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Stochastic PDE models for chiral media: Well posedness, singular limits and a priori estimates for their validity

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Abstract

In the present work we consider the problem of evolution of electromagnetic fields in a random chiral medium, with random external excitations. We establish the well posedness of the resulting stochastic model with additive noise, based on the Maxwell equations with nonlocal in time linear constitutive relations, as well as of a singular limit (local in time) approximative model which formally resembles the Drude-Born-Fedorov constitutive relations for chiral media. We also provide a priori estimates of the difference of the fields as predicted by both models, depending on properties of the stochastic term, the time horizon, properties of the domain, properties of the random initial data and source terms and the chirality measure of the approximative model.

Keywords: Stochastic PDE models, random chiral media, nonlocal in time constitutive laws, singular limits, local in time constitutive laws, time domain analysis

1. Introduction

The mathematical modeling of general linear homogeneous causal media with time invariant response characteristics (such media are usually called *bianisotropic*) is done through the modification of the constitutive relations for the well known Maxwell's equations in a region $\Omega \subset \mathbb{R}^3$, $t > 0$:

$$\left.\begin{aligned} \frac{\partial D}{\partial t} - curl H &= -J_e \\ \frac{\partial B}{\partial t} + curl E &= -J_m \end{aligned}\right\} \qquad (1)$$

where E is the electric field, H is the magnetic field, D is the electric displacement, B is the magnetic induction and J_e, J_m are the densities of the electric and magnetic current, respectively.

Maxwell's equations must be endowed with a set of constitutive relations; these model the dynamics of the charged constituents of materials. Under the assumption of linear response to the applied fields, causality, invariance under time translation and continuity, the general form of the constitutive relations for chiral media in the time domain (see [9], [12]), have the integral form ($\star$ stands for convolution in time)

$$\left.\begin{array}{rcl} D & = & \varepsilon E + c_e \star E + \alpha \star H \\ B & = & \mu H - \alpha \star E + \sigma_m \star H \end{array}\right\} \tag{2}$$

where $\varepsilon > 0$ is the electric permittivity, $\mu > 0$ is the magnetic permeability and α, c_e, σ_m respond to the chiral properties of the medium. We shall refer to Maxwell's system (1), under the constitutive relations (2), as the *nonlocal system* for chiral media. A time–domain analysis of the nonlocal system can be found in [11], [12], [17]. Related work can be found in [3], [7].

Though the mathematical treatment of nonlocal problems is feasible, in a number of important applications this may be cumbersome to handle, due to the integrodifferential character of the governing equations. Thus, local approximations to the nonlocal constitutive relations have been proposed, that are capable of keeping the general features of chiral media, without the mathematical complications introduced by the convolution (integral) terms. In practice, a common approximation scheme to the nonlocal constitutive relations for chiral media, is the Drude–Born–Fedorov (DBF) approximation which leads to the constitutive relations

$$\left.\begin{array}{rcl} D & = & \varepsilon\,(E + \beta curl E) \\ B & = & \mu\,(H + \beta curl H) \end{array}\right\} \tag{3}$$

where $\beta \neq 0$ is the chirality measure. The DBF approximation is "universally" used to model chiral media in the frequency domain (see [8], [9], [13], as well as all publications treating mathematical problems for chiral media when the involved fields have harmonic time dependence). The DBF constitutive relations have been used in the time domain, as well, see e.g. [1], [6], [7]; F. I. Fedorov has also used these constitutive relations in the time domain in his 1959 pioneering publications.

One can refer to the introduction in [2] for certain reasons that clarify the implication of stochastic terms in the equations of mathematical physics. For the use of stochastic models in electromagnetic theory, containing also engineering literature, one can refer to [14], [16] and references therein. In a number of applications it is of interest to study phenomena where the densities of the electric and magnetic currents J_e and J_m are assumed to be stochastic. These can be modeled as random fields, i.e. as random variables indexed by spatial and time coordinates. We will consider Gaussian random fields which may be modeled as an infinite dimensional Wiener process. It is the aim of this paper (i) to clarify the functional background where well–posedness results for the corresponding stochastic Cauchy problems for Maxwell's equations, under the nonlocal constitutive relations and the DBF ones, can be established and (ii) using this background to provide an estimate for the error induced when the nonlocal constitutive relations are replaced by the DBF ones.

The paper is organized as follows: in Section 2 we list the functional spaces and operators needed in this paper. In Sections 3 and 4 we establish the well–posedness of the deterministic and the stochastic nonlocal problem with additive noise respectively, both in the L^2 setting and in the divergence free setting. In Section 5 we examine the same questions for the DBF model. Finally, in Section 6 we study the error induced when approximating nonlocal constitutive relations by the local (DBF) ones.

2. Spaces and Operators

Let Ω be a bounded and simply connected domain of $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. By n we denote the unit outward normal vector to $\partial\Omega$. We shall make use of the following function spaces: let $L^2(\Omega)$ be the usual space of square integrable functions in Ω. Let us recall the "standard" function spaces employed in electromagnetics (see [5])

$$H(curl;\Omega) = \left\{U \in L^2(\Omega)^3 : \ curlU \in L^2(\Omega)^3\right\},$$
$$H_0(curl;\Omega) = \{U \in H(curl;\Omega) : \ U \times n = 0 \ \ in \ \ \partial\Omega\},$$
$$H(div;\Omega) = \left\{U \in L^2(\Omega)^3 : \ divU \in L^2(\Omega)\right\},$$
$$H_0(div;\Omega) = \{U \in H(div;\Omega) : \ U \cdot n = 0 \ \ in \ \ \partial\Omega\},$$
$$H(div0;\Omega) = \left\{U \in L^2(\Omega)^3 : \ divU = 0\right\}.$$

Further, let
$$\mathbf{X} := L^2(\Omega)^3 \times L^2(\Omega)^3.$$

This is is a Hilbert space when equipped with the inner product (the overbar denotes complex conjugation)

$$\left(\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} \right)_{\mathbf{X}} = \int_\Omega (\varepsilon\phi_1 \cdot \overline{\phi_2} + \mu\psi_1 \cdot \overline{\psi_2}) \ \ dx.$$

Now define the spaces

$$\mathbf{X}_{curl} := H_0(curl;\Omega) \times H(curl;\Omega),$$
$$\mathbf{W} := H(div0;\Omega) \cap H_0(div;\Omega), \qquad \mathbf{H} := \mathbf{W} \times \mathbf{W}$$
$$\mathbf{W}_{curl} := \{U \in \mathbf{W} \ : curlU \in \mathbf{W}\}, \qquad \mathbf{H}_{curl} := \mathbf{W}_{curl} \times \mathbf{W}_{curl}.$$

Finally, for a linear operator $N : \mathcal{X} \supset D(N) \to \mathcal{Y}$, ($\mathcal{X}, \mathcal{Y}$ being normed spaces) we set

$$\mathbf{H}_N := (D(N), \ \| \cdot \|_N),$$

where $\| \cdot \|_N$ is the graph–norm of N. For the reader's convenience, we now collect the main operators that we shall use in the sequel. Here we just give their forms, while their domains are various function spaces, specified accordingly in the text whenever these operators appear. So, we define

$$A = \begin{pmatrix} \varepsilon I_3 & O_3 \\ O_3 & \mu I_3 \end{pmatrix}, \qquad K = \begin{pmatrix} c_e\, I_3 & \alpha\, I_3 \\ -\alpha\, I_3 & \sigma_m\, I_3 \end{pmatrix}, \qquad M = \begin{pmatrix} O_3 & curl \\ -curl & O_3 \end{pmatrix},$$

$$Q = \begin{pmatrix} curl & O_3 \\ O_3 & curl \end{pmatrix}, \qquad \mathscr{M} = A^{-1}M = \begin{pmatrix} O_3 & \frac{1}{\varepsilon}\, curl \\ -\frac{1}{\mu}\, curl & O_3 \end{pmatrix},$$

$$\mathscr{K} = -A^{-1}\frac{dK}{dt} = \begin{pmatrix} -\frac{1}{\varepsilon}\frac{dc_e}{dt}\, I_3 & -\frac{1}{\mu}\frac{d\alpha}{dt}\, I_3 \\ \frac{1}{\varepsilon}\frac{d\alpha}{dt}\, I_3 & -\frac{1}{\mu}\frac{d\sigma_m}{dt}\, I_3 \end{pmatrix},$$

and

$$L = I_6 + \beta Q = \begin{pmatrix} I_3 + \beta curl & O_3 \\ O_3 & I_3 + \beta curl \end{pmatrix},$$

$$R = \beta I_6 + Q^{-1} = \begin{pmatrix} \beta I_3 + curl^{-1} & O_3 \\ O_3 & \beta I_3 + curl^{-1} \end{pmatrix}.$$

Recall that M is called the Maxwell operator, while A is usually called the optical response matrix and K the susceptibility matrix.

3. The nonlocal deterministic problem

In this section we follow [12], for the formulation of the nonlocal problem in order to get a Cauchy problem for an integrodifferential equation of Volterra type. We assume that Maxwell's equations (1) hold in Ω, for $t > 0$.

3.1. The L^2–case.

Equations (1), supplemented with the initial data $E(0,x) = E_0(x),\ H(0,x) = H_0(x)$, and the boundary condition of a perfect conductor $E \times n = 0,\ on\ \partial\Omega$, under the complete constitutive relations lead to the following initial–boundary value problem for E, H:

$$\left.\begin{array}{lll} \frac{\partial}{\partial t}(\varepsilon E + c_e \star E + \alpha \star H) - curl H = -J_e & \text{in} & \Omega,\quad t>0, \\ \frac{\partial}{\partial t}(\mu H - \alpha \star E + \sigma_m \star H) + curl E = -J_m & \text{in} & \Omega,\quad t>0, \\ E \times n = 0, & \text{on} & \partial\Omega,\quad t>0, \\ E(\cdot,0) = E_0,\quad H(\cdot,0) = H_0 & \text{in} & \Omega. \end{array}\right\} \tag{4}$$

Using the so–called six vector notation,

$$\mathscr{E} = \begin{pmatrix} E \\ H \end{pmatrix}, \quad \mathscr{E}_0 = \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}, \qquad F = \begin{pmatrix} -J_e \\ -J_m \end{pmatrix},$$

system (4) takes the form of a Cauchy problem for a pseudoparabolic equation in $\mathbf{X}$:

$$\left.\begin{array}{c} \frac{d}{dt}(A\mathscr{E} + K \star \mathscr{E}) = M\mathscr{E} + F \\ \mathscr{E}(0) = \mathscr{E}_0 \end{array}\right\} \tag{5}$$

Assumption 3.1 We assume that:

1 ε and μ are positive constants,

2 $c_e, \sigma_m, \alpha \in W^{1,\infty}([0,T])$ with $c_e(0) = \sigma_m(0) = \alpha(0) = 0$,

3 $M : D(M) = \mathbf{X}_{curl} \to \mathbf{X}$,

4 $F \in L^1([0,T];\ \mathbf{X})$.

3.2. **The divergence–free case**

We now consider the appropriate formulation for the determination of the suitable functional background for the comparison of the nonlocal problem with the DBF model. We need the following

Assumption 3.2 We assume that:

(i) $J_e \cdot n = J_m \cdot n = 0$, on $\partial\Omega \times \{t > 0\}$,

(ii) $E_0 \cdot n = H_0 \cdot n = 0$ on $\partial\Omega$.

Maxwell equations (1), under the nonlocal constitutive relations lead to the following initial–boundary value problem for E, H:

$$\left.\begin{array}{ccc} \dfrac{\partial}{\partial t}(\varepsilon E + c_e \star E + \alpha \star H) - curl H = -J_e & \text{in} & \Omega, \quad t > 0, \\ \dfrac{\partial}{\partial t}(\mu H - \alpha \star E + \sigma_m \star H) + curl E = -J_m & \text{in} & \Omega, \quad t > 0, \\ div E = 0 & \text{in} & \Omega, \quad t \geq 0, \\ div H = 0 & \text{in} & \Omega, \quad t \geq 0, \\ E \cdot n = H \cdot n = curl E \cdot n = curl H \cdot n = 0 & \text{on} & \partial\Omega, \quad t > 0, \\ E(\cdot, 0) = E_0, \quad H(\cdot, 0) = H_0 & \text{in} & \Omega. \end{array}\right\} \tag{6}$$

Using the six vector notation, system (6) takes, too, the form of a Cauchy problem for a pseudoparabolic equation in $\mathbf{H}$:

$$\left.\begin{array}{ccc} \frac{d}{dt}(A\mathscr{E} + K \star \mathscr{E}) & = & M\mathscr{E} + F \\ \mathscr{E}(x, 0) & = & \mathscr{E}_0 \end{array}\right\} \tag{7}$$

Assumption 3.3 We assume that:

1 ε and μ are positive constants,

2 $c_e, \sigma_m, \alpha \in W^{1,\infty}([0,T])$ with $c_e(0) = \sigma_m(0) = \alpha(0) = 0$,

3 $M : D(M) = \mathbf{H}_{curl} \to \mathbf{H}$

4 $F \in L^1([0,T];\ \mathbf{H})$.

3.3. Well posedness of the models

We will treat hereafter both cases with the same notation, see [12] studying them as abstract Cauchy problems. In this subsection $\mathfrak{H}$ will be used as a generic notation for either $\mathbf{X}$ or $\mathbf{H}$, depending on whether we consider problem (4) or (6); both these problems take the form of a Cauchy problem for an integrodifferential equation of Volterra type in $\mathfrak{H}$:

$$\left.\begin{array}{c} \frac{d}{dt}\mathscr{E} = \mathscr{M}\mathscr{E} + \mathscr{K} \star \mathscr{E} + \mathscr{F} \\ \mathscr{E}(0) = \mathscr{E}_0 \end{array}\right\} \tag{8}$$

where $\mathscr{F} = A^{-1}F$. Clearly $D(\mathscr{M})$, as well, is either $\mathbf{X}_{curl}$ or $\mathbf{H}_{curl}$ depending on whether we work for the L^2–case or the divergence–free case, respectively, and $\mathfrak{H}_{\mathscr{M}}$ is then defined accordingly.

It has been proved for both the L^2-case and the divergence–free case (see e.g. [10]) that the the densely defined operator $i\mathscr{M}$ is selfadjoint, therefore by Stone's theorem, $\mathscr{M}$ generates a unitary group $(\mathscr{T}(t))_{t\in\mathbb{R}}$ on $\mathfrak{H}$.

Lemma 3.1 *Assumptions 3.1, 3.3 yield*

(i) The operator $\mathscr{M} : D(\mathscr{M}) \to \mathfrak{H}$ is the infinitesimal generator of a C_0-group of unitary operators $\mathscr{T}(t)$, $t \in \mathbb{R}$, in $\mathfrak{H}$, i.e. $||\mathscr{T}(t)||_{\mathcal{L}(\mathfrak{H})} = 1$, for every $t \in [0,T]$.

(ii) $\{\mathscr{K}(t)\}_{t\geq 0}$ is a family of bounded operators in $\mathfrak{H}$, which satisfies

$$\sup_{t\in[0,T]} \|\mathscr{K}(t)\|_{\mathcal{L}(\mathfrak{H})} \leq C_{\mathcal{K}}, \quad for\ some \quad C_{\mathcal{K}} > 0.$$

(iii) $\mathscr{F} \in L^1([0,T];\ \mathfrak{H})$.

(iv) $\mathscr{E}_0 \in \mathfrak{H}$.

The proof of (i), (iii) and (iv) is straightforward while in (ii) the uniform boundedness principle is needed.

We will give now the definitions of mild, strong and classical solutions of (8).

Definition 3.1 A function $\mathscr{E} \in C([0,T];\ \mathfrak{H})$ is called a mild solution of (8), if for all $t \in [0,T]$ we have

$$\mathscr{E}(t) = \mathscr{T}(t)\mathscr{E}_0 + \int_0^t \mathscr{T}(t-s) \int_0^s \mathscr{K}(s-r)\mathscr{E}(r)dr\ ds + \int_0^t \mathscr{T}(t-s)\mathscr{F}(s)\ ds.$$

Definition 3.2 An $\mathfrak{H}-$valued function $\mathscr{E}$ is called a strong solution of (8) if:

1 $\mathscr{E}(t) \in D(\mathscr{M})$, a.e. on $[0,T]$

2 $\displaystyle\int_0^T \left[\ \|\mathscr{M}\mathscr{E}(s)\|_{\mathfrak{H}} + \|\int_0^s \mathscr{K}(s-r)\mathscr{E}(r)\ dr\|_{\mathfrak{H}}\ \right]\ ds < \infty$

3 $\mathscr{E}(t)$ satisfies the equation of (8) a.e. on $[0,T]$.

Definition 3.3 An $\mathfrak{H}-$valued function $\mathscr{E}$ is called a classical solution of (8) if:

1 $\mathscr{E}(t) \in D(\mathscr{M})$, $t \in [0,T]$

2 $\mathscr{M}\mathscr{E}(t)$ and $\displaystyle\int_0^t \mathscr{K}(t-s)\mathscr{E}(s)\ ds$ are continuous in $[0,T]$.

3 $\mathscr{E}(t)$ satisfies the equation of (8) for all $t \in [0,T]$.

By [11], applying a fixed point theorem technique in the appropriate Banach spaces, we have the following results for the weak strong and classical well–posedness of (8):

Theorem 3.4 *Under Assumptions 3.1, 3.3, Problem* (8) *is mildly well–posed.*

Assumption 3.5 Suppose that

1 $\mathscr{K}(t)y \in D(\mathscr{M})$ *for every* $y \in D(\mathscr{M})$, a.e. on $[0,T]$ and there is a constant $C_{\mathscr{M}\mathcal{K}} > 0$ such that $\|\mathscr{M}\mathscr{K}(t)y\|_{\mathfrak{H}} \leq C_{\mathscr{M}\mathcal{K}}\|y\|_{\mathfrak{H}_{\mathscr{M}}}$, $t \in [0,T]$.

2 $\mathscr{F}(t) \in D(\mathscr{M})$ a.e. on $[0,T]$ and $\mathscr{M}\mathscr{F} \in L^1([0,T];\ \mathfrak{H})$

3 $\mathscr{E}_0 \in D(\mathscr{M})$.

Theorem 3.6 *Under Assumption 3.5, Problem* (8) *is strongly well–posed.*

Theorem 3.7 *Assume that Assumption 3.5 holds for every* $t \in [0,T]$ *and that:*

(i) The family of operators $\{\mathscr{K}(t)\}_{t\geq 0}$ *is continuous on* $[0,T]$*,*

(ii) $\mathscr{F}$ *is continuous on* $[0,T]$.

Then Problem (8) *is classically well–posed.*

4. The nonlocal stochastic model

Stochastic models allow us to describe phenomena (clearly not covered by deterministic models) arising from various forms of uncertainty in space and time. This uncertainty may be related to stochastic densities of electric and magnetic currents J_e and J_m respectively, which may depend nonlinearly on the electromagnetic field. If we assume that the evolution of electromagnetic fields in a chiral medium is disturbed by some electromagnetic noise, an extra term containing the stochastic effects, modeled by functionals of a Wiener process, has to be added in equation (8). This noise may either be of the additive or the multiplicative type. With the help of [11] one can provide well posedness results evolving predictable processes.

4.1. The abstract Cauchy problem

Let U be a real separable and infinite dimensional Hilbert space and consider the real and separable Hilbert space $\mathfrak{H}$ of 3.3, the probability space $(\Omega, \mathscr{F}, P)$ with a normal filtration $\mathscr{F}_t,\ t \geq 0$, and the predictable σ-field $\mathscr{P}_T$ in the space $\Omega_T = [0,T] \times \Omega$. Consider also the measurable spaces $(U, \mathscr{B}(U))$, $(\mathfrak{H}, \mathscr{B}(\mathfrak{H}))$, $(\Omega_T \times \mathfrak{H}, \mathscr{P}_T \times \mathscr{B}(\mathfrak{H}))$ (as usual $\mathscr{B}$ is the Borrel σ-field) and $(L_2^0, \mathscr{B}(L_2^0))$, where by L_2^0 we denote the space of all Hilbert-Schmidt operators in $L_2(U_0, H)$ with $U_0 = Q^{1/2}(U)$, and $Q \in \mathcal{L}(U)$ is a nonnegative, nuclear operator $(Tr[Q] < \infty)$.

A non linear stochastic model, with additive noise, for Problem (8), is described as the Cauchy problem for a stochastic integrodifferential equation of the form:

$$\left.\begin{aligned} d\mathscr{E}_t &= [\mathscr{M}\mathscr{E}_t + \int_0^t \mathscr{K}(t-s)\mathscr{E}_s \, ds + F(t,\ \mathscr{E}_t)] \ \ dt + B \ \ dW_t, \quad t \geq 0 \\ \mathscr{E}_0 &= \xi \end{aligned}\right\} \tag{9}$$

where $W_t,\ t \geq 0$ is a $U-$valued Q-Wiener process in the probability space $(\Omega, \mathscr{F}, P)$.

Assumption 4.1 We assume the following:

1 The operator $\mathscr{M}$ and the family of bounded operators $\{\mathscr{K}(t)\}_{t\geq 0}$, satisfy the assumptions of Lemma 3.1 of Section 3.3.

2 $B \in \mathcal{L}(U, \mathfrak{H})$.

3 The function $F:\ \Omega_T \times \mathfrak{H} \to \mathfrak{H}$ with $(t, \omega, x) \to F(t, \omega, x)$ is measurable from $(\Omega_T \times \mathfrak{H}, \mathbb{P}_T \times \mathscr{B}(\mathfrak{H}))$ to $(\mathfrak{H}, \mathscr{B}(\mathfrak{H}))$ and there exist $C,\ C_F > 0$, such that:

   (a) $\|F(t,\omega,x)\|_{\mathfrak{H}} \ \leq \ C\|x\|_{\mathfrak{H}}$ where $x \in \mathfrak{H},\ \ t \in [0,T],\ \ \omega \in \Omega$,

   (b) $\|F(t,\omega,x) - F(t,\omega,y)\|_{\mathfrak{H}} \ \leq \ C_F\|x-y\|_{\mathfrak{H}}$ where $x, y \in \mathfrak{H},\ \ t \in [0,T]$, $\omega \in \Omega$.

4 ξ is an $\mathfrak{H}-$valued, $\mathscr{F}_0-$measurable, square integrable random variable, i.e. $E[\|\xi\|_{\mathfrak{H}}^2] < \infty$.

We will employ the space of all continuous (in mean square) and square integrable predictable processes

$$\mathcal{C}([0,T];\mathfrak{H}) = \{Y \in C([0,T]; L^2(\Omega,\mathfrak{H}) : Y \text{ is predictable}\}.$$

This space equipped with the norm

$$\|Y\|_{\mathcal{C}} = \sup_{t\in[0,T]} \left(E\left[\|Y_t\|^2_{\mathfrak{H}}\right]\right)^{1/2}$$

is a Banach space. We note that an adapted stochastic process which is continuous its self is predictable. We will give now the definitions of mild and strong solutions for the stochastic Problem (9).

Definition 4.1 A stochastic process $\mathscr{E}_t \in \ \mathcal{C}([0,T];\mathfrak{H})$ is called a mild solution of problem (9) if:

$$\mathscr{E}_t = \mathscr{T}(t)\xi \ + \ \int_0^t \mathscr{T}(t-s)\int_0^s \mathscr{K}(s-r)\mathscr{E}_r \ drds \ + \ \int_0^t \mathscr{T}(t-s)F(s,\mathscr{E}_s) \ ds \ + \\ \int_0^t \mathscr{T}(t-s)B \ dW_s, \quad t\in[0,T], \quad \text{P-as.}$$

Definition 4.2 An $\mathfrak{H}-$valued predictable process $\mathscr{E}_t$, $t \in [0,T]$, is called a weak solution of problem (9) if:

1 $\displaystyle\int_0^T \|\mathscr{E}_s\|_{\mathfrak{H}} \ \ ds < \infty$, P-a.s.

2 For every $\zeta \in D(\mathscr{M}^*) = D(\mathscr{M})$ holds

$$\langle \mathscr{E}_t \ , \ \zeta\rangle = \langle \xi \ , \ \zeta\rangle + \int_0^t -\langle \mathscr{E}_s \ , \ \mathscr{M}\zeta\rangle + \langle F(s,\mathscr{E}_s) \ , \ \zeta\rangle \ ds \ + \\ \int_0^t \langle \int_0^s \mathscr{K}(s-r)\mathscr{E}_r \ dr \ , \ \zeta\rangle \ \ ds \ + \ \int_0^t \langle BdW_s \ , \ \zeta\rangle, \quad t\in[0,T], \ \text{P-as.}$$

Definition 4.3 An $\mathfrak{H}-$valued predictable process $\mathscr{E}_t$, $t \in [0,T]$, is called a strong solution of problem (9) if:

1 $\mathscr{E}_t \in D(\mathscr{M})$, P-a.s., a.e. on $[0,T]$

2 $\displaystyle\int_0^T [\ \|\mathscr{M}\mathscr{E}_s\|_{\mathfrak{H}} + \|\int_0^s \mathscr{K}(s-r)\mathscr{E}_r \ dr\|_{\mathfrak{H}} \ ] \ ds < \infty$, P-a.s.

3 $$\mathscr{E}_t = \xi + \int_0^t \mathscr{M}\mathscr{E}_s \ ds \ + \ \int_0^t \int_0^s \mathscr{K}(s-r)\mathscr{E}_r dr \ ds \ + \int_0^t \mathscr{F}(s,\mathscr{E}_s) \ ds \ + \\ \int_0^t B \ dW_s, \qquad t\in[0,T], \ \text{P-as.}$$

We have the following result:

Theorem 4.2 *Under the Assumption 4.1, Problem* (9) *is weakly well-posed.*

Remark 4.1 We observe that the hypothesis of Theorem 6.10 in [4] of the generation of a contraction semigroup by an operator A (i.e. $< Ax, x >\leq 0$ for every $x \in \mathfrak{H}$), is fulfilled, hence $\mathscr{E}_t,\ t \in [0,T]$, has a continuous modification.

Assumption 4.3 Suppose that the assumptions 4.1.1-4.1.4 and 3.5.1 hold. Suppose also that

1 $\xi \in D(\mathscr{M}),\ F(t,x) \in D(\mathscr{M})$ and $BQ^{1/2}h \in D(\mathscr{M})$ P-as for all $t \in [0,T]$, $x \in \mathfrak{H}$, $h \in U$

2 $\|\mathscr{M}F(t,x)\|_{\mathfrak{H}} \leq g_1(t)\|x\|_{\mathfrak{H}},\ g_1 \in L^1([0,T];\mathbb{R}),\ x \in \mathfrak{H}$.

Note that since the operator $\mathscr{M}B : U \to \mathfrak{H}$, due to the closed graph theorem, is bounded, one can prove that $\|\mathscr{M}B\|_{L_2^0} < \infty$.

Theorem 4.4 *Under the Assumptions 4.3, Problem* (9) *is strongly well posed.*

5. The DBF Model

We will study now the well–posedness of the DBF model in the time domain, using the same assumptions made for the nonlocal problem.

5.1. The L^2-case

Maxwell's equations (1) under the DBF constitutive relations, supplemented with the initial data $E(0,x) = E_0(x)$, $H(0,x) = H_0(x)$, and the boundary condition of a perfect conductor, lead to the following Cauchy problem for a pseudoparabolic (Sobolev type) equation in the Hilbert space $\mathbf{X}$:

$$\left.\begin{array}{c} \frac{d}{dt}(L\mathscr{E}) = \mathscr{M}\mathscr{E} + \mathscr{F} \\ \mathscr{E}(0) = \mathscr{E}_0 \end{array}\right\}, \tag{10}$$

with

$$D(L) = D(\mathscr{M}) = \mathbf{X}_{curl}.$$

Since the operators $curl : H_0(curl;\Omega) \to L^2(\Omega)^3$ and $curl : H(curl;\Omega) \to L^2(\Omega)^3$ have spectrum $\sigma(curl) = \mathbb{C}$ (see [15], [18]), the operator $L : D(L) \to \mathbf{X}$ does not have a bounded inverse. Thus,

Proposition 5.1 *The Cauchy problem* (10) *is degenerate in* $\mathbf{X}$.

5.2. **The divergence–free case**

In [10], where a stochastic DBF model is considered, it is shown that the initial-boundary value problem for E, H, with the same initial-boundary conditions as in Problem (6), in a bounded and simply connected domain $\Omega \subseteq \mathbb{R}^3$ with smooth boundary $\partial\Omega$, can be written in the form of a Cauchy problem for a pseudoparabolic equation in the Hilbert space $\mathbf{H}$:

$$\left.\begin{array}{c} \frac{d}{dt}(L\mathscr{E}) = \mathscr{M}\mathscr{E} + \mathscr{F} \\ \mathscr{E}(x,0) = \mathscr{E}_0 \end{array}\right\} \tag{11}$$

with

$$D(L) = D(\mathscr{M}) = \mathbf{H}_{curl}.$$

If $\beta \neq -\dfrac{1}{\lambda}$, for $\lambda \in \sigma_\rho(curl)$, then the operator $L : D(L) = D(\mathscr{M}) \to \mathbf{H}$ is invertible and the operator $L^{-1} : \mathbf{H} \to D(\mathscr{M})$ is bounded. So, problem (7) can be written as:

$$\left.\begin{array}{c} \frac{d}{dt}(\mathscr{E}) = L^{-1}\mathscr{M}\mathscr{E} + L^{-1}\mathscr{F} \\ \mathscr{E}(x,0) = \mathscr{E}_0 \end{array}\right\} \tag{12}$$

which is a standard Cauchy problem in the Hilbert space $\mathbf{H}_{\mathscr{M}}$. It has been proven in [12] that the bounded operator $L^{-1}\mathscr{M} : \mathbf{H}_{\mathscr{M}} \to \mathbf{H}_{\mathscr{M}}$ is skewadjoint, therefore is the infinitesimal generator of a group of unitary operators $\mathscr{S}(t), t \in \mathbb{R}$, hence

$$\|\mathscr{S}(t)\|_{\mathcal{L}(\mathbf{H}_{\mathscr{M}})} = 1,\ t \in \mathbb{R}.$$

For $\beta \neq -\dfrac{1}{\lambda}$, $\lambda \in \sigma_\rho(curl)$, problem (12) is well–posed in $\mathbf{H}_{\mathscr{M}}$ and the unique solution is given by

$$\mathscr{E}(t) = \mathscr{S}(t)\mathscr{E}_0 + \int_0^t \mathscr{S}(t-s)L^{-1}\mathscr{F}(s)\ \ ds, \quad t \in [0,T].$$

5.3. **The stochastic DBF model**

Let U be a real separable and infinite dimensional Hilbert space and consider the real and separable Hilbert space $\mathbf{H}_{\mathscr{M}}$ and the measurable spaces $(\mathbf{H}_{\mathscr{M}}, \mathscr{B}(\mathbf{H}_{\mathscr{M}}))$, $(\Omega_T \times \mathbf{H}_{\mathscr{M}}, \mathscr{P}_T \times \mathscr{B}(\mathbf{H}_{\mathscr{M}}))$, $(L_2^0, \mathscr{B}(L_2^0))$, but hereafter we denote by L_2^0 the space of all Hilbert-Schmidt operators in $L_2(U_0, \mathbf{H}_{\mathscr{M}})$, instead of $L_2(U_0, \mathbf{H})$. The Cauchy problem for a non linear stochastic model of the equation (12), with additive noise, has the form:

$$\left.\begin{array}{rl} d\mathscr{E}_t & = [L^{-1}\mathscr{M}\mathscr{E}_t + L^{-1}F(t,\mathscr{E}_t)]\ \ dt + L^{-1}B\ dW_t, \quad t \geq 0 \\ \mathscr{E}_0 & = \xi \end{array}\right\} \tag{13}$$

where $W_t,\ t \geq 0$, is the $U-$valued Q-Wiener process, with $Tr[Q] < \infty$.

Assumption 5.1 For the abstract problem (13) we assume the following:

1 The bounded operator $L^{-1}\mathscr{M} : \mathbf{H}_{\mathscr{M}} \to \mathbf{H}_{\mathscr{M}}$ is skewadjoint, hence is the infinitesimal generator of a group of unitary operators $\mathscr{S}(t), t \in \mathbb{R}$.

2 Suppose that for B assumption 4.1.2 hold. One can prove that for the operator $L^{-1}B : \ U \to \mathbf{H}_{\mathscr{M}}$, holds: $E\left[\int_0^T \|L^{-1}B\|^2_{L_2^0}\ ds\right] < \infty$.

3 Suppose that for F assumption 4.1.3 hold. One can prove that the function $L^{-1}F : \ \Omega_T \times \mathbf{H} \to \mathbf{H}_{\mathscr{M}}$ with $(t, \omega, x) \to L^{-1}F(t, \omega, x)$ is measurable from $(\Omega_T \times \mathbf{H}, \mathbb{P}_T \times \mathscr{B}(\mathbf{H}))$ to $(\mathbf{H}_{\mathscr{M}}, \mathscr{B}(\mathbf{H}_{\mathscr{M}}))$ and there exist $C_L,\ C_{LF} > 0$, such that:

(a) $\|L^{-1}F(t, \omega, x)\|_{\mathbf{H}_{\mathscr{M}}} \ \leq \ C_L\|x\|_{\mathbf{H}}$ where $x \in \mathbf{H}$, $t \in [0, T]$, $\omega \in \Omega$,

(b) $\|L^{-1}F(t, \omega, x) - L^{-1}F(t, \omega, y)\|_{\mathbf{H}_{\mathscr{M}}} \ \leq \ C_{LF}\|x - y\|_{\mathbf{H}}$ where $x, y \in \mathbf{H}$, $t \in [0, T]$, $\omega \in \Omega$.

4 ξ is an $\mathbf{H}_{\mathscr{M}}-$valued, $\mathscr{F}_0-$measurable, square integrable random variable, i.e. $E[\|\xi\|^2_{\mathbf{H}_{\mathscr{M}}}] < \infty$ and $\mathscr{E}_t,\ t \geq 0$, is the unknown $\mathbf{H}_{\mathscr{M}}-$valued process.

We now employ the space of all continuous (in mean square) and square integrable predictable processes

$$\mathcal{C}([0, T]; \mathbf{H}_{\mathscr{M}}) = \{Y \in C([0, T]; L^2(\Omega, \mathbf{H}_{\mathscr{M}}) : Y \text{ is predictable}\},$$

which is a Banach space. For the strong well posedness of (13), we have the following definition and result:

Definition 5.1 An $\mathbf{H}_{\mathscr{M}}-$valued predictable process $\mathscr{E}_t$, $t \in [0, T]$, is called a strong solution of problem (13) if:

1 $\int_0^T \|L^{-1}\mathscr{M}\mathscr{E}_s\|_{\mathbf{H}_{\mathscr{M}}}\ ds < \infty$, P-a.s.

2 $$\mathscr{E}_t = \xi + \int_0^t L^{-1}\mathscr{M}\mathscr{E}_s\ ds + \int_0^t L^{-1}\mathscr{F}(s, \mathscr{E}_s)\ ds\ + \int_0^t L^{-1}B\ dW_s, \qquad t \in [0, T], \quad \text{P-as.}$$

Theorem 5.2 *Under the assumptions 5.1 problem* (13) *is strongly well-posed.*

6. The Error Induced when Approximating the Nonlocal Constitutive Laws by the (Local) DBF Ones

For the study of the error in the deterministic case we refer to [12]. For the stochastic case, we have seen that the nonlocal stochastic problem (9), under appropriate conditions, has a unique strong solution, which belongs to the space

$$\mathcal{C}([0, T]; \mathbf{H}_{\mathscr{M}}) = \{Y \in C([0, T]; L^2(\Omega, \mathbf{H}_{\mathscr{M}}) : Y \text{ is predictable}\},$$

and for $t \in [0,T]$ has the form

$$\mathscr{E}_1(t) = \mathscr{T}(t)\xi \ + \ \int_0^t \mathscr{T}(t-s)\int_0^s \mathscr{K}(s-r)\mathscr{E}_1(r)\ drds \ + \int_0^t \mathscr{T}(t-s)F(s,\mathscr{E}_1(s))\ ds \ + \int_0^t \mathscr{T}(t-s)B\ dW(s), \quad \text{P-as},$$

where $\mathscr{T}(t),\ t \in \mathbb{R}$, is the C_0-group of unitary operators in $\mathbf{H}$, generated by the operator $\mathscr{M} : D(\mathscr{M}) \to \mathbf{H}$.

The stochastic DBF model (Problem (13)), under appropriate conditions, has also a unique strong solution, which belongs to the space $\mathcal{C}([0,T];\mathbf{H}_{\mathscr{M}})$, and for $t \in [0,T]$ has the form

$$\mathscr{E}_2(t) = \mathscr{S}(t)\xi \ + \ \int_0^t \mathscr{S}(t-s)L^{-1}F(s,\mathscr{E}_2(s))\ ds \ + \int_0^t \mathscr{S}(t-s)L^{-1}B\ dW(s),$$

P-as, where $\mathscr{S}(t),\ t \in \mathbb{R}$ is the uniformly continuous group of unitary operators in $\mathbf{H}_{\mathscr{M}}$, generated by the bounded operator $L^{-1}\mathscr{M} : \mathbf{H}_{\mathscr{M}} \to \mathbf{H}_{\mathscr{M}}$.

We will estimate the $\mathcal{C}([0,T];\mathbf{H})$-norm of $\mathfrak{E}$, using the $\mathcal{C}([0,T];\mathbf{H}_{\mathscr{M}}))$-norm for the processes $\mathscr{E}_1,\ \mathscr{E}_2$.

The stochastic error $\mathfrak{E}$ satisfies the following stochastic Cauchy problem in $\mathbf{H}$:

$$\left.\begin{aligned} d\mathfrak{E} &= \left[\mathscr{M}\mathfrak{E} + \mathscr{K}\star\mathscr{E}_1 + \beta R^{-1}\mathscr{M}\mathscr{E}_2 + \beta R^{-1}F(t,\mathscr{E}_2) + (F(t,\mathscr{E}_1) - F(t,\mathscr{E}_2))\right] dt \\ &\qquad + \beta R^{-1}B \ \ dW_t \\ \mathfrak{E}(0) &= 0 \end{aligned}\right\} \tag{14}$$

The unique strong solution of (14), whose existence is guaranteed by the chosen functional environment, is given for $t \in [0,T]$, by

$$\mathfrak{E}(t) = \int_0^t \mathscr{T}(t-s)\int_0^s \mathscr{K}(s-r)\mathscr{E}_1(r)dr\ ds + \beta\int_0^t \mathscr{T}(t-s)R^{-1}\mathscr{M}\mathscr{E}_2(s)\ ds \ +$$
$$\beta\int_0^t \mathscr{T}(t-s)R^{-1}F(s,\mathscr{E}_2(s))\ ds \ + \ \int_0^t \mathscr{T}(t-s)(F(s,\mathscr{E}_1(s)) - F(s,\mathscr{E}_2(s))\ ds \ +$$
$$\beta\int_0^t \mathscr{T}(t-s)R^{-1}B\ dW(s),$$

where $\mathscr{T}(t),\ t \in \mathbb{R}$, is the C_0-group of unitary operators in $\mathbf{H}$, generated by the operator $\mathscr{M} : D(\mathscr{M}) \to \mathbf{H}$.

In [12] one can find that for the operator $R : \mathbf{H} \to \mathbf{H}$, holds:

$$\|R^{-1}\|_{\mathcal{L}(\mathbf{H})} \leq \frac{1}{|\beta| - \lambda_0}\ , \quad for \quad |\beta| > \lambda_0.$$

Applying the Cauchy-Schwartz inequality for the Riemann integrals and the Ito isometry for the stochastic convolutions, by Gronwall's inequality we obtain:

Lemma 6.1 *Suppose that Assumption 4.3 holds with the additional hypothesis that* $g_1 \in L^2([0,T];\mathbb{R})$*, so that* $\mathscr{E}_1 \in \mathcal{C}([0,T];\ \mathbf{H}_{\mathscr{M}})$*. Then*

1 $\|\mathscr{E}_1\|^2_{\mathcal{C}([0,T];\ \mathbf{H}_{\mathscr{M}})} \leq 2(TC^2_{\mathscr{M}B}Tr[Q] + \|\xi\|^2_{L^2(\Omega,H_{\mathscr{M}})})\ \exp\left[2\left(T^4\Lambda^2 + TM_{F,g_1}\right)\right],$

$$\text{where}\ \ M_{F,g_1} = \max\left\{C^2T, \int_0^T g_1^2(t)dt\right\},\ C^2_{\mathscr{M}B} = \|B\|^2_{\mathcal{L}(U,\mathbf{H})} + \|\mathscr{M}B\|^2_{\mathcal{L}(U,\mathbf{H})}$$

and $\Lambda = C_{\mathcal{K}} + C_{\mathscr{M}\mathcal{K}}$.

2 $\|\mathscr{E}_2\|^2_{\mathcal{C}([0,T];\ \mathbf{H}_{\mathscr{M}})} \leq 2(T\|L^{-1}B\|^2_{\mathcal{L}(U,\mathbf{H}_{\mathscr{M}})}Tr[Q] + \|\xi\|^2_{L^2(\Omega,H_{\mathscr{M}})})\ \exp\left(2C_L^2T^2\right).$

Theorem 6.1 *Under the assumptions of Lemma 6.1, we have the following estimate for the* $\mathcal{C}([0,T];\mathbf{H})$*–norm of the difference* $\mathfrak{E}$ *of the solutions of the nonlocal stochastic problem (*$\mathscr{E}_1$*) and the stochastic DBF problem (*$\mathscr{E}_2$*), respectively:*

$$\|\mathfrak{E}\|^2_{\mathcal{C}([0,T];\ \mathbf{H})} \leq$$

$$\tfrac{1}{1-2T^2C_F^2}\left[4T^4\Lambda^2\left(TC^2_{\mathscr{M}B}Tr[Q] + \|\xi\|^2_{L^2(\Omega,H_{\mathscr{M}})}\right)e_1 + \right.$$

$$\left.\tfrac{4\beta^2T^2(C^2+1)}{(|\beta|-\lambda_0)^2}\left(T\|L^{-1}B\|^2_{\mathcal{L}(U,\mathbf{H}_{\mathscr{M}})}Tr[Q] + \|\xi\|^2_{L^2(\Omega,H_{\mathscr{M}})}\right)e_2 + \tfrac{2\beta^2T}{(|\beta|-\lambda_0)^2}\|B\|^2_{\mathcal{L}(U,\mathbf{H})}Tr[Q]\right],$$

for $|\beta| > \lambda_0$, *where* $e_1 = \exp[2(T^4\Lambda^2 + TM_{F,g_1})]$, $e_2 = \exp[2\left(C_L^2T^2\right)]$.

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