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Uniqueness theorems for thermo-electro-magneto-elasticity problems

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Abstract

We prove the uniqueness theorems for interior and exterior basic boundary value problems of statics of the thermo-electro-magneto-elasticity theory.

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1. Introduction

Although natural materials rarely show full coupling between elastic, electric, magnetic and thermal fields, some artificial materials do. In the reference [15] it is reported that the fabrication of $BaTiO_3$-$CoFe_2O_4$ composite had the magnetoelectric effect not existing in either constituent. Other examples of similar complex coupling can be found in the references [1]-[6], [8], [11], [12], [14], [16].

The mathematical model of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with the appropriate boundary and initial conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations.

In this paper we prove the uniqueness theorems of solutions for basic and mixed interior and exterior boundary value problems of statics. We note that the most important question in the study of exterior BVPs is to find appropriate asymptotic behaviour of solutions at infinity which guarantee the uniqueness.

2. Basic equations and formulation of boundary value problems

2.1. Field equations. Throughout the paper $u=(u_1,u_2,u_3)^\top$ denotes the displacement vector, σ_{ij} is the mechanical stress tensor, $\varepsilon_{kj}=2^{-1}(\partial_k u_j+\partial_j u_k)$ is the strain tensor, $E=(E_1,E_2,E_3)^\top=-\operatorname{grad}\varphi$ and $H=(H_1,H_2,H_3)^\top=-\operatorname{grad}\psi$ are electric and magnetic fields respectively, $D=(D_1,D_2,D_3)^\top$ is the electric displacement vector and $B=(B_1,B_2,B_3)^\top$ is the magnetic induction vector, φ and ψ stand for the electric and magnetic potentials, ϑ is the temperature increment, $q=(q_1,q_2,q_3)^\top$ is the heat flux vector, and $\mathcal{S}$ is the entropy density. We employ the notation $\partial=(\partial_1,\partial_2,\partial_3)$, $\partial_j=\partial/\partial x_j$, $\partial_t=\partial/\partial t$; the superscript $(\cdot)^\top$ denotes transposition operation; the summation over the repeated indices is meant from 1 to 3, unless stated otherwise.

In this subsection we collect the field equations of the linear theory of thermo-electro-magneto-elasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators.

Constitutive relations:

$$\begin{aligned}
&\sigma_{rj}=\sigma_{jr}=c_{rjkl}\,\varepsilon_{kl}-e_{lrj}E_l-q_{lrj}H_l-\lambda_{rj}\vartheta, \qquad r,j=1,2,3,\\
&D_j=e_{jkl}\,\varepsilon_{kl}+\varkappa_{jl}\,E_l+a_{jl}\,H_l+p_j\,\vartheta, \qquad j=1,2,3,\\
&B_j=q_{jkl}\,\varepsilon_{kl}+a_{jl}\,E_l+\,\mu_{jl}H_l+m_j\,\vartheta, \qquad j=1,2,3,\\
&\mathcal{S}=\lambda_{kl}\,\varepsilon_{kl}+p_k\,E_k+m_k\,H_k+\gamma\,\vartheta.
\end{aligned}$$

Fourier Law: $q_j=-\eta_{jl}\,\partial_l\vartheta, \quad j=1,2,3.$

Equations of motion: $\partial_j\sigma_{rj}+X_r=\varrho\;\partial_t^2u_r, \qquad r=1,2,3.$

Quasi-static equations for electro-magnetic fields where the rate of magnetic field is small (electric field is curl free) and there is no electric current (magnetic field is curl free): $\partial_jD_j=\varrho_e\,, \qquad \partial_jB_j=0\,.$

Linearized equation of the entropy balance: $T_0\,\partial_t\mathcal{S}-Q=-\partial_jq_j.$

Here ϱ is the mass density, ϱ_e is the electric density, c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, $\gamma=\varrho\,c\,T_0^{-1}$ is the thermal constant, T_0 is the initial reference temperature, c is the specific heat per unit mass, $X=(X_1,X_2,X_3)^\top$ is a mass force density, Q is a heat source intensity. The constants involved in these equations satisfy the symmetry conditions:

$$\begin{aligned}
&c_{rjkl}=c_{jrkl}=c_{klrj},\;\; e_{klj}=e_{kjl},\;\; q_{klj}=q_{kjl},\;\; \varkappa_{kj}=\varkappa_{jk},\\
&\lambda_{kj}=\lambda_{jk},\;\; \mu_{kj}=\mu_{jk},\;\; \eta_{kj}=\eta_{jk},\;\; a_{kj}=a_{jk},\;\; r,j,k,l=1,2,3.
\end{aligned} \tag{2.1}$$

From physical considerations it follows that (see, e.g., [7], [13]):

$$c_{rjkl}\,\xi_{rj}\,\xi_{kl} \geq c_0\,\xi_{kl}\,\xi_{kl},\ \varkappa_{kj}\,\xi_k\,\xi_j \geq c_1\,|\xi|^2,\ \mu_{kj}\,\xi_k\,\xi_j \geq c_2\,|\xi|^2,\ \eta_{kj}\,\xi_k\,\xi_j \geq c_3\,|\xi|^2, \tag{2.2}$$

for all $\xi_{kj} = \xi_{jk} \in \mathbb{R}$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, where c_0, c_1, c_2, and c_3 are positive constants. More careful analysis related to the positive definiteness of the potential energy and thermodynamical laws insure positive definiteness of the matrix

$$\Xi := \begin{bmatrix} [\,\varkappa_{kj}\,]_{3\times 3} & [\,a_{kj}\,]_{3\times 3} & [\,p_j\,]_{3\times 1} \\ [\,a_{kj}\,]_{3\times 3} & [\,\mu_{kj}\,]_{3\times 3} & [\,m_j\,]_{3\times 1} \\ [\,p_j\,]_{1\times 3} & [\,m_j\,]_{1\times 3} & \gamma \end{bmatrix}_{7\times 7}. \tag{2.3}$$

Further we introduce the following generalized stress operator

$$\mathcal{T}(\partial, n) := \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times 3} & [e_{lrj}\,n_j\,\partial_l]_{3\times 1} & [q_{lrj}\,n_j\,\partial_l]_{3\times 1} & [-\lambda_{rj}\,n_j]_{3\times 1} \\ [-e_{jkl}\,n_j\,\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & -p_j\,n_j \\ [-q_{jkl}\,n_j\,\partial_l]_{1\times 3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & -m_j\,n_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l \end{bmatrix}_{6\times 6}.$$

Evidently, for a six vector $U := (u, \varphi, \psi, \vartheta)^\top$ we have

$$\mathcal{T}(\partial, n)\,U = (\,\sigma_{1j}\,n_j,\ \sigma_{2j}\,n_j,\ \sigma_{3j}\,n_j,\ -D_j\,n_j,\ -B_j\,n_j,\ -q_j\,n_j)^\top. \tag{2.4}$$

The components of the vector $\mathcal{T}U$ given by (2.4) have the physical sense: the first three components correspond to the mechanical stress vector in the theory of thermo-electro-magneto-elasticity, the forth, fifth and sixth ones are respectively the normal components of the electric displacement vector, magnetic induction vector and heat flux vector with opposite sign.

From the above equations we derive the following *equations of statics*:

$$A(\partial)\,U(x) = \Phi(x)\,, \tag{2.5}$$

where $U = (u_1, \cdots, u_6)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought for vector function and $\Phi = (\Phi_1, \cdots, \Phi_6)^\top := (-X_1, -X_2, -X_3, -\varrho_e, 0, -Q)^\top$ is a given vector function; $A(\partial) = [A_{pq}(\partial)]_{6\times 6}$ is the matrix differential operator

$$A(\partial) = \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l]_{3\times 3} & [e_{lrj}\,\partial_j\partial_l]_{3\times 1} & [q_{lrj}\,\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\,\partial_j]_{3\times 1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l & -p_j\,\partial_j \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l & -m_j\,\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,\partial_j\,\partial_l \end{bmatrix}_{6\times 6}.$$

From the symmetry conditions (2.1), inequalities (2.2) and positive definiteness of the matrix (2.3) it follows that $A(\partial)$ is a nonselfadjoint strongly elliptic operator.

2.2. Green's formulas. Let Ω^+ be a bounded 3-dimensional domain in $\mathbb{R}^3$ with a smooth boundary $S = \partial\Omega^+$. Throughout the paper we assume that the origin of the co-ordinate system belongs to Ω^+. Assume that the domain $\overline{\Omega^+}$ is filled with an anisotropic homogeneous material with the above described thermo-electro-magneto-elastic properties. By L_2, W_2^s and H_2^s with $s \in \mathbb{R}$ we denote the well-known Lebesgue, Sobolev-Slobodetski and Bessel potential function spaces, respectively (see, e.g., [9]). For arbitrary vector-functions $U \in \big[C^2(\overline{\Omega^+})\big]^6$ and $U' \in \big[C^2(\overline{\Omega^+})\big]^6$ we can derive the following Green's identity:

$$\int_{\Omega^+} \big[A(\partial)\, U \cdot U' + \mathcal{E}(U,U')\big]\, dx = \int_{\partial\Omega^+} \{\mathcal{T}(\partial,n)U\}^+ \cdot \{U'\}^+ dS\,, \tag{2.6}$$

where the symbol $\{\,\cdot\,\}^+$ denotes the one sided limit (the trace operator) on $\partial\Omega^+$ from Ω^+, $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outward unit normal vector with respect to Ω^+ at the point $x \in \partial\Omega^+$, the central dot denotes the scalar product and

$$\begin{aligned} &\mathcal{E}(U,U') = c_{rjkl}\, \partial_l u_k\, \partial_j u'_r + e_{lrj}\, (\partial_l\varphi\, \partial_j u'_r - \partial_j u_r\, \partial_l \varphi') \\ &+ q_{lrj}\, (\partial_l\psi\, \partial_j u'_r - \partial_j u_r\, \partial_l\psi') + \varkappa_{jl}\, \partial_l\varphi\, \partial_j\varphi' + a_{jl}\, (\partial_l\varphi\, \partial_j\psi' + \partial_j\psi\, \partial_l\varphi') \\ &+ \mu_{jl}\, \partial_l\psi\, \partial_j\psi' - \vartheta\, \partial_j u'_k) + \vartheta\, \partial_l\varphi') + \vartheta\, \partial_l\psi') + \eta_{jl}\, \partial_l\vartheta\, \partial_j\vartheta'\,. \end{aligned} \tag{2.7}$$

Remark that the above Green's formula by standard limiting procedure can be generalized to Lipschitz domains and to vector-functions $U \in [W_2^1(\Omega^+)]^6$ and $U' \in [W_2^1(\Omega^+)]^6$ with $A(\partial)U \in \big[L_2(\Omega^+)\big]^6$.

With the help of this Green's formula we can correctly determine a *generalized trace vector* $\{\mathcal{T}(\partial,n)U\}^+ \in [H_2^{-1/2}(\partial\Omega^+)]^6$ for a function $U \in [W_2^1(\Omega^+)]^6$ with $A(\partial)U \in [L_2(\Omega^+)]^6$ by the following relation

$$\langle\, \{\mathcal{T}(\partial,n)U\}^+\,,\, \{U'\}^+ \rangle_{\partial\Omega^+} := \int_{\Omega^+} [\, A(\partial)\, U \cdot U' + \mathcal{E}(U,\overline{U'})\,]\, dx, \tag{2.8}$$

where $U' \in [W_2^1(\Omega^+)]^6$ is an arbitrary vector-function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ denotes the duality between the function spaces $[H_2^{-\frac{1}{2}}(\partial\Omega^+)]^6$ and $[H_2^{\frac{1}{2}}(\partial\Omega^+)]^6$ which extends the usual L_2 scalar product.

2.3. Formulation of boundary value problems. As above let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a smooth simply connected boundary $S = \partial\Omega^+$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domains $\overline{\Omega^\pm}$ are filled with an anisotropic homogeneous material with the above described thermo-electro-magneto-elastic properties. The symbols $\{\,\cdot\,\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$, while $n = (n_1, n_2, n_3)$ stands for the outward unit normal vector on S with respect to Ω^+. Further, let S_D and S_N denote two disjoint sub-manifolds of S such that $S = \overline{S}_D \cup \overline{S}_N$. Put $\partial S_D = \partial S_N =: \ell_m$. In what follows, for simplicity we assume that S, S_D, S_N, ℓ_m are Lipschitz if not otherwise stated. Now we formulate the basic interior and exterior boundary value problems of the thermo-electro-magneto-elasticity theory.

Dirichlet problem $(D)^\pm$: *Find a solution vector* $U = (u,\varphi,\psi,\vartheta)^\top \in [W_2^1(\Omega^+)]^6$ (*respectively* $U = (u,\varphi,\psi,\vartheta)^\top \in [W^1_{2,\,loc}(\Omega^-)]^6$) *to the system of equations of statics*

$$A(\partial)\, U = \Phi \quad \text{in} \quad \Omega^\pm \tag{2.9}$$

satisfying the Dirichlet type boundary condition

$$\{U\}^{\pm} = g \quad on \quad S. \tag{2.10}$$

Neumann problem $(N)^{\pm}$: *Find a solution vector* $U = (u, \varphi, \psi, \vartheta)^{\top} \in [W_2^1(\Omega^+)]^6$ *(respectively* $U = (u, \varphi, \psi, \vartheta)^{\top} \in [W^1_{2,\,loc}(\Omega^-)]^6$*) to the system of equations* (2.9) *satisfying the Neumann type boundary condition*

$$\{\mathcal{T}(\partial, n)U\}^{\pm} = G \quad on \quad S. \tag{2.11}$$

Mixed problem $(M)^{\pm}$: *Find a solution vector* $U = (u, \varphi, \psi, \vartheta)^{\top} \in [W_2^1(\Omega^+)]^6$ *(respectively* $U = (u, \varphi, \psi, \vartheta)^{\top} \in [W^1_{2,\,loc}(\Omega^-)]^6$*) to the system of equations* (2.9) *satisfying the mixed Dirichlet-Neumann type boundary conditions*

$$\{U\}^{\pm} = g^{(D)} \quad on \quad S_D, \qquad \{\mathcal{T}(\partial, n)U\}^{\pm} = G^{(N)} \quad on \quad S_N. \tag{2.12}$$

We require that the data involved in the above setting possess the natural smoothness properties associated with the trace theorems, more precisely, we assume that $\Phi \in [L_2(\Omega^+)]^6$, $\Phi \in [L_{2,\,comp}(\Omega^-)]^6$, $g \in [H_2^{\frac{1}{2}}(S)]^6$, $G \in [H_2^{-\frac{1}{2}}(S)]^6$, $g^{(D)} \in [H_2^{\frac{1}{2}}(S_D)]^6$, $G^{(N)} \in [H_2^{-\frac{1}{2}}(S_N)]^6$. Thus, in the case of the exterior problems the right hand side vector in the differential equation (2.9) is assumed to be compactly supported. In addition, in this case we have to require some decay conditions for the components of solution vectors and its derivatives at infinity. These asymptotic conditions will be specified later.

3. Uniqueness theorems for interior static problems

Note that the differential equation for the temperature function and the corresponding boundary conditions are decoupled and we obtain a separated BVPs for ϑ, since $[A(\partial)U]_6 = \eta_{jl}\partial_j\partial_l\vartheta$ and $\{\mathcal{T}(\partial, n)U\}_6^+ = \{\eta_{jl}n_j\partial_l\vartheta\}^+$.

Theorem 3.1 *The homogeneous boundary value problems of statics* $(D)^+$ *and* $(M)^+$ *have only the trivial solution in the space* $\left[W_2^1(\Omega^+)\right]^6$.

Proof. Let $U = (u, \varphi, \psi, \vartheta)^{\top}$ be a solution to the homogeneous BVP $(M)^+$. Then ϑ solves the following decoupled mixed BVP

$$\eta_{jl}\,\partial_j\partial_l\vartheta = 0 \text{ in } \Omega^+, \quad \{\vartheta\}^+ = 0 \text{ on } S_D, \quad \{\eta_{jl}\ n_j\,\partial_l\vartheta\}^+ = 0 \text{ on } S_N. \tag{3.13}$$

By Green's formula

$$\textstyle\int_{\Omega^+} \eta_{jl}\,\partial_l\vartheta\,\partial_j\vartheta\,dx = \langle\{\eta_{jl}\ n_j\,\partial_l\vartheta\}^+\,, \{\vartheta\}^+\rangle_S \tag{3.14}$$

and with the help of the homogeneous boundary conditions we derive $\vartheta = const$ in Ω^+, since the right hand side duality expression in (3.14) vanishes and the matrix

$[\eta_{jl}]_{3\times3}$ is positive definite. Consequently, $\vartheta = 0$ in Ω^+ due to the homogeneous Dirichlet condition in (3.13). Therefore, the five dimensional vector $V = (u, \varphi, \psi)^\top$, constructed by the first five components of the solution vector U, solves the following homogeneous mixed BVP

$$\widetilde{A}(\partial)V = 0 \text{ in } \Omega^+, \quad \{V\}^+ = 0 \text{ on } S_D, \quad \{T(\partial, n)V\}^+ = 0 \text{ on } S_N, \tag{3.15}$$

where $\widetilde{A}(\partial) = [\widetilde{A}_{pq}(\partial)]_{5\times5}$ is the 5×5 differential operator of statics of the electro-magneto-elasticity theory without taking into account thermal effects

$$\widetilde{A}(\partial) := \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l]_{3\times3} & [e_{lrj}\,\partial_j\partial_l]_{3\times1} & [q_{lrj}\,\partial_j\partial_l]_{3\times1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l \end{bmatrix}_{5\times5}, \tag{3.16}$$

and $T(\partial, n) = \big[\, T_{pq}(\partial, n)\, \big]_{5\times5}$ is the corresponding 5×5 generalized stress operator

$$T(\partial, n) = \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times3} & [e_{lrj}\,n_j\,\partial_l]_{3\times1} & [q_{lrj}\,n_j\,\partial_l]_{3\times1} \\ [-e_{jkl}\,n_j\,\partial_l]_{1\times3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l \\ [-q_{jkl}\,n_j\,\partial_l]_{1\times3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l \end{bmatrix}_{5\times5}. \tag{3.17}$$

In this case, Green's identity reads as

$$\int_{\Omega^+} [\,\widetilde{A}(\partial)\,V \cdot V' + \widetilde{\mathcal{E}}(V, V')\,]\,dx = \big\langle \{TV\}^+,\, \{V'\}^+ \big\rangle_{\partial\Omega^+}, \tag{3.18}$$

$$\begin{aligned}\widetilde{\mathcal{E}}(V, V') = {} & c_{rjkl}\,\partial_l u_k\,\partial_j u'_r + e_{lrj}\,(\partial_l\varphi\,\partial_j u'_r - \partial_j u_r\,\partial_l\varphi') + \mu_{jl}\,\partial_l\psi\,\partial_j\psi' \\ & + \varkappa_{jl}\,\partial_l\varphi\,\partial_j\varphi' + a_{jl}\,(\partial_l\varphi\,\partial_j\psi' + \partial_j\psi\,\partial_l\varphi') + q_{lrj}\,(\partial_l\psi\,\partial_j u'_r - \partial_j u_r\,\partial_l\psi')\,. \end{aligned} \tag{3.19}$$

From the above Green's formula with a solution V of the problem (3.15) and $V' = V$, with the help of the inequalities (2.2) and positive definiteness of the matrix (2.3) we conclude that

$$\widetilde{\mathcal{E}}(V, V) = c_{rjkl}\,\partial_l u_k\,\partial_j u_r + \varkappa_{jl}\,\partial_l\varphi\,\partial_j\varphi + 2\,a_{jl}\,\partial_l\varphi\,\partial_j\psi + \mu_{jl}\,\partial_l\psi\,\partial_j\psi = 0. \tag{3.20}$$

Consequently $\partial_j\varphi = 0$ and $\partial_j\psi = 0$ in Ω^+ for $j = 1, 2, 3$, and $c_{rjkl}\,\partial_l u_k\,\partial_j u_r = 0$ in Ω^+. The general solution of the last equation is a rigid displacement vector $\chi(x) = a \times x + b$, where $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$ are arbitrary real constant vectors and the symbol " $\times$ " denotes the cross product. Therefore, from (3.20) it follows that $u(x) = \chi(x)$, $\varphi = b_4$, $\psi = b_5$, where $\chi(x)$ is an arbitrary rigid displacement vector and b_4 and b_5 are arbitrary real constants. Now, the homogeneous Dirichlet condition in (3.15) implies $u = 0$, $\varphi = 0$, and $\psi = 0$ in Ω^+, which proves the uniqueness theorem for the homogenous problem $(M)^+$. It is clear that the proof for the problem $(D)^+$ is word for word. □

Further, we analyse the homogenous Neumann type boundary value problem $(N)^+$. Let a vector $U = (u, \varphi, \psi, \vartheta)^\top$ solve the homogenous problem $(N)^+$. In this case the temperature function ϑ solves the following decoupled problem

$$\eta_{jl}\,\partial_j\partial_l\vartheta = 0 \;\text{ in }\; \Omega^+, \quad \{\eta_{jl}\;n_j\,\partial_l\vartheta\}^+ = 0 \;\text{ on }\; S = \partial\Omega^+. \tag{3.21}$$

Whence, by (3.14), we get $\vartheta = b_6 = const$ in Ω^+. Therefore, the vector $V = (u, \varphi, \psi)^\top$ solves then the nonhomogeneous BVP

$$\widetilde{A}(\partial)V = 0 \ \text{ in } \ \Omega^+, \quad \{T(\partial, n)V\}^+ = b_6\, G^* \ \text{ on } \ S, \tag{3.22}$$

where $\widetilde{A}(\partial)$ and $T(\partial, n)$ are defined by (3.16) and (3.17), and G^* is the following five dimensional vector function

$$G^* = (\lambda_{1j}n_j,\, \lambda_{2j}n_j,\, \lambda_{3j}n_j,\, p_j\, n_j,\, m_j\, n_j)^\top. \tag{3.23}$$

Due to Green's formula (3.18) we easily derive that a solution to the BVP (3.22) is defined modulo the summand

$$\widetilde{V} = (\chi(x),\, b_4,\, b_5)^\top, \tag{3.24}$$

where $\chi(x)$ is an arbitrary rigid displacement vector and b_4 and b_5 are arbitrary real constants. Therefore, an arbitrary solution to the homogeneous Neumann BVP (3.22) is represented as $V = \widetilde{V} + b_6\, V^*$, where $\widetilde{V}$ is given by (3.24) and $V^* = (u^*, \varphi^*, \psi^*)^\top$ is a particular solution to the BVP

$$\widetilde{A}(\partial)V^* = 0 \ \text{ in } \ \Omega^+, \quad \{T(\partial, n)V^*\}^+ = G^* \ \text{ on } \ S, \tag{3.25}$$

with G^* defined by (3.23). Now, we show that the vector V^* can be constructed explicitly in terms of linear functions for arbitrary domain Ω^+. Namely, let

$$V^* = (u^*, \varphi^*, \psi^*)^\top, \quad u_k^* = \widetilde{b}_{kq}^* x_q,\ k = 1,2,3, \quad \varphi^* = \widetilde{c}_q^* x_q, \quad \psi^* = \widetilde{d}_q^* x_q, \tag{3.26}$$

where $\widetilde{b}_{kq}^* = \widetilde{b}_{qk}^*$, $\widetilde{c}_q^*$ and $\widetilde{d}_q^*$, $k, q = 1,2,3$, are unknown real coefficients. The vector V^* solves the differential equation (3.25). In view of (3.17) the boundary condition in (3.25) leads to the equations

$$\begin{aligned}
&c_{rjkl}\, n_j\, \widetilde{b}_{kl}^* + e_{lrj}\, n_j\, \widetilde{c}_l^* + q_{lrj}\, n_j\, \widetilde{d}_l^* = \lambda_{rj}\, n_j, \quad r = 1,2,3,\\
&-e_{jkl}\, n_j\, \widetilde{b}_{kl}^* + \varkappa_{jl}\, n_j\, \widetilde{c}_l^* + a_{jl}\, n_j\, \widetilde{d}_l^* = p_j\, n_j,\\
&-q_{jkl}\, n_j\, \widetilde{b}_{kl}^* + a_{jl}\, n_j\, \widetilde{c}_l^* + \mu_{jl}\, n_j\, \widetilde{d}_l^* = m_j\, n_j.
\end{aligned} \tag{3.27}$$

Further, we equate the expressions which stand at the components n_j of the normal vector to obtain 12 linear equations with 12 unknown coefficients

$$\begin{aligned}
&c_{rjkl}\, \widetilde{b}_{kl}^* + e_{lrj}\, \widetilde{c}_l^* + q_{lrj}\, \widetilde{d}_l^* = \lambda_{rj}, \quad r, j = 1,2,3,\\
&-e_{jkl}\, \widetilde{b}_{kl}^* + \varkappa_{jl}\, \widetilde{c}_l^* + a_{jl}\, \widetilde{d}_l^* = p_j, \quad j = 1,2,3,\\
&-q_{jkl}\, \widetilde{b}_{kl}^* + a_{jl}\, \widetilde{c}_l^* + \mu_{jl}\, \widetilde{d}_l^* = m_j, \quad j = 1,2,3.
\end{aligned} \tag{3.28}$$

Due to the first inequality in (2.2) and positive definiteness of the matrix (2.3), and since $\widetilde{b}_{kq}^* = \widetilde{b}_{qk}^*$, it follows that the homogeneous version of the system (3.28) possesses only the trivial solution, i.e., the determinant of the system is different from zero.

Therefore, the nonhomogeneous system (3.28) is uniquely solvable and we can define the twelve unknown coefficients $\widetilde{b}^*_{kq} = \widetilde{b}^*_{qk}$, $\widetilde{c}^*_q$ and $\widetilde{d}^*_q$, $k, q = 1, 2, 3$. It is clear that then the boundary conditions (3.27) are satisfied and, consequently, the vector V^* solves the BVP (3.25) for arbitrary domain Ω^+. Thus, we have the following uniqueness theorem.

Theorem 3.2 *A general solution to the homogeneous Neumann type boundary value problem of statics* $(N)^+$ *in the space* $\big[W^1_2(\Omega^+)\big]^6$ *reads as* $U = (\widetilde{V}, 0)^\top + b_6\,(V^*, 1)^\top$, *where* $\widetilde{V} = (a \times x + b,\, b_4,\, b_5)^\top$ *with* $a = (a_1, a_2, a_3)^\top$ *and* $b = (b_1, b_2, b_3)^\top$ *and* V^* *is given by* (3.26) *with coefficients* $\widetilde{b}^*_{kq} = \widetilde{b}^*_{qk}$, $\widetilde{c}^*_q$, $\widetilde{d}^*_q$, $k, q = 1, 2, 3$, *defined by the uniquely solvable system* (3.28), *and where* a_1, a_2, a_3, *and* $b_1, \cdots, b_6$ *are arbitrary real constants.*

4. Uniqueness theorems for exterior static problems

4.1. Auxiliary material. Here first we prove several technical lemmas.

Lemma 4.1 *Let* $U = (u_1, u_2, \cdots, u_N)^\top$ *be a bounded solution to the homogeneous differential equation*

$$L(\partial)U(x) = 0, \quad x \in \Omega^-, \tag{4.29}$$

where $\Omega^- \subset \mathbb{R}^3$ *is a complement of a bounded region* $\overline{\Omega^+}$ *with a compact boundary and* $L(\partial) = [L_{kj}(\partial)]_{N\times N}$ *is a strongly elliptic second order matrix differential operator with constant coefficients,* $L_{kj}(\partial) = \sum\limits_{p,q=1}^{3} a^{kj}_{pq}\partial_p\partial_q, \quad k, j = \overline{1, N}$.
Then $U(x) = C + \mathcal{O}(|x|^{-1})$ *as* $|x| \to +\infty$, *where* $C = (C_1, \cdots, C_N)^\top$ *is a constant vector.*

Proof. Let U be a bounded solution to equation (4.29) and $B(O, R)$ be a ball centered at the origin and radius R, such that $\overline{\Omega^+} \subset B(O, R)$. Clearly, $U \in [C^\infty(\Omega^-)]^N$ due to the ellipticity of the operator $L(\partial)$. Let $V = (v_1, \cdots, v_N)^\top \in [C^\infty(\mathbb{R}^3)]^N$ be a vector whose restriction on $\Omega^-_R := \Omega^- \setminus \overline{B(O, R)}$ coincides with U, i.e,

$$V(x) = U(x) \ \text{ for } \ x \in \Omega^-_R\,. \tag{4.30}$$

Due to (4.29) and (4.30) the vector V solves the nonhomogeneous differential equation

$$L(\partial)V(x) = \Phi(x), \quad x \in \mathbb{R}^3, \tag{4.31}$$

with $\Phi = (\Phi_1, \cdots, \Phi_N)^\top \in [C^\infty_{comp}(\mathbb{R}^3)]^N$ having a compact support, $\operatorname{supp}\Phi \subset \overline{B(O, R)}$. Keeping in mind that V is bounded, we can apply the generalized Fourier transform to equation (4.31) to obatin

$$L(-i\,\xi)\,\widehat{V}(\xi) = \widehat{\Phi}(\xi), \quad \xi \in \mathbb{R}^3, \tag{4.32}$$

where $\widehat{V} = \mathcal{F}[V]$ and $\widehat{\Phi} = \mathcal{F}[\Phi] \cap C^\infty(\mathbb{R}^3)$. This equation is understood in the sense of tempered distributions. Since $\det L(-i\,\xi) \neq 0$ for $\xi \neq 0$ and the entries of the inverse matrix $[L(-i\,\xi)]^{-1}$ are C^∞-smooth homogeneous functions of order -2 in $\mathbb{R}^3 \setminus \{0\}$, from (4.32) we conclude

$$\widehat{V}(\xi) = [L(-i\,\xi)]^{-1}\,\widehat{\Phi}(\xi) + \sum_{|\alpha|\leq M} C_\alpha\, \delta^{(\alpha)}(\xi), \tag{4.33}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $C_\alpha = (C_{\alpha,1}, \cdots, C_{\alpha,N})^\top$ are arbitrary constant vectors, M is a nonnegative integer, $\delta(\cdot)$ is Dirac's distribution and $\delta^{(\alpha)} = \partial^\alpha \delta$.

By applying the inverse Fourier transform to (4.33) we get

$$V(x) = \mathcal{F}^{-1}_{\xi\to x}\Big([L(-i\,\xi)]^{-1}\,\widehat{\Phi}(\xi)\Big) + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha. \tag{4.34}$$

Denote by $\Gamma_L(x)$ the fundamental matrix of the operator $L(\partial)$ whose entries are homogeneous functions of order -1,

$$\Gamma_L(x) := \mathcal{F}^{-1}_{\xi\to x}\Big([L(-i\,\xi)]^{-1}\Big), \quad \Gamma_L \in C^\infty(\mathbb{R}^3 \setminus \{0\}), \quad L(\partial)\Gamma_L(x) = \delta(x)\, I_N. \tag{4.35}$$

Then (4.34) can be rewritten as follows

$$V(x) = \mathcal{F}^{-1}_{\xi\to x}\mathcal{F}\Big(\Gamma_L * \Phi\Big) + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha = \Big(\Gamma_L * \Phi\Big)(x) + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha, \tag{4.36}$$

where $*$ denotes the convolution operator. Therefore,

$$V(x) = \int_{\mathbb{R}^3} \Gamma_L(x-y)\, \Phi(y)\, dy + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha. \tag{4.37}$$

Since $\operatorname{supp}\Phi \subset \overline{B(O,R)}$ is compact, the first summand in the right hand side in (4.37) decays at infinity as $\mathcal{O}(|x|^{-1})$. Then it follows that $C_\alpha = 0$ for $|\alpha| \geq 1$ due to boundedness of V at infinity. Finally, we get $V(x) = C + \mathcal{O}(|x|^{-1})$, where $C = (C_1, \cdots, C_N)^\top$ is an arbitrary constant vector. □

Lemma 4.2 *Let $L(\partial)$ be as in Lemma 4.1 and $P \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^N$ be an odd homogeneous vector function of order -2. Then the equation*

$$L(\partial)U(x) = P(x), \quad x \in \mathbb{R}^3 \setminus \{0\}, \tag{4.38}$$

possesses a unique homogeneous solution $U^{(0)} \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^N$ of zero order satisfying the condition

$$\int_{|x|=1} U^{(0)}(x)\, dS = 0. \tag{4.39}$$

Proof. From (4.38) by the Fourier transform we get $L(-i\,\xi)\widehat{U}(\xi) = \widehat{P}(\xi)$, $x \in \mathbb{R}^3$, where $\widehat{P}(\xi)$ is an odd homogeneous vector function of order -1, $\det L(-i\,\xi) \neq 0$ for $\xi \neq 0$ and the entries of the inverse matrix $[\,L(-i\,\xi)\,]^{-1}$ are even, C^∞-smooth homogeneous functions of order -2. This equation is understood in the sense of the space of tempered distributions and as in the proof of Lemma 4.1 we have

$$\widehat{U}(\xi) = [L(-i\,\xi)]^{-1}\,\widehat{P}(\xi) + \sum_{|\alpha|\leq M} C_\alpha\,\delta^{(\alpha)}(\xi) \tag{4.40}$$

with the same α, C_α and M as in (4.33). Note that the first summand in the right hand side is an odd homogeneous function of order -3 satisfying the condition $\int_{|\xi|=1}[L(-i\,\xi)]^{-1}\,\widehat{P}(\xi)\,dS = 0$. Therefore, we can regularize the first summand in the Principal Value (v.p.) sense. Then the corresponding inverse Fourier transform

$$U^{(0)}(x) := \mathcal{F}^{-1}_{\xi\to x}\Big(\text{v.p.}\,[L(-i\,\xi)]^{-1}\,\widehat{P}(\xi)\Big) \tag{4.41}$$

is a homogeneous vector function of order zero satisfying the condition

$$\int_{|x|=1} U^{(0)}(x)\,dS = 0. \tag{4.42}$$

Moreover, $U^{(0)} \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^N$ (see, e.g., [10], Assertion 2.13 and Theorem 2.16). Now, from (4.40) we get $U(x) = U^{(0)}(x) + \sum_{|\alpha|\leq M} C_\alpha\, x^\alpha$. Since U should be a homogeneous vector function of order zero satisfying condition (4.39) we conclude that $C_\alpha = 0$ for all α in view of (4.42), and $U(x) = U^{(0)}(x)$, which completes the proof. □

Lemma 4.3 *Let $L(\partial)$ be as in Lemma 4.1, $\Gamma_L(x)$ be the fundamental solution of the operator $L(\partial)$ defined by* (4.35)*, and $Q = (Q_1, Q_2, \cdots, Q_N)^\top \in [C^\infty(\overline{\Omega^-})]^N$ with $\partial^\alpha Q_j(x) = \mathcal{O}(|x|^{-3-|\alpha|})$ as $|x| \to \infty$, $j = \overline{1,N}$, for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Then the vector $V(x) = \int_{\Omega^-} \Gamma_L(x-y)\,Q(y)\,dy$, is a particular solution of the equation $L(\partial)U = Q$ in Ω^-. Moreover, $V \in [C^\infty(\Omega^-)]^N \cap [C^2(\overline{\Omega^-})]^N$ and $\partial^\alpha V(x) = \mathcal{O}(|x|^{-1-|\alpha|}\,\ln|x|\,)$ as $|x| \to \infty$ for any multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.*

Proof. The prof of the equation $L(\partial)V = Q$ in Ω^- and the inclusion $V \in [C^\infty(\Omega^-)]^N \cap [C^2(\overline{\Omega^-})]^N$ is standard. The estimate for $\partial^\alpha V(x)$ follows from the relation

$$|\partial^\alpha V(x)| \leq c_1 \int_{\Omega^-} \tfrac{1}{|x-y|\,|y|^{3+|\alpha|}}\,dy \leq c_1 \sum_{k=1}^{4} \int_{\Omega_k} \tfrac{1}{|x-y|\,|y|^{3+|\alpha|}}\,dy\,, \tag{4.43}$$

where c_1 is a positive constant, $r = |x|$ is sufficiently large and $\Omega_1 = \Omega^- \cap B(O, \frac{r}{2})$, $\Omega_2 = \Omega^- \cap B(x, \frac{r}{2})$, $\Omega_3 = B(O, \frac{3r}{2}) \setminus [B(x, \frac{r}{2}) \cup \Omega_2]$, $\Omega_4 = \Omega^- \setminus B(O, \frac{3r}{2})$. We recall that the origin of the coordinate system belongs to the domain Ω^+. □

Corollary 4.1 *Let $\Phi \in [L_{2,\,comp}(\Omega^-)]^N$ and $L(\partial)$, Ω^-, P, and Q be as in Lemmas 4.1-4.3. Further, let $U \in [W^1_{2,\,loc}(\Omega^-)]^N$ be a solution of the equation*

$$L(\partial)U(x) = P(x) + Q(x) + \Phi(x), \quad x \in \Omega^-, \tag{4.44}$$

satisfying the condition $U(x) = \mathcal{O}(1)$ as $|x| \to \infty$. Then U can be represented as $U(x) = C + U^{(0)}(x) + U^{(1)}(x)$, where $C = (C_1, \cdots, C_N)^\top$ is a constant vector, $U^{(0)}$ is given by (4.41) and $U^{(1)} \in [W^1_{2,\,loc}(\Omega^-)]^N \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp} \Phi)]^N$ possesses the following asymptotic at infinity $\partial^\alpha U^{(1)}(x) = \mathcal{O}(|x|^{-1-|\alpha|} \ln |x|)$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Proof. Let $\Gamma_L(x)$ be the fundamental matrix of the operator $L(\partial)$ defined by (4.35). Note that the Newtonian potential

$$N_{\Omega^-}(\Phi)(x) := \int_{\Omega^-} \Gamma_L(x-y)\,\Phi(y)\,dy = \int_{\Omega^- \cap \operatorname{supp}\,\Phi} \Gamma_L(x-y)\,\Phi(y)\,dy$$

belongs to $[W^2_{2,\,loc}(\Omega^-)]^N \cap [C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\,\Phi)]^N$, solves the equation $L(\partial)N_{\Omega^-}(\Phi) = \Phi$ in Ω^-, and at infinity has the property $\partial^\alpha N_{\Omega^-}(\Phi)(x) = \mathcal{O}(|x|^{-1-|\alpha|})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Then it is clear that the vector $U^*(x) := U^{(0)}(x) + N_{\Omega^-}(Q)(x) + N_{\Omega^-}(\Phi)(x)$ is bounded at infinity and solves the nonhomogeneous equation (4.44) by Lemmas 4.2-4.3. Now, Lemma 4.1 completes the proof. □

4.2. Asymptotic behaviour of the temperature field at infinity. As we have mentioned above, in the case of static problems the differential equation

$$A_{66}(\partial)\,\vartheta \equiv \eta_{jl}\partial_j\partial_l\vartheta = \Phi_6 \;\text{ in }\; \Omega^- \tag{4.45}$$

and the corresponding boundary conditions for temperature field are separated. Here the right hand side function Φ_6 has a compact support. Therefore, one can easily prove the corresponding uniqueness theorems for the homogenous BVPs for the temperature function $\vartheta \in W^1_{2,\,loc}(\Omega^-)$ satisfying the decay condition $\vartheta = o(1)$ at infinity. This decay condition automatically implies that $\partial^\alpha \vartheta(x) = \mathcal{O}(|x|^{-|\alpha|-1})$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

For such solutions to the differential equation (4.45) we have the following integral representation formula for the domain Ω^-

$$\vartheta(x) = \int_S \Gamma^{(0)}_{66}(x-y)\,\{\partial_{n(y)}\vartheta(y)\}^-\,dS_y - \int_S \partial_{n(y)}\Gamma^{(0)}_{66}(x-y)\,\{\vartheta(y)\}^-\,dS_y$$
$$+ \int_{\Omega^-} \Gamma^{(0)}_{66}(x-y)\,\Phi_6(y)\,dy, \quad x \in \Omega^-, \tag{4.46}$$

where $\partial_{n(y)} = \mathcal{T}_{66}(\partial_y, n(y)) = \eta_{kj} n_j(y) \partial_k$ denotes the co-normal derivative. Here $\Gamma^{(0)}_{66}(x)$ is the fundamental solution of the operator $A_{66}(\partial)$ defined by formula

$$\Gamma^{(0)}_{66}(x) = -\frac{\alpha_0}{4\,\pi\,(D\,x\cdot x)^{1/2}} = -\frac{\alpha_0}{4\,\pi\,[d_{kj}\,x_k\,x_j]^{1/2}}\,, \quad \alpha_0 = (\det D)^{1/2}, \tag{4.47}$$

where $D = D^\top = [d_{kj}]_{3\times 3}$ is the inverse to the positive definite matrix $[\eta_{kj}]_{3\times 3}$.

Applying (4.46) we derive the following asymptotic relation

$$\vartheta(x) = \frac{\theta_0}{(Dx \cdot x)^{1/2}} + \mathcal{O}(|x|^{-2}) \ \text{ as } \ |x| \to \infty, \tag{4.48}$$

where $D = [d_{kj}]_{3\times 3}$ is defined in (4.47), θ_0 is a real constant

$$\theta_0 = \lim_{|x|\to\infty} (D\,x \cdot x)^{1/2}\,\vartheta(x) = -\tfrac{\alpha_0}{4\,\pi}\Big[\int_S \{\partial_{n(y)}\vartheta(y)\}^- \,dS_y + \int_{\Omega_0} \Phi(y)\,dy\Big] \tag{4.49}$$

with α_0 as in (4.47) and $\Omega_0 = \operatorname{supp}\Phi_6 \subset \Omega^-$ being a compact. Note that (4.48) can be differentiated any times with respect to x_j, $j = 1, 2, 3$. In particular,

$$\partial_j\vartheta(x) = -\frac{\theta_0\, d_{jl}\, x_l}{(D\,x \cdot x)^{3/2}} + \mathcal{O}(|x|^{-3}) \ \text{ as } \ |x| \to \infty, \quad j = 1, 2, 3. \tag{4.50}$$

4.3. General uniqueness results. First, let us consider the exterior Dirichlet problem of statics of thermo-electro-magneto-elasticity:

$$A(\partial)U = \Phi \ \text{ in } \ \Omega^-, \quad \{U\}^- = g \ \text{ on } \ S = \partial\Omega^-, \tag{4.51}$$

where $U = (u, \varphi, \psi, \vartheta)^\top \in [W^1_{2,\,loc}(\Omega^-)]^6$ is a sought for vector and $\Phi = (\Phi_1, \cdots, \Phi_6)^\top \in [L_{2,\,comp}(\Omega^-)]^6$, $g = (g_1, \cdots, g_6)^\top \in [H_2^{\frac{1}{2}}(S)]^6$. Our goal is to establish asymptotic conditions at infinity which guarantee the uniqueness for the BVP (4.51).

For the temperature function ϑ we have the separated exterior Dirichlet problem

$$A_{66}(\partial)\vartheta = \eta_{kj}\partial_k\partial_j\vartheta = \Phi_6 \ \text{ in } \ \Omega^-, \quad \{\vartheta\}^+ = g_6 \ \text{ on } \ S = \partial\Omega^-, \tag{4.52}$$

$$\vartheta(x) = \mathcal{O}(|x|^{-1}) \ \text{ as } \ |x| \to \infty. \tag{4.53}$$

Then the BVP (4.52)-(4.53) is uniquely solvable for arbitrary Φ_6 and g_6, and there holds the asymptotic relation (4.48) with θ_0 given by (4.49). Since Φ_6 has a compact support, the temperature function ϑ is C^∞-smooth outside $\operatorname{supp}\Phi_6$.

Thus, assuming that the temperature function is known we can substitute it into the first five equations in (4.51). Then we obtain the following BVP for the unknown vector function $\widetilde{U} = (u, \psi, \varphi)^\top \in [W^1_{2,\,loc}(\Omega^-)]^5$

$$\widetilde{A}(\partial)\widetilde{U} = \widetilde{\Psi} + \widetilde{\Phi} \ \text{ in } \ \Omega^-, \quad \{\widetilde{U}\}^- = \widetilde{g} \ \text{ on } \ S = \partial\Omega^-, \tag{4.54}$$

where $\widetilde{\Phi} = (\Phi_1, \cdots, \Phi_5)^\top \in [L_{2,\,comp}(\Omega^-)]^5$, $\widetilde{g} = (g_1, \cdots, g_5)^\top \in [H_2^{\frac{1}{2}}(S)]^5$, and

$$\widetilde{\Psi} = (\lambda_{1j}\,\partial_j\vartheta,\ \lambda_{2j}\,\partial_j\vartheta,\ \lambda_{3j}\,\partial_j\vartheta,\ p_j\,\partial_j\vartheta,\ m_j\,\partial_j\vartheta)^\top \in [L_2(\Omega^-)]^5. \tag{4.55}$$

Note that $\widetilde{\Psi}$ has not a compact support and due to formulas (4.50), $\widetilde{\Psi}(x) = \theta_0\,\widetilde{P}(x) + \widetilde{Q}(x)$, where $\widetilde{Q} \in [L_2(\Omega^-)]^5 \cap C^\infty(\mathbb{R}^3 \setminus \operatorname{supp}\Phi_6)$ and $\widetilde{Q}(x) = \mathcal{O}(|x|^{-3})$ as $|x| \to \infty$, while $\widetilde{P}(x)$ is an odd, C^∞-smooth homogeneous vector function of order -2,

$$\widetilde{P}(x) = -\tfrac{1}{(D\,x,\,x)^{3/2}}\,(\lambda_{1j}\,d_{jl}\,x_l,\ \lambda_{2j}\,d_{jl}\,x_l,\ \lambda_{3j}\,d_{jl}\,x_l,\ p_j\,d_{jl}\,x_l,\ m_j\,d_{jl}\,x_l)^\top. \tag{4.56}$$

Therefore, it is easy to see that in a vicinity of infinity, more precisely, outside of $\operatorname{supp}\Phi$ the solution vector $\widetilde{U}$ of equation (4.54) is C^∞-smooth but we can not assume that $\widetilde{U}$ decays at infinity, in general.

Now, we establish asymptotic properties of $\widetilde{U}(x)$ as $|x| \to \infty$. To this end, let us consider the equation $\widetilde{A}(\partial)\widetilde{U} = \theta_0\,\widetilde{P}$ in $\mathbb{R}^3 \setminus \{0\}$, where θ_0 is given by (4.49). In view of (4.56) and in accordance with Lemma 4.2, this equation possesses a unique solution $\widetilde{W}^{(0)} \in [C^\infty(\mathbb{R}^3 \setminus \{0\})]^5$ in the space of zero order homogeneous vector functions satisfying the condition

$$\int_{|x|=1} \widetilde{W}^{(0)}(x)\, dS = 0. \tag{4.57}$$

This solution reads as (cf. (4.41))

$$\widetilde{W}^{(0)}(x) = \theta_0\,\widetilde{U}^{(0)}(x) \;\text{ with }\; \widetilde{U}^{(0)}(x) := \mathcal{F}^{-1}_{\xi\to x}\Big(\text{v.p.}\,[\widetilde{A}(-i\,\xi)]^{-1}\,\mathcal{F}\widetilde{P}(\xi)\Big)\,. \tag{4.58}$$

Equation (4.54) can be rewritten as

$$\widetilde{A}(\partial)\widetilde{U} = \theta_0\,\widetilde{P} + \widetilde{Q} + \widetilde{\Phi} \;\text{ in }\; \Omega^-, \tag{4.59}$$

and by Lemmas 4.1-4.3 and Corollary 4.1 we conclude that a solution of (4.59), which is bounded at infinity, has the form

$$\widetilde{U}(x) = C + \theta_0\,\widetilde{U}^{(0)}(x) + \widetilde{U}^*(x) \;\; x \in \Omega^-, \tag{4.60}$$

where $C = (C_1, \cdots, C_5)^\top$ is an arbitrary constant, $\widetilde{U}^{(0)}$ is given by (4.58) and satisfies the condition (4.57), $\widetilde{U}^* \in [W^1_{2,\,loc}(\Omega^-)]^5 \cap [C^\infty(\mathbb{R}^3 \setminus \text{supp}\ \Phi)]^5$ and $\partial^\alpha \widetilde{U}^*(x) = \mathcal{O}(|x|^{-1-|\alpha|}\ln|x|)$ as $|x| \to \infty$ for arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Along with the boundedness at infinity, if we require that the mean value of a solution vector $\widetilde{U}$ over the sphere $\Sigma(O, R)$ tends to zero as $R \to \infty$, i.e.,

$$\lim_{R\to\infty} \tfrac{1}{4\pi R^2} \int_{\Sigma(O,R)} \widetilde{U}(x)\, d\Sigma(O, R) = 0\,, \tag{4.61}$$

then the constant summand C in the formula (4.60) vanishes and we arrive at the following assertion.

Lemma 4.4 *Let* $\widetilde{U} \in [W^1_{2,\,loc}(\Omega^-)]^5$ *be a solution of equation* (4.59)*, i.e., equation* (4.54)*, which is bounded at infinity and satisfies the condition* (4.61).

Then $\widetilde{U}(x) = \theta_0\,\widetilde{U}^{(0)}(x) + \widetilde{U}^*(x)$, $x \in \Omega^-$, *where* $\widetilde{U}^{(0)}$ *is given by* (4.58) *and* $\widetilde{U}^*$ *is as in* (4.60).

Now, we are in a position to prove the following theorem.

Theorem 4.1 *The exterior Dirichlet boundary value problem* (4.51) *has at most one solution* $U = (u, \varphi, \psi, \vartheta)^\top$ *in the space* $[W^1_{2,\,loc}(\Omega^-)]^6$*, provided* $\vartheta(x) = \mathcal{O}(|x|^{-1})$ *as* $|x| \to \infty$*, and* $\widetilde{U} = (u, \varphi, \psi)^\top$ *is bounded at infinity and satisfies the condition* (4.61).

Proof. Let $U^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)}, \vartheta^{(1)})^\top$ and $U^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)}, \vartheta^{(2)})^\top$ be two solutions of the problem under consideration with properties indicated in the theorem. It is evident that the difference $V = (u', \varphi', \psi', \vartheta')^\top = U^{(1)} - U^{(2)}$ solves then the corresponding homogeneous problem.

Therefore, for the temperature function ϑ' we get the homogeneous Dirichlet problem of type (4.52) and since ϑ' satisfies the decay condition at infinity, it is identical zero in Ω^-. Consequently, the vector $\widetilde{V} = (u', \varphi', \psi')^\top$ is a solution of the homogeneous exterior Dirichlet problem

$$A^{(0)}(\partial)\widetilde{V} = 0 \ \text{ in } \ \Omega^-, \quad \{\widetilde{V}\}^- = 0 \ \text{ on } \ S = \partial\Omega^-. \tag{4.62}$$

Moreover, $\widetilde{V}$ satisfies the condition (4.61) with $\widetilde{V}$ for $\widetilde{U}$ since both vectors $\widetilde{U}^{(1)} = (u^{(1)}, \varphi^{(1)}, \psi^{(1)})^\top$ and $\widetilde{U}^{(2)} = (u^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top$ satisfy the same condition.

By Lemma 4.4 then $\widetilde{V}$ is representable in the form $\widetilde{V}(x) = \theta'_0\,\widetilde{U}^{(0)}(x) + \widetilde{V}^*(x)$, $x \in \Omega^-$, where $\widetilde{U}^{(0)}$ is given by (4.58), $\partial^\alpha \widetilde{V}^*(x) = \mathcal{O}(|x|^{-1-|\alpha|}\ln|x|)$ as $|x| \to \infty$ for arbitrary multi-index α and $\theta'_0 = \lim_{|x|\to\infty}(D\,x\cdot x)^{1/2}\,\vartheta'(x) = 0$ since $\vartheta' = 0$ in Ω^- (cf. (4.49)). Therefore, $\partial^\alpha \widetilde{V} = \mathcal{O}(|x|^{-1-|\alpha|}\ln|x|)$ as $|x| \to \infty$. For vectors satisfying these decay conditions, we can easily derive the following Green's formula (cf. (3.18))

$$\int_{\Omega^-}[\,\widetilde{A}(\partial)\,\widetilde{V}\cdot\widetilde{V} + \widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})\,]\,dx = -\big\langle\{T\widetilde{V}\}^-,\,\{\widetilde{V}\}^-\big\rangle_{\partial\Omega^-}\,, \tag{4.63}$$

where $T(\partial, n)$ is given by (3.17) and $\widetilde{\mathcal{E}}(\widetilde{V},\widetilde{V})$ as in (3.20). From (4.62) and (4.63) along with the inequalities (2.2) we get $u'(x) = a \times x + b$, $\varphi'(x) = b_4$, $\psi'(x) = b_5$, where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ are arbitrary constant vectors, and b_4 and b_5 are arbitrary constants. Now, in view of the decay conditions at infinity we arrive at the equalities $u'(x) = 0$, $\varphi'(x) = 0$ and $\psi'(x) = 0$ for $x \in \Omega^-$. Consequently, $U^{(1)} = U^{(2)}$ in Ω^-. □

The proof of the following theorem is word for word.

Theorem 4.2 *The exterior Neumann and mixed boundary value problems of statics of thermo-electro-magneto-elasticity have at most one solution* $U = (u, \varphi, \psi, \vartheta)^\top$ *in the space* $[W^1_{2,\,loc}(\Omega^-)]^6$*, provided* $\vartheta(x) = \mathcal{O}(|x|^{-1})$ *as* $|x| \to \infty$*, and* $\widetilde{U} = (u, \varphi, \psi)^\top$ *is bounded at infinity and satisfies the condition* (4.61)*.*

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