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The Moore-Penrose Inverse in Portfolio Selection of Exchange Rates

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Abstract

In this work we use the generalized inverse in the classical Portfolio Selection Problem, a portfolio consisting of 7 intraday exchange rates which is updated every minute. In this case, since the variance covariance matrix is very close to singular and numerically ill-conditioned, the problem is tackled in an efficient way to benefit from the numerical tractability of the Moore-Penrose inverse.

Keywords: Covariance matrix, Portfolio selection, Moore-Penrose inverse matrix, ill-conditioned matrix.

1. Introduction

The objective of this work is to present an application of the generalized inverse matrix to Portfolio Selection Problem (Markowitz (1952) [9]). A central issue in empirical finance with implications to portfolio selection problems is the invertibility of the covariance matrix of returns. Consider a universe of N exchange whose returns are distributed with mean vector μ and covariance matrix Σ. The classical problem of portfolio selection is, as defined by Markowitz:

$$\min_{w} w'\Sigma w$$

$$\begin{aligned} s.t. \quad & w'\mathbf{1} = 1 \\ & w'\mu = q, \end{aligned}$$

where $\mathbf{1}$ denotes a conformable vector of ones, signing we invest 100% of available cash in the portfolio and q is the expected rate of return that is required on the portfolio. The required rate of return is set by investors according to their preferences (see eg. Danthine and Donaldson [5]), while negative elements of w denote short positions.
The analytical solution of this problem is given by:

$$\hat{w} = \frac{C - qB}{AC - B^2}\Sigma^{-1}\mathbf{1} + \frac{qA - B}{AC - B^2}\Sigma^{-1}\mu \tag{1}$$

where $A = \mathbf{1}'\Sigma^{-1}\mathbf{1}, \quad B = \mathbf{1}'\Sigma^{-1}\mu, \quad C = \mu'\Sigma^{-1}\mu.$

This equation shows that optimal portfolio weights depend on the inverse of the covariance matrix. Difficulties arise when the covariance matrix estimator is not invertible, or numerically ill-conditioned, which means that inverting it amplifies estimation error tremendously. Various attempts to tackle this problem appear in the literature; see Bengtsson and Holst [2], Buser [3], Ledoit and Wolf [7], Lefoll [8]. One possible solution to get around this problem is to use the Moore-Penrose inverse. We show that by replacing the inverse of the sample covariance matrix by the pseudo-inverse into equation (1) yields well-defined portfolio weights and makes portfolio optimization attainable.

2. Theoretical framework

Let A be a $r \times m$ real matrix. When A is singular, then its unique generalized inverse $A^\dagger$ (known as the Moore-Penrose inverse) is defined. Denote by $\mathbb{R}^{r\times m}$ the linear space of all $r \times m$ real matrices. The generalized inverse $A^\dagger$ is the unique matrix that satisfies the following four Penrose equations:

$$AA^\dagger = (AA^\dagger)^*, \qquad A^\dagger A = (A^\dagger A)^*, \qquad AA^\dagger A = A, \qquad A^\dagger AA^\dagger = A^\dagger,$$

where A^* denotes the transpose matrix of A.

Let us consider the equation $Ax = b, A \in \mathbb{R}^{r\times m}, b \in \mathbb{R}^r$, where A is singular. If $b \notin R(A)$, then the equation has no solution. Therefore, instead of trying to solve the equation $\|Ax-b\| = 0$, we must find a vector u that minimizes the norm $\|Ax-b\|$. Note that the vector u is unique. So, in this case we consider the equation $Ax = P_{R(A)}b$, where $P_{R(A)}$ is the orthogonal projection on the range of A. Standard references on generalized inverses are the books of Ben-Israel and Greville [1], Campbell and Meyer [4] and Groetsch [6].
The following two propositions can be found in [6]:

Proposition 2.1 *Let $A \in \mathbb{R}^{r\times m}$ and $b \in \mathbb{R}^r, b \notin R(A)$. Then, for $u \in \mathbb{R}^m$, the following are equivalent:*

(i) $Au = P_{R(A)}b$

(ii) $\|Au - b\| \leq \| Ax - b\|, \forall x \in \mathbb{R}^m$

(iii) $A^*Au = A^*b$

Let $\mathbb{B} = \{u \in \mathbb{R}^m | A^*Au = A^*b\}$. This set of solutions (known as the set of the generalized solutions) is closed and convex, therefore, it has a unique vector with minimal norm.

Proposition 2.2 *Let $A \in \mathbb{R}^{r\times m}$ and $b \in \mathbb{R}^r, b \notin R(A)$, and the equation $Ax = b$. Then, if $A^\dagger$ is the generalized inverse of A, we have that $A^\dagger b = u$, where u is the minimal norm solution.*

We shall make use of the latter property of the Moore-Penrose Inverse to minimize the risk in portfolio selection. Under the Markowitz's problem, the conditions for minimizing the Lagrange equation, give the following system of equations in matrix form:

$$\Sigma w = -\frac{1}{2}(\lambda_1 \mu + \lambda_2 \mathbf{1}). \tag{2}$$

When the covariance matrix Σ is singular, the solution stated by Markowitz in equation (1) is no longer valid. Then, we can consider two cases:
In the case when this equation has infinite solutions, then the Moore-Penrose inverse gives the minimum norm solution between them.
In the case when equation (2) has no solution, then we propose a solution from the set of the generalized solutions, using the minimal norm solution among them.

In both cases from proposition 2.2, the minimum norm solution (i.e the optimal portfolio positions) in order to minimize $w'\Sigma w$ is

$$\hat{w} = -\frac{1}{2}(\lambda_1 \Sigma^\dagger \mu + \lambda_2 \Sigma^\dagger \mathbf{1}), \tag{3}$$

where λ_1 and λ_2 are the Lagrange multipliers. The uniqueness of the solution is due to the uniqueness of the Moore-Penrose Inverse.

In Pappas [10] it is shown that in the case of linearly constrained optimization of a quadratic form, like in this case, the use of the Moore-Penrose inverse gives the minimum for all vectors $x \in \mathcal{N}(\Sigma)^\perp = \mathcal{R}(\Sigma^*) = \mathcal{R}(\Sigma)$. In other words, in the case of a singular quadratic form the minimization takes place for all its non zero values. In addition, when the covariance matrix is close to singular or numerically ill-conditioned, the Moore-Penrose inverse gives much better and accurate results, while the use of the inverse matrix fails to compute the portfolio weights. The reason for this, is that the Moore-Penrose inverse calculation is done using the Singular Value Decomposition (SVD) of the matrix. SVD calculation algorithms are rather stable (i.e. error accumulates no faster than the condition number would seem to warrant), since this method is based on unitary matrices that do not increase the norm of the error matrix (see eg. Campbell and Meyer [4]). Moreover, no matrix inversion is involved and therefore the use of the Moore-Penrose inverse is much more accurate in the ill-conditioned case.

The analytical form of (3) gives the optimal portfolio weights under variance-covariance singularity:

$$\hat{w} = \frac{C - qB}{AC - B^2}\Sigma^\dagger \mathbf{1} + \frac{qA - B}{AC - B^2}\Sigma^\dagger \mu \tag{4}$$

where $A = \mathbf{1}'\Sigma^\dagger \mathbf{1}, \quad B = \mathbf{1}'\Sigma^\dagger \mu, \quad C = \mu'\Sigma^\dagger \mu$. Replacing these optimal weights on the portfolio variance equation, $\sigma_p^2 = \hat{w}'\Sigma\hat{w}$, we obtain the Efficient Portfolio Frontier

(EPF), that represents, in the mean-standard deviation space, the geometric location of the optimal portfolios (in terms of minimum variance) that can be formed for each level of required return q.

3. The dataset and the underlying problem

In this study, we consider a portfolio consisting of 7 intraday exchange rates which is updated every minute. The exchange rates are the following: Euro/ Swiss Franc (EURCHF), Euro/British Pound (EURGBP), Euro/Japanese Yen (EURJPY), Euro/US Dollar (EURUSD), British Pound/US Dollar (GBPUSD), US Dollar/Swiss Franc (USDCHF) and US Dollar/Japanese Yen (USDJPY). Our data cover from 21 October 2008, time 00.00 to 22 October 2008, time 23.55, a total of 2880 intervals.

Our target is to perform portfolio optimization intertemporally every minute, based on the available history of the last 5 minutes. That is, at every minute we form a new portfolio, recalculating the optimal portfolio weights, based on the returns observed the last 5 minutes. This pattern, although may seem rather intricate, is not far away from the conditions pertaining in the exchange rates market. In fact several players in these markets consider portfolios that are updated in even smaller intervals, while remarkable changes in volumes and variances take place within milliseconds. In such extreme conditions of uncertainty, the pattern of returns that are as close as possible to the current instance is of outmost importance. Thus the hypothesis to use the information incumbent in the past 5 minutes to form a new portfolio, does not seem unreasonable.

In this case however, the determinant of the corresponding covariance matrices (using a 5-minute interval) is very often equal to zero. We show that even when the covariance matrix Σ is singular, the Moore-Penrose inverse can give us satisfying results for optimal portfolio weights. In the present work, all the computations have been performed using Matlab programming language. All the data used in this paper are provided by Bloomberg.

4. Empirical analysis

4.1. Intertemporal Portfolio Optimization for a given rate of expected return

In this section we perform intertemporal portfolio optimization for a given level of expected return. That is, we start with the first 5 minutes and calculate the optimal portfolio variance for expected return 1.1428e-4 (i.e. 0.0114%), in 1 minute basis. To have a reasonable level of required return, this is calculated as the average of absolute 1 minute returns over all exchange rates. Note that generally the investor sets the required return according to its preferences. Then, after calculating the optimal weights $\hat{w}$ the portfolio variance is given by $\hat{w}'\Sigma\hat{w}$. To perform the intertemporal optimization at every step we remove a minute from our data and add the next one. In this way, we construct 2875 optimal portfolios. In many of these cases, the variance-covariance matrix is close to singular, or singular. This can be seen from Figure 1 where the determinant of Σ is often zero or negative.

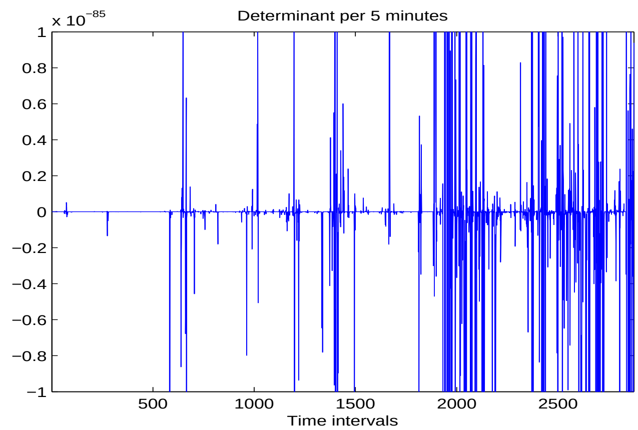

Figure 1: Time intervals vs the Determinant of the covariance matrix

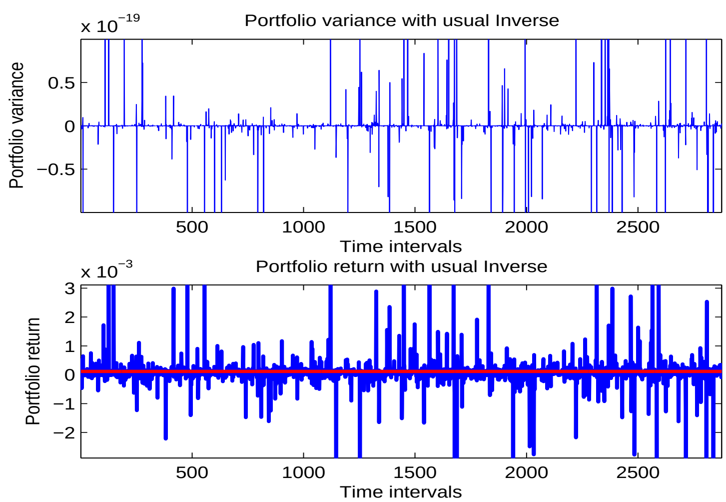

Figure 2: Time intervals vs the portfolio variance and expected return using the usual Inverse Σ^{-1} for expected 1 minute return 0.0114%

When using the regular inverse, we cannot perform the optimization in several instances. As can be seen in Figure 2, the produced portfolio variance is often either very large or even negative. Furthermore, the optimization fails to meet the level of 0.0114% required return (red line). However, making use of the Moore-Penrose Inverse in Figure 3, the target return is obtained while the portfolio variance takes reasonable values. Thus, portfolio optimization is attainable.

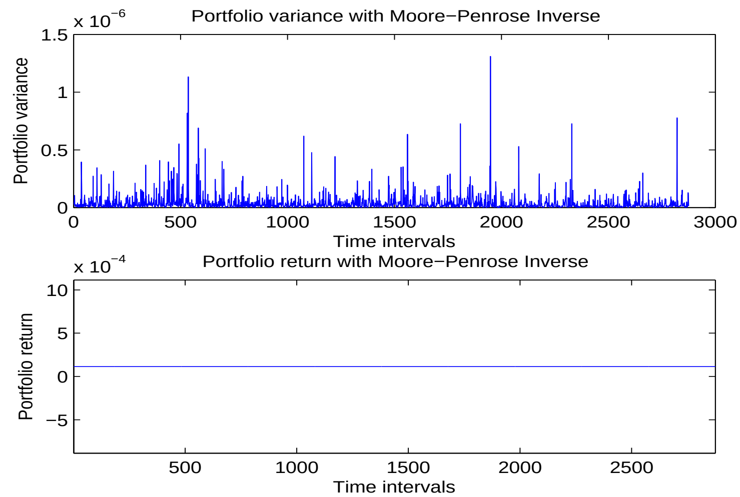

Figure 3: Time intervals vs the portfolio variance and expected return using the Moore-Penrose Inverse $\Sigma^{\dagger}$ for expected 1 minute return 0.0114%

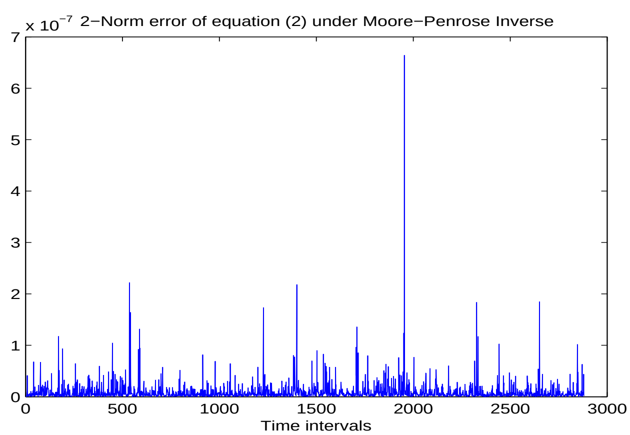

Figure 4: Error of equation (2) for each time interval

In addition we are examining the 2-norm error of equation (2) using the Moore-Penrose inverse only, since the usual inverse fails to perform the optimization. In

Figure 4, is provided the 2-norm error of equation (2) , that is, $\| \Sigma\hat{w} + \frac{1}{2}(\lambda_1\mu + \lambda_2\mathbf{1}) \|$ for each time interval. We observe that remains in reasonable levels (i.e. maximum value equals 6.6396e-7).

4.2. **Intertemporal Efficient Portfolio Frontier**

In this section we extend the analysis, where for each time interval we perform portfolio optimization for a vector of expected returns, so as to construct the Efficient Portfolio Frontier intertemporally. This cannot be performed when using the regular inverse, but only the Moon-Penrose Inverse. Furthermore, this enables us to show the applicability of the method to a more broad range of singular and close to singular matrices.

We consider for each time instance a vector of 100 required returns (taken at equal distances), with values from $1e-4$ up to $2e-4$ and calculate the optimal portfolio weights for each of these cases, so as to draw the EPF. The EPF produced for the first time interval (5 minutes) is given in Figure 5. The results produced seem logical, since the portfolio variance increases along with the expected return.

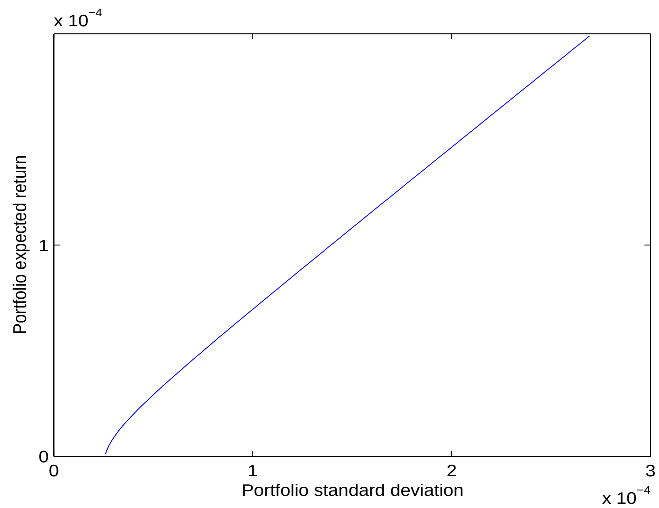

Figure 5: EPF for the first 5 minutes

The EPF for the first 10 time intervals, we perform the optimization, is given in Figure 6. The more bold the graph the more recent the EPF. Again the Moore-Penrose inverse enables us to perform all the optimizations efficiently. Finally the evolution of the EPF for all the time intervals in our sample (2875) is given in Figure 7. The maximum 2-norm error of equation (2) considering all required returns and time intervals equals 9.9358e-7.

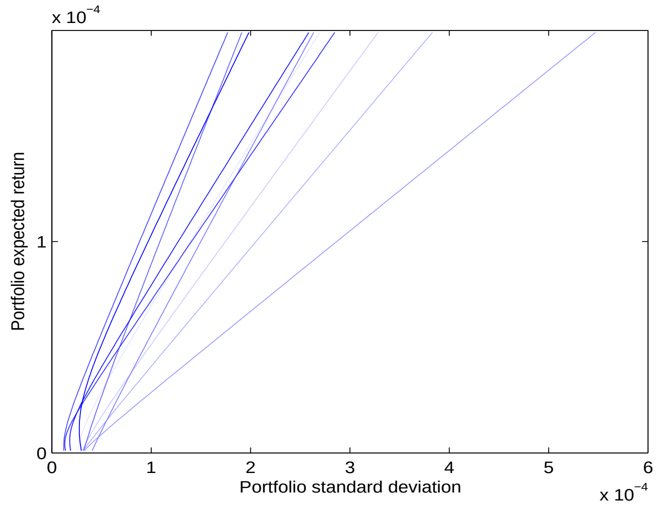

Figure 6: EPF for the first 10 time intervals

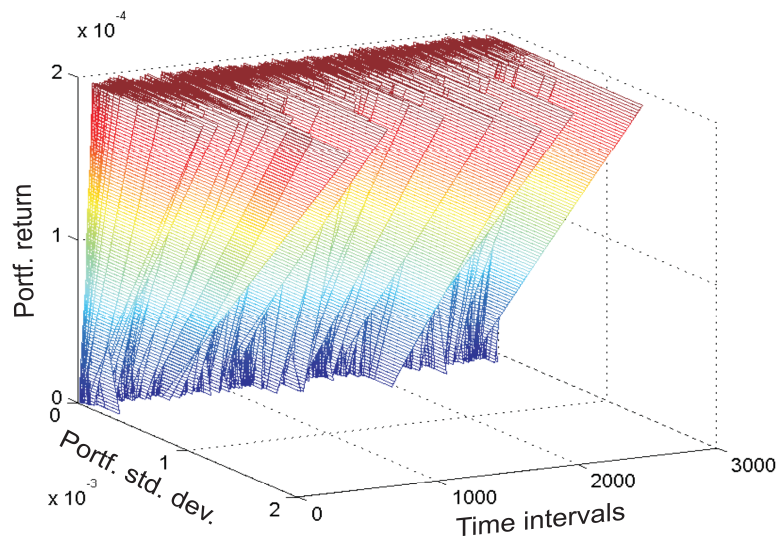

Figure 7: Intertemporal EPF for the whole sample

5. Concluding remarks

In this study we showed how portfolio optimization may take place when the variance-covariance matrix is close to singular or singular. Such a case is when the portfolio is updated in very short-time intervals and only the most recent historical information is used. The use of the Moore-Penrose inverse enables us to acquire the optimal weights, while satisfying the optimization constraints. This was illustrated by calculating the optimal portfolio weights over a sample of 2875 time-intervals, while moreover the EPF for each of these instances was constructed. Straightforward extentions of this method are the cases we include a risk-free asset, or given the EPF try to find the market protfolio according to the investor's expected utility function.

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