultants. This is now considered here and this provides the basis for the study of the properties of the family of

Generalised Resultants. We consider the general (m;d) set $\mathcal{K}$ that leads to a combinant of degree p which is defined by:

$$\{\mathcal{K}_p^d\} = \{k_1(s) : \partial[k_1(s)] = p - n = \widetilde{d}, k_2(s) : \partial[k_2(s)] = d,$$
$$\widetilde{d} \leq d \leq d^* = p - q, k_i(s) : \partial[k_i(s)] \leq d, i = 3, \dots, m\} \tag{54}$$

The above set $\{\mathcal{K}_p^d\}$, $p \geq n$ and with d taking values as above, represents the general set generating dynamic combinants a given degree d and order p. Note that in the above expression we consider all $k_i(s)$, $i = 3, \dots, m$ as polynomials with reference degree d ($\partial[k_i(s)] \leq d$) and thus we can express them as

$$k_1(s) = k_{1,\widetilde{d}} s^{\widetilde{d}} + \cdots + k_{1,1}s + k_{1,0} = [k_{1,\widetilde{d}}, \dots, k_{1,1}, k_{1,0}]\underline{\widetilde{e}}_{\widetilde{d}}(s) = \underline{k}^t_{1,\widetilde{d}}\underline{\widetilde{e}}_{\widetilde{d}}(s)$$
$$k_i(s) = k_{i,d}s^d + \cdots + k_{i,1}s + k_{i,0} = [k_{i,d}, \dots, k_{i,1}, k_{i,0}]\underline{\widetilde{e}}_d(s) = \underline{k}_{i,\widetilde{d}}\underline{\widetilde{e}}_{\widetilde{d}}(s). \tag{55}$$
$$\underline{\widetilde{e}}_\mu(s)^t = [s^\mu, s^{\mu-1}, \dots, s, 1].$$

Using this representation for $\{\mathcal{K}_p^d\}$ the corresponding combinant becomes

$$f_d(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^m k_i(s)p_i(s) =$$
$$= \underline{k}^t_{1,\widetilde{d}} \begin{bmatrix} s^{\widetilde{d}}p_1(s) \\ \vdots \\ sp_1(s) \\ p_1(s) \end{bmatrix} + \sum_{i=2}^m \underline{k}^t_{i,\widetilde{d}} \begin{bmatrix} s^d p_i(s) \\ \vdots \\ sp_i(s) \\ p_i(s) \end{bmatrix} \tag{56}$$

Proposition 5.1 *The dynamic combinant $f_d(s, \mathcal{K}_d^d, \mathcal{P})$ is equivalent to a constant combinant of degree p that is generated by the set:*

$$\mathcal{P}_p^d = \{s^{\widetilde{d}}p_1(s), \dots, sp_1(s), p_1(s); s^d p_2(s), \dots, sp_2(s), p_2(s); \dots;$$
$$s^d p_m(s), \dots, sp_m(s), p_m(s)\} \tag{57}$$

where $\widetilde{d} = p - n$, $\widetilde{d} \leq d \leq p - q = d^*$.

The set $\mathcal{P}_p^d$ is the (p,d)-power of $\mathcal{P}$ and has degree p. The polynomial vector representative is

$$\underline{p}_{p,d}(s) = \begin{bmatrix} \underline{p}_{1,\widetilde{d}}(s) \\ \underline{p}_{2,d}(s) \\ \vdots \\ \underline{p}_{m,d}(s) \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{n,\widetilde{d}}(p_1) \\ \mathcal{S}_{n,d}(p_2) \\ \vdots \\ \mathcal{S}_{n,d}(p_m) \end{bmatrix} \underline{\widetilde{e}}_p(s) = \mathcal{S}_{p,d}\underline{\widetilde{e}}_p(s) \tag{58}$$

where the structure of the Toeplitz type blocks above $\mathcal{S}_{n,\widetilde{d}}(p_1)$, $\mathcal{S}_{q,d}(p_i)$ $i = 2, \dots, m$ defining the corresponding Generalised Resultants is given below

Proposition 5.2 *The Generalised Resultants corresponding to the parameterized set* $\{\mathcal{K}_p^d\}$ *are defined by:*

(i) *Given that* $\underline{p}_{1,\widetilde{d}}(s)$ *has degree* $\widetilde{d}+n = p-n+n = p$ *then*

$$\mathcal{S}_{n,\widetilde{d}}(p_1) = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_1 & a_0 \end{bmatrix} \in \mathbb{R}^{(\widetilde{d}+1)\times(p+1)} \tag{59}$$

(ii) *Given that* $\underline{p}_{1,d}(s)$ *has degree* $d+q$ *which satisfies the inequality* $p-(n-q) \leq d+q \leq p$ *and thus* $d+q+1 \leq p+1$. *The structure of* $\mathcal{S}_{q,d}(p_i)$ *is defined for all* $i = 2, \ldots, m$ *and* $\forall d : p-n \leq d \leq p-q$ *by*

$$\mathcal{S}_{q,d}(p_i) = \begin{bmatrix} 0 & \ldots & 0 & b_{i,q} & \ldots & b_{i,1} & b_{i,0} & 0 & \ldots & \ldots & 0 \\ 0 & \ldots & 0 & 0 & b_{i,q} & \ldots & b_{i,1} & b_{i,0} & 0 & \ldots & 0 \\ \vdots & & 0 & \vdots & \ddots & \ddots & & & & & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 & b_{i,q} & \ldots & \ldots & b_{i,1} & b_{i,0} \end{bmatrix} \in \mathbb{R}^{(d+1)\times(p+1)} \tag{60}$$

Clearly in the boundary case $d = p-q$, there is no zero block and when $d = p-n$, then the zero block takes its maximal dimension $n-q$. The matrix $\mathcal{S}_{p,d} \in \mathbb{R}^{\sigma\times(p+1)}, \sigma = p-n-d+m(d+1)$ will be called the (p,d)-generalised resultant of the set $\mathcal{P}$ where $p-n \leq d \leq p-q$. Clearly the $\mathcal{S}_{p,d}$ is the basis matrix of the (p,d) power of $\mathcal{P}, \mathcal{P}_p^d$.

Remark 5.1 For the given $(m; n(q))$ set $\mathcal{P}$ we can parameterise all dynamic combinants in terms of the degree p and the corresponding order d as

(a) $p = n$: then $0 \leq d \leq n-q$

(b) $p = n+1$: then $1 \leq d \leq n-q+1$

(c) $p > n+1$: then $p-n \leq d \leq p-q$

and their properties are defined by the properties of corresponding (p,d)-generalised resultants $\mathcal{S}_{p,d}(\mathcal{P})$.

In the following we will investigate the properties of all dynamic combinants by considering the corresponding family

$$S(\mathcal{P}) = \{\mathcal{S}_{p,d}\ \forall\, p \geq n \text{ and } \forall\ d\ : p-n \leq d \leq p-q\} \tag{61}$$

which will be referred to as the *family of generalised resultants* of the set $\mathcal{P}$. Amongst the elements of $S(\mathcal{P})$ we distinguish a special element that corresponds to $p = n+q-1$, $d = n-1$ and thus $\partial[k_1(s)] = p-n = q-1$. This generalised resultant $\mathcal{S}_{n+q-1,n-1}(\mathcal{P})$

is denoted in short as $\widetilde{\mathcal{S}}_{\mathcal{P}}$ and it is referred to as the *Sylvester Resultant* of the set $\mathcal{P}$ and has the following form

$$\widetilde{\mathcal{S}}_{\mathcal{P}} = \begin{bmatrix} \mathcal{S}_{n,q-1}(p_1) \\ \mathcal{S}_{q,n-1}(p_2) \\ \vdots \\ \mathcal{S}_{q,n-1}(p_m) \end{bmatrix} \in \mathbb{R}^{\tau\times(n+q)}, \tau = [q+(m-1)n] \tag{62}$$

where $\mathcal{S}_{n,q-1}(p_1) \in \mathbb{R}^{q\times(n+q)}, \mathcal{S}_{q,n-1}(p_i) \in \mathbb{R}^{q\times(n+q)}, j = 2, \ldots, m$ and $\tau = [q+(m-1)n]$. The characteristic of this is that none of the blocks $\mathcal{S}_{n,q-1}(p_1), \mathcal{S}_{q,n-1}(p_i)$ have zero columns blocks and that the rank of $\widetilde{\mathcal{S}}_{\mathcal{P}}$ is clearly related to algebraic properties of $\mathcal{P}$, as it will be seen subsequently.

6. Spectrum Assignment of Dynamic Combinants and the Sylvester Resultant

We have described the link of dynamic combinants to Generalised Resultants, the structure of the family $S(\mathcal{P})$ of all generalised resultants, and we now consider the problem of arbitrary assignment of the spectrum of dynamic combinants for some appropriate order and degree. This is part of the more general problem dealing with the parameterisation of all possible degree and order combinants for which assignment may be achieved. Given that problems of spectrum assignment of dynamic combinants are always reduced to equivalent problems of constant combinants, we start our study by reviewing the basic results from the theory of constant combinants

Spectral Properties and Assignability of Constant Polynomial Combinants

Consider the $(m; n(q))$ set $\mathcal{P}$ as described previously, with a polynomial vector representative

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ p_2(s) \\ \vdots \\ p_m(s) \end{bmatrix} = [\underline{p}_n, \underline{p}_{n-1}, \ldots, \underline{p}_1, \underline{p}_0] \begin{bmatrix} s^n \\ s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix} = \widetilde{P}\underline{\widetilde{e}}_n(s) \tag{63}$$

where $\widetilde{P} \in \mathbb{R}^{m\times(n+1)}$ is the basis matrix of $\mathcal{P}$ with respect to the vector $\underline{\widetilde{e}}_n(s)$. The constant polynomial combinant $f_d(s, \mathcal{K}, \mathcal{P})$ is defined by

$$f_0(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^{m} k_i p_i(s) = [k_1, k_2, \ldots, k_m]\widetilde{P}\underline{\widetilde{e}}_n(s) \tag{64}$$

where $\mathcal{K} = \{k_i \in \mathbb{R},\ i \in \widetilde{m}\}$ is an arbitrary set. Clearly this is a polynomial of maximal degree n and if $k_1 \neq 0$ then it has degree n. We may thus write

$$f_0(s, \mathcal{K}, \mathcal{P}) = \underline{k}^t\widetilde{P}\underline{\widetilde{e}}_n(s) = \phi(s) = [\phi_n, \ldots, \phi_1, \phi_0]\underline{\widetilde{e}}_n(s) \tag{65}$$

The above suggests that study of properties of $f_0(s, \mathcal{K}, \mathcal{P})$ is equivalent to a study of properties of degree n polynomials with real coefficients defined by a vector $\underline{\phi}\ of\ \mathbb{R}^{n+1}$ which are defined by:

$$[k_1, k_2, \ldots, k_m][\underline{p}_n, \underline{p}_{n-1}, \ldots, \underline{p}_1, \underline{p}_0] = [\phi_n, \ldots, \phi_1, \phi_0] \tag{66}$$

or

$$\underline{k}^t \widetilde{P} = \underline{\phi}^t, \;\; \widetilde{P} \in \mathbb{R}^{m\times(n+1)} \tag{67}$$

Lemma 6.1 *For the set $\mathcal{P}$ with a basis matrix $\widetilde{P} \in \mathbb{R}^{m\times(n+1)}$ the constant combinant $f_0(s,\mathcal{K},\mathcal{P})$ is arbitrarily assignable if and only if*

$$rank\{\widetilde{P}\} = n+1 \tag{68}$$

Clearly, if $f_0(s,\mathcal{K},\mathcal{P})$ is assignable a necessary condition is that $m > n$. The study of constant combinants has been given in [2], where also some classification of the sets has been given according to their spectra assignability properties.

Definition 6.1 If for a set $\mathcal{P}$ there exists $\underline{k}$ such that $f_0(s,\mathcal{K},\mathcal{P}) = \phi_0 \in \mathbb{R}, \neq 0$, then the n-th degree combinant has all its roots at $s=\infty$ and $\mathcal{P}$ may be referred to as $\infty -$ *assignable* set. In the case where there is no $\underline{k}$ such that $f_0(s,\mathcal{K},\mathcal{P}) = \phi_0 \in \mathbb{R}$ then $f_0(s,\mathcal{K},\mathcal{P})$ has effective degree at least one and the set $\mathcal{P}$ will be called strongly non - assignable. For strongly non assignable sets, for all $\underline{k}$ at least one of the roots of $f_0(s,\mathcal{K},\mathcal{P})$ is finite.

Proposition 6.1 *Consider the set $\mathcal{P}$ with a basis matrix $\widetilde{P} = [\underline{p}_n, \underline{p}_{n-1}, \ldots, \underline{p}_1, \underline{p}_0] \in \mathbb{R}^{m\times(n+1)}$. The following properties hold true:*

(i) *The set $\mathcal{P}$ is $\infty -$ assignable if and only if*

$$\mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, \ldots, \underline{p}_1]\} \neq \{0\} \tag{69}$$

(ii) *The set $\mathcal{P}$ is strongly nonassignable if and only if*

$$\mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, \ldots, \underline{p}_1]\} = \{0\} \tag{70}$$

Furthermore, $f_0(s,\mathcal{K},\mathcal{P})$ has at least ν finite roots for all $\mathcal{K}$ if and only if

$$\mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, \ldots, \underline{p}_\nu]\} = \{0\} \tag{71}$$

If we denote by $\widetilde{P}^{(\nu)} = [\underline{p}_n, \underline{p}_{n-1}, \ldots, \underline{p}_\nu]$ the submatrix of $\widetilde{P}$, then if $\mathcal{N}_\ell\{\widetilde{P}^\nu\} = \{0\}$ and $\mathcal{N}_r\{\widetilde{P}^{(\nu+1)}\} \neq \{0\}$ then ν will be called the *index* of $\mathcal{P}$ and denotes the least number of finite zeros of $f_0(s,\mathcal{K},\mathcal{P})$ for all $\mathcal{K}$. The existence of finite roots for all $\underline{k}$ when $\nu \geq 1$ raises the question of whether there exists a region Ω of $\mathbb{C}$ that contains the ν finite roots. Such a problem has been investigated [14]. We consider next the spectrum assignment case for the dynamic case.

Spectral Assignability of Dynamic Combinants

We start our investigation of assignability by using the previous Lemma that establishes assignability for the case of constant combinants. This result together with the reduction of dynamic combinants to equivalent constant formulation leads to the following result:

Proposition 6.2 *Given the $(m;n(q))$ set $\mathcal{P}$, then the combinant $f_d(s,\mathcal{K},\mathcal{P})$ generated by the $(m;d)$ set $\mathcal{K}$ is assignable, if and only if the $m(d+1)\times(d+n+1)$ Toeplitz representation $\mathcal{Q}_{\mathcal{P},d}$ defined by (37) satisfies the condition*

$$rank\{\mathcal{Q}_{\mathcal{P},d}\} = n+d+1 \tag{72}$$

The link of coprimeness of $\mathcal{P}$ to the assignability is considered next

Proposition 6.3 *If the set $\mathcal{P}$ is not coprime and $\phi(s)$ is its GCD, then for all d and all $\mathcal{K}$ sets the combinant $f_d(s,\mathcal{K},\mathcal{P})$ is not completely assignable and*

$$rank\{\mathcal{Q}_{\mathcal{P},d}\} < n+d+1 \tag{73}$$

Proof:

If $\mathcal{P}$ is not coprime and $\phi(s)$ is its GCD, then if $\mathcal{P}=\{p_i(s), i\in\widetilde{m}\}$ we may write $p_i(s)=\phi(s)\widetilde{p}_i(s)$, $i\in\widetilde{m}$ and thus $f_d(s,\mathcal{K},\mathcal{P})=\sum_{i=1}^m k_i(s)p_i(s)=\phi(s)\{\sum_{i=1}^m k_i(s)\widetilde{p}_i(s)\}$. Clearly $f_d(s,\mathcal{K},\mathcal{P})$ has all zeros of $\phi(s)$ as fixed zeros and thus for all $\mathcal{K}$ we do not have assignability. For such sets $\mathcal{P}$ ($\phi(s)$ nontrivial gcd), we have that

$$rank\{\mathcal{Q}_{\mathcal{P},d}\} \leq min(m(d+1), n+d+1) \tag{74}$$

and thus $rank\{\mathcal{Q}_{\mathcal{P},d}\}\leq n+d+1$. If equality holds true, then by the previous Lemma we have assignability of $f_d(s,\mathcal{K},\mathcal{P})$ which contradicts the assumption made above

Corollary 6.1 *Necessary condition for complete assignability of $f_d(s,\mathcal{K},\mathcal{P})$ for some d is that $\mathcal{P}$ is coprime.*

We consider next sufficient conditions for the assignability of combinants for some appropriate order d. This study involves an extensive use of generalised resultants. For the special case of resultants with $p=n+q-1$, $d=n-1$ the so called Sylvester resultant $\widetilde{\mathcal{S}}_{\mathcal{P}}=\mathcal{S}_{n+q-1,n-1}(\mathcal{P})$ we have the following well known property [3],[4].

Lemma 6.2 *Let $\mathcal{P}$ be an (m,n(q)) set with Sylvester Resultant $\widetilde{\mathcal{S}}_{\mathcal{P}}$. The set $\mathcal{P}$ is coprime, if and only if $\widetilde{\mathcal{S}}_{\mathcal{P}}$ has full rank*

We may now state the main result on the assignability of dynamic combinants:

Theorem 6.1 *Let $\mathcal{P}$ be an (m,n(q)) set. There exists a d such that $f_d(s,\mathcal{K},\mathcal{P})$ is completely assignable, if and only if the set $\mathcal{P}$ is coprime.*

Proof:

The necessity has already been established by the previous proposition. To prove sufficiency, we consider $d=n-1$. We consider a special combinant of degree $p=n+q-1$ and order $n-1$ such as

$$\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s) \tag{75}$$

where

$$\partial[k_1(s)] = q-1,\ \partial[k_i(s)] = n-1,\ i=2,3,\ldots,m. \tag{76}$$

If we denote

$$k_1(s) = \underline{\widetilde{k}}_1^t \underline{\widetilde{e}}_{q-1}(s), k_i(s) = \underline{\widetilde{k}}_i^t \underline{\widetilde{e}}_{n-1}(s), i = 2, 3, \ldots, m \tag{77}$$

then

$$\widetilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P}) = [\underline{\widetilde{k}}_1^t, \underline{\widetilde{k}}_2^t, \ldots, \underline{\widetilde{k}}_m^t] \begin{bmatrix} \mathcal{S}_{n,q-1}(p_1) \\ \mathcal{S}_{q,n-1}(p_2) \\ \vdots \\ \mathcal{S}_{q,n-1}(p_m) \end{bmatrix} \underline{\widetilde{e}}_{n+q-1}(s) \tag{78}$$

$$\underline{\widetilde{k}}^t \widetilde{\mathcal{S}}_{n+q-1,n-1}(\mathcal{P}) \underline{\widetilde{e}}_p(s) = \underline{\widetilde{k}}^t \widetilde{\mathcal{S}}_{\mathcal{P}} \underline{\widetilde{e}}_p(s)$$

However, $\widetilde{\mathcal{S}}_{\mathcal{P}}$ is the Sylvester resultant and by the previous Lemma it has full rank, since the set $\mathcal{P}$ is coprime. Therefore, $rank\{\widetilde{\mathcal{S}}_{\mathcal{P}}\} = n+q$ and given that $\widetilde{\mathcal{S}}_{\mathcal{P}}$ and $\mathcal{Q}_{\mathcal{P},d}$ are equivalent under column - row permutations, then assignability is established.

Corollary 6.2 *From the (m,n(q)) coprime set $\mathcal{P}$ the following properties hold true:*

(i) There exists a combinant $\widetilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$ of degree $p = n+q-1$ and order $d = n-1$ which is completely assignable

(ii) All combinants $f_{n-1}(s, \mathcal{K}, \mathcal{P})$ of order d=n-1 and degree $p:\ n+q-1 \leq p \leq 2n-1$ are also completely assignable.

Proof:

Part(i) follows from Theorem (6.1) proof by the construction of the Sylvester resultant which leads to the definition of the combinant $\widetilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$ with $\partial[k_1(s)] = q-1$ and $\partial[k_i(s)] = n-1,\ i = 2, \ldots, m$. Consider now the general combinant of order $d = n-1$ which has maximal degree $p = 2n-1$. We can then express $k_1(s)$ as

$$k_1(s) = k_{n-1,1}s^{n-1} + \ldots + k_{q,1}s^q + k_{q-1,1}s^{q-1} + \ldots + k_{1,1}s + k_{0,1} =$$

$$= [k_{n-1,1}, \ldots, k_{q,1}; k_{q-1,1}, k_{1,1}, k_{0,1}] \underline{\widetilde{e}}_{n-1}(s) =$$

$$= [\underline{\widehat{k}}_1^t; \underline{\widetilde{k}}_1^t] \underline{\widetilde{e}}_{n-1}(s) \tag{79}$$

Then $f_{n-1}(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^m k_i(s)p_i(s),\ \partial[k_i(s)] = n-1$ can be expressed as

$$f_{n-1}(s, \mathcal{K}, \mathcal{P}) = [\underline{\widehat{k}}_1^t; \underline{\widetilde{k}}_1^t, \underline{\widetilde{k}}_2^t, \ldots, \underline{\widetilde{k}}_m^t] \mathcal{S}_{2n-1,n-1}(\mathcal{P}) \underline{\widetilde{e}}_{2n-1}(s) \tag{80}$$

where the generalised resultant $\mathcal{S}_{2n-1,n-1}(\mathcal{P}) = \widehat{\mathcal{S}}_{\mathcal{P}}$ may be partitioned according to the partitioning of $[\underline{\widehat{k}}_1^t; \underline{\widetilde{k}}^t]$ and it is expressed as

$$\widehat{\mathcal{S}}_{\mathcal{P}} = \left[\begin{array}{cccccccc:ccc} 1 & a_{n-1} & & & \cdots & \cdots & & & x & \cdots & x \\ 0 & 1 & & a_{n-1} & \cdots & \cdots & & & x & \cdots & x \\ \vdots & \ddots & \ddots & & & & & & x & \cdots & x \\ \vdots & & \ddots & \ddots & & & & & \vdots & & \\ \vdots & & & \ddots & \ddots & & & & \vdots & & \\ 0 & & & & 0 & 1 & & & x & \cdots & x \\ \hdashline & & & 0 & & & & & & \widetilde{\mathcal{S}}_p & \end{array}\right] \tag{81}$$

The upper block diagonal structure of $\widehat{\mathcal{S}}_{\mathcal{P}}$ and the full rank property of the Sylvester Resultant $\widetilde{\mathcal{S}}_{\mathcal{P}}$ implies that $\widehat{\mathcal{S}}_{\mathcal{P}}$ has full rank since $rank\{\widehat{\mathcal{S}}_{\mathcal{P}}\} = n - q + rank\{\widetilde{\mathcal{S}}_{\mathcal{P}}\} = 2n-1$. The proof for any degree $p = n + q - 1 \leq p < 2n - 1$ follows along similar lines.

The matrix $\widehat{\mathcal{S}}_{\mathcal{P}}$ defined above is an extension of the Sylvester Resultant and may be referred to as n-order extended Sylvester Resultant. The special combinant of order $d = n - 1$ and degree $p = n + q - 1$ will be referred to as the *Sylvester combinant* of the set $\mathcal{P}$.

Remark 6.1 For the Sylvester combinant $\widetilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$:

$$\widetilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^{m} k_i(s) p_i(s) \tag{82}$$

$\partial[k_1(s)] = q - 1$, $\partial[k_i(s)] = n - 1$, $i = 2, \ldots, m$ the zero assignment problem is equivalent to making $\widetilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$ be an arbitrary polynomial $\alpha(s)$ of degree n+q-1, i.e. $\alpha(s) = \underline{\alpha}^t \underline{\widetilde{e}}_{n+q-1}(s)$. This is equivalent to solving the equation

$$[\underline{\widetilde{k}}_1^t; \underline{\widetilde{k}}_2^t; \ldots; \underline{\widetilde{k}}_m^t] \begin{bmatrix} \mathcal{S}_{n,q-1}(p_1) \\ \mathcal{S}_{q,n-1}(p_2) \\ \vdots \\ \mathcal{S}_{q,n-1}(p_m) \end{bmatrix} = \underline{\alpha}^t \tag{83}$$

or

$$\underline{\widetilde{k}}^t \widetilde{\mathcal{S}}_{\mathcal{P}} = \underline{\alpha}^t. \tag{84}$$

Under coprimeness assumption the above equation has always a solution and the number of degrees of freedom is $\rho_s = mn + 1 - 2n$. For the case $m = 2$ the assignment problem has a unique solution.

From corollary (6.2) it is clear that the two combinants of the same order $d = n-1$ and different degree may be both assignable. In fact, under the coprimeness assumption, both combinants $\widetilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$, $f_{n-1}(s, \mathcal{K}, \mathcal{P})$ of degree respectively $n + q - 1$ and $2n - 1$ are assignable This raises the following important questions of investigating the assignability of all combinants $f_d(s,\mathcal{K},\mathcal{P})$ with $d < n-1$ and parameterize all combinants $\hat{f}_d(s,\mathcal{K},\mathcal{P})$ of order d, $d \le n-1$ and degree $p \le n+q-1$ which are assignable.

7. Conclusions

The fundamentals of the theory of dynamic polynomial combinants have been introduced and their representation in terms of Generalized Resultants has been established. The parameterization of combinants in terms of order and degree has been introduced and this lays the foundations for investigating the properties of the family of Generalised Resultants. The current framework allows the development of the theory of dynamic combinants that may answer questions related to zero distribution of combinants, and its links to the existence of a nontrivial GCD, as well as "approximate GCD". The conditions for existence of spectrum assignable combinants have been established and these are equivalent to the coprimeness of the generating set $\mathcal{P}$. Amongst the problems under current investigation is the minimal design problem dealing with finding the least order and degree for which spectrum assignability may be guaranteed.

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