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Large amplitude internal waves

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Abstract

We consider here models for internal waves of large amplitude with emphasize on the "shallow-water/shallow-water" asymptotic model obtained in [5] from the two-layer system with rigid lid, for the description of large amplitude internal waves. For one-dimensional interfaces, this system is of hyperbolic type and its local wellposedness does not raise serious difficulties, though other issues (blow -up, loss of hyperbolicity,..) turn out to be delicate. For two- dimensional interfaces, the system turns out to be nonlocal. We prove that it conserves some properties of "hyperbolic type" and prove that the associated Cauchy problem is locally well posed in suitable Sobolev classes provided some natural restrictions are imposed on the data.

1. Introduction

The generation of gravity internal waves at the interface of two ocean layers of different densities (this stratification could be due to differences of salinity or temperature) has for a long time been recognized as a major problem in oceanography. As expressed by Helland-Hansen and Nansen in 1909,

> The knowledge of the exact nature and causes of these "waves" and their movements would, in our opinion, be of signal importance to Oceanography, and as far as we can see, it is one of the greatest problems that most urgently calls for a solution.

On the other hand, the mathematical theory of waves on the interface between two layers of immiscible fluid of different densities has attracted interest because it is the simplest idealization for internal wave propagation and because of the challenging modeling, mathematical and numerical issues that arise in the analysis of this system. The recent survey article of Helfrich and Melville [14] provides a rather extensive bibliography and a good overview of the properties of steady internal solitary waves in such systems as well as for more general density stratifications. The compendium [17] of field observations comprised of synthetic aperture radar (SAR) images of large-amplitude internal waves in different oceans together with associated physical data shows just how varied can be the propagation of internal waves. This variety is reflected in the mathematical models for such phenomena. Because of the range of scaling regimes that come to the fore in real environments, the literature on internal wave models is markedly richer in terms of different types of model equations than is the case for surface wave propagation (see, e.g. [3, 4] and the references therein).

We review in this article recent modelling and mathematical progress on the theory of *large* internal waves. In particular, we will describe a work in collaboration with Philippe Guyenne and David Lannes [13], which deals with the modelling of large internal gravity waves at the interface of two layers of incompressible fluids of different (constant) densities. We will study the Cauchy problem for an asymptotic system derived in [5] which generalizes the classical Saint-Venant (Shallow-water) system for surface water waves.

This system is one of those derived in [5], for a wide class of scaling regimes, to describe the propagation of internal waves at the interface of two layers of immiscible fluids of different densities under the assumption of flat bottom and rigid lid.

More precisely, we consider an homogeneous fluid of depth d_1 and density ρ_1 located on the top of another homogeneous fluid of depth d_2 and density $\rho_2 > \rho_1$. The two flows are potential; the top and bottom are rigid, flat and impenetrable. Surface tension effects are neglected, which is physically reasonable.

One can they reduce the Euler system (using Bernoulli formulation, natural boundary conditions and the fact that the interface is bording : the fluid particles are transported along it) to a system of evolution equations in the spatial domain $\mathbb{R}^d$, $d = 1, 2$, which involves two nonlocal operators, a classical Dirichlet-Neumann operator and an interface operator which couples the traces of the two layers potentials (see [5] for details).

The asymptotic models are obtained in expanding the nonlocal operators with respect to suitable small parameters depending on the wave amplitude, the wavelengths and the ratio of the two depths. The asymptotic models are shown to be *consistent* with the Euler system in a sense which will be precised later on.

In some details, denoting a a typical amplitude of the interface deformation and λ a typical horizontal wavelength, we inroduce adimensionned parameters

$$\gamma := \frac{\rho_1}{\rho_2} \in [0,1], \quad \delta := \frac{d_1}{d_2}, \quad \varepsilon := \frac{a}{d_1}, \quad \mu := \frac{d_1^2}{\lambda^2}.$$

It is always useful to introduce two other parameters ε_2 and μ_2 defned by

$$\varepsilon_2 = \frac{a}{d_2} = \varepsilon\delta, \qquad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

The paameters ε_2 and μ_2 correspond to ε et μ, with d_2 rather than d_1 taken as unit of length in the vertical direction.

The two layer system with rigid lid and flat bottom can be written on a nondimensional form involving the parameters γ, δ, ϵ and μ (see [5]). A key point is that this system can be reduced to a system of two evolution equations coupling the free surface ζ to the velocity potential of the upper layer evaluated at the interface namely ψ_1. Such a system generalizes the Zakharov, Craig-Sulem-Sulem formulation ([24, 10]) of the classical water waves. It writes :

$$\begin{cases} \partial_{\widetilde{t}}\widetilde{\zeta} - \frac{1}{\mu}G^{\mu}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 & = 0, \\ \partial_{\widetilde{t}}\big(\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1 - \gamma\nabla\widetilde{\psi}_1\big) + (1-\gamma)\nabla\widetilde{\zeta} & \\ \quad + \frac{\varepsilon}{2}\nabla\big(|\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]\widetilde{\psi}_1|^2 - \gamma|\nabla\widetilde{\psi}_1|^2\big) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\widetilde{\zeta},\widetilde{\psi}_1) & = 0, \end{cases} \tag{1}$$

where $\mathcal{N}^{\mu,\delta}$ is defined for all pairs (ζ,ψ) smooth enough by

$$\mathcal{N}^{\mu,\delta}(\zeta,\psi) := \mu\frac{\gamma\big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\nabla\psi\big)^2 - \big(\frac{1}{\mu}G^{\mu}[\zeta]\psi + \nabla\zeta\cdot\mathbf{H}^{\mu,\delta}[\zeta]\psi\big)^2}{2(1+\mu|\nabla\zeta|^2)}.$$

Here $G^{\mu}[\varepsilon\widetilde{\zeta}]$ is a Dirichlet-Neumann operator for the upper fluid while $\mathbf{H}^{\mu,\delta}[\varepsilon\widetilde{\zeta}]$ is the (nonlocal) interface operator which links the trace at the interface of the velocity potential for the upper layer to the trace on the interface of the velocity potential of the lower fluid.

The asymptotic models in [5] couple the elevation ζ of the interface to a variable $\mathbf{v}$ defined by

$$\mathbf{v} := \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1, \tag{2}$$

and which is essentially the difference of the horizontal component of the velocities of the two fluids at the interface and also the gradient of the second canonical variable in the hamiltonian formulation of the two layer system (1), (see [2, 9]).

Remark 1.1 Linearizing the equations (1) around the rest state $\widetilde{\zeta} = 0$, $\widetilde{\psi}_1 = 0$ one finds the equations

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}G^{\mu}[0]\psi_1 & = & 0, \\ \partial_t\big(\mathbf{H}^{\mu,\delta}[0]\psi_1 - \gamma\nabla\psi_1\big) + (1-\gamma)\nabla\zeta & = & 0. \end{cases}$$

Since $G^{\mu}[0]$ and $\mathbf{H}^{\mu,\delta}[0]$ are easily computed in Fourier variables , this leads to the dispersion relation

$$\omega^2 = (1-\gamma)\frac{|\mathbf{k}|}{\sqrt{\mu}}\frac{\tanh(\sqrt{\mu}|\mathbf{k}|)\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}{\tanh(\sqrt{\mu}|\mathbf{k}|) + \gamma\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}; \tag{3}$$

corresponding to plane waves solutions $e^{i\mathbf{k}\cdot X - i\omega t}$.

The linearization around the null solution is therefore well-posed in the d'Hadamard sense when $\gamma \leq 1$ but the linearization around a state presenting a discontinuity of the horizontal velocities at the interface leads, in absence of surface tension, to Kelvin-Helmholtz instabilities (see [7, 15]), possibly suppressed by surface tension (see for instance [16]).

A similar situation, (voir [7], §100), is that of plane shear flows with constant horizontal velocities U_1 et U_2. The plane interface develops instabilities for perturbations with wave numbers $\mathbf{k}$ such that (in nondimensional variables)

$$|\mathbf{k}| \geq k_{min} = \frac{g(\rho_2^2 - \rho_1^2)}{\rho_1\rho_2(U_2 - U_1)^2} \tag{4}$$

This formula shows in particular the stabilizing role of gravity on long waves.

Coming back to internal waves, one has in fact an apparent paradox. Kelvin-Helmholtz instabilities appear in the two-layer system in absence of surface tension. On the other hand, surface tension effects are physically negligeable in oceanographic and (often) experimental situations, while one observes in the ocean and in the laboratory "stable" structures (in he sense "observable on sufficienkly large time scales") which do not seem to display instabilities.

An answer to this paradox has been recently given by D. Lannes [19] who shows that the "stable" (in the sense "observable") part of the wave is stabilized by gravity. The low frequencies are thus stabilized by gravity while a (small) surface tension stabilizes the high frequencies but does not affect the "principal part" of the wave.

1.1. The asymptotic models

The following table from [5] summarize, in function of the parameters ϵ, μ et δ the different regimes for which one can derive asymptotic models.

	$\varepsilon = O(1)$	$\varepsilon \ll 1$
$\mu = O(1)$	Full equations	$\delta \sim 1$: FD/FD eq'ns
$\mu \ll 1$	$\delta \sim 1$: SW/SW $\delta^2 \sim \mu \sim \varepsilon_2^2$: SW/FD	$\mu \sim \varepsilon$ and $\delta^2 \sim \varepsilon$: B/FD $\mu \sim \varepsilon$ and $\delta \sim 1$: B/B $\delta^2 \sim \mu \sim \varepsilon^2$: ILW $\delta = 0$ and $\mu \sim \varepsilon^2$: BO

One recover here various classical systems of Boussinesq, KdV or Benjamin-Ono type but also many other ones. For all the asymptotic models one can prove the consistency in the following sense.

Definition 1.1 The internal wave equations (1) are *consistent* with a system S of $d+1$ equations for ζ and $\mathbf{v}$ if for all sufficiently smooth solutions (ζ, ψ_1) of (1) satisfying a non cavitation property, the pair $(\zeta, \mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1)$ solves S up to a small residual called the *precision* of the asymptotic model.

We will focus here to the **Shallow water/Shallow water (SW/SW)** regime, that is to say we suppose that $\mu \ll 1$, and $\mu_2 \ll 1$. In this regime we allow large amplitude waves relatively to the upper layer ($\varepsilon = O(1)$) and to the lower layer ($\varepsilon_2 = O(1)$). *To simplify we will take from now on $\varepsilon = 1$ (and thus $\varepsilon_2 = \delta$).*

The model writes in dimension $d = 1$,

$$\begin{cases} \partial_t\zeta + \partial_x\big(\dfrac{h_1 h_2}{\delta h_1 + \gamma h_2}\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\partial_x\zeta + \dfrac{1}{2}\partial_x\big(\dfrac{(\delta h_1)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2}\mathbf{v}^2\big) = 0, \end{cases} \tag{5}$$

where $h_1 = 1 - \zeta$ and $h_2 = 1 + \delta\zeta$ (this system is also formally derived in [9] and in a slightly different form in [8] but apparently it has not been studied previously as an hyperbolic system.

It is proved in [5] that system (1) is consistent with (5).

The two-dimensional version ($d = 2$) of (5) was established for the first time in [5]. It involves a nonlocal term and writes

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(h_1\mathfrak{R}[\zeta]\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \dfrac{1}{2}\nabla\Big(\big|\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2\Big) = 0, \end{cases} \tag{6}$$

where $h_1 = 1 - \zeta$, $h_2 = 1 + \delta\zeta$, and the operator $\mathfrak{R}[\zeta]$, which contains the nonlocal effects, is defined as follows.

Definition 1.2 Let $\gamma \in [0, 1)$, , $\delta > 0$ and $\zeta \in L^\infty(\mathbb{R}^d)$ such that

$$\big|\frac{1-\gamma}{\gamma+\delta}\delta\zeta\big|_\infty < 1.$$

The operator $\mathfrak{R}[\zeta]$ is then defined by

$$\mathfrak{R}[\zeta]: \begin{array}{lcl} L^2(\mathbb{R}^d)^d & \to & L^2(\mathbb{R}^d)^d \\ \mathbf{u} & \mapsto & \mathfrak{R}[\zeta]\mathbf{u} := \frac{1}{\gamma+\delta}\big(1-\Pi(\frac{1-\gamma}{\gamma+\delta}\delta\zeta\Pi\cdot)\big)^{-1}\Pi\big(h_2\mathbf{u}\big), \end{array}$$

where $h_2 = 1+\delta\zeta$, and $\Pi := \frac{\nabla\nabla^T}{\Delta}$ denotes the orthogonal projector in $L^2(\mathbb{R}^d)^d$ on gradient vector fields.

Remark 1.2 The hypothesis $\big|\frac{1-\gamma}{\gamma+\delta}\delta\zeta\big|_\infty < 1$ allows to define $(1-\Pi(\frac{1-\gamma}{\gamma+\delta}\delta\zeta\Pi\cdot))^{-1}$ by its Neumann series:

$$\mathfrak{R}[\zeta]\mathbf{u} = \frac{1}{\gamma+\delta}\sum_{n=0}^{\infty}\big(\Pi(\frac{1-\gamma}{\gamma+\delta}\delta\zeta\Pi\cdot)\big)^n\Pi(h_2\mathbf{u}). \tag{7}$$

Note also that when $d=1$, one has $\Pi = 1$ et

$$\mathfrak{R}[\zeta]\mathbf{u} = \frac{h_2}{\delta h_1+\gamma h_2}\mathbf{u},$$

so that (6) and (5) coïncide as expected.

In fact, the operator $\mathfrak{R}[\zeta]$ appears when one looks (by a BKW method after reduction to a flat domain by a sutable diffeomorphism) an approximate solution :

$$\underline{\Phi}_{app} = \Phi^{(0)} + \mu_2\Phi^{(1)},$$

to the linear elliptic problem which leads to the definition of $\mathbf{H}^{\mu,\delta}$.

Matching the boundary condition at $z=-1$ leads to the restriction

$$\nabla\cdot(h_2\nabla\Phi^{(0)}) = -\delta\nabla\cdot(h_1\nabla\psi_1),$$

which leads to $\Pi(h_2\nabla\Phi^{(0)}) = \Pi(-\delta h_1\nabla\psi_1)$.

Remark 1.3 One can shows that (see [5]) $\nabla\cdot(h_1\mathfrak{R}[\zeta]\mathbf{v}) = \nabla\cdot(h_2\mathbf{v})$ when $\gamma = 0$ and $\delta = 1$. The system (6) degenerates then into the Saint-Venant (shallow-water) system in $2D$ which does not involve any nonlocal terms. Those are therefore specific of *internal* waves. Moreover, V. Duchêne has recently shown [12] that there are a consequence of the rigid lid assumption. For a free upper surface, he gets a *local* hyperbolic (under suitable conditions) system which formally converges to (6) as the amplitude of the upper free surface tends to zero.

2. The Cauchy problem for the SW/SW system

2.1. The case $d=1$

Let $f(\zeta)$ be defined by

$$f(\zeta) = \frac{h_1h_2}{\delta h_1+\gamma h_2}. \tag{8}$$

One checks that (5) writes in the conservative form

$$\begin{cases} \partial_t\zeta + \partial_x\big(f(\zeta)\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\partial_x\zeta + \frac{1}{2}\partial_x\big(f'(\zeta)\mathbf{v}^2\big) = 0. \end{cases} \tag{9}$$

A simple but tedious computation shows on the other hand that (5) can be also be rewritten in the "quasilinear" form :

$$\partial_t U + A(U)\partial_x U = 0, \qquad U = (\zeta, \mathbf{v})^T, \tag{10}$$

with

$$A(U) = \begin{pmatrix} a(U) & b(\zeta) \\ c(U) & a(U) \end{pmatrix}$$

et

$$a(U) = f'(\zeta)\mathbf{v}, \qquad b(\zeta) = f(\zeta), \qquad c(U) = (1-\gamma) + \frac{1}{2}f''(\zeta)\mathbf{v}^2, \tag{11}$$

The two formulations are equivalent.
One checks that (10) is strictly hyperbolic provided

$$\begin{cases} \inf_{\mathbb{R}}(1-\zeta) > 0, \\ \inf_{\mathbb{R}}(1+\delta\zeta) > 0, \\ \inf_{\mathbb{R}}\big[1-\gamma(1+\delta\frac{(1+\delta)^2}{(\delta+\gamma-\delta(1-\gamma)\zeta)^3}\mathbf{v}^2)\big] > 0. \end{cases} \tag{12}$$

The first two conditions (no cavitation) are the exact counterparts of the hyperbolic condition of the Saint-Venant system. The condition may be seen as a "trace" of the possible Kelvin-Helmholtz instabilities in the two layer system (see [19] and [20]).
The following theorem is proven by using classical methods of quasiliner symmetrizable hyperbolic systems, but we nevertheless indicate the main steps of the proof which serve as a framework of the two-dimensuional case (see [13] for details).

Theorem 2.1 *Let $\delta > 0$ and $\gamma \in [0,1)$. Let also $t_0 > 1/2$, $s \geq t_0 + 1$ et $U^0 = (\zeta^0, \mathbf{v}^0)^T \in H^s(\mathbb{R})^2$ such that (12) is satisfied. Then*

— *There exist $T_{max} > 0$ and a unique maximal solution $U = (\zeta, \mathbf{v})^T \in C([0, T_{max}); H^s(\mathbb{R})^2)$ of (5) satisfying (12) on $[0, T_{max})$ and initial condition U^0;*

— *This solution satisfies the conservation of energy on $[0, T_{max})$:*

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}[(1-\gamma)\zeta^2 + v^2 f(\zeta)]dx = 0,$$

with $f(\zeta)$ as in (8).

— *Si $T_{max} < \infty$ then $\lim_{t\to T_{max}}|U(t)|_{W^{1,\infty}} = \infty$ or one of the three conditions of (12) ceases to be satisfied when $t \to T_{max}$.*

Proof.

Step 1. Construction of a regularized system of equations by truncating the large frequencies. Let χ a even function, smooth, compactly supported, defined on $\mathbb{R}$ with values into $[0,\infty)$, equal to 1 in a neighborhood of the origin. For all $\iota > 0$, one defines the operator χ_ι by

$$\chi_\iota = \chi(\iota D);$$

χ_ι is thus a smoothing operator sending continuously H^s into H^r for all $s, r \in \mathbb{R}$. The regularization of (5) is thus defined by

$$\partial_t U^\iota + \chi_\iota\big(A(U^\iota)\chi_\iota(\partial_x U^\iota)\big) = 0. \tag{13}$$

Since U^0 satisfies (12), the map $U \mapsto \chi_\iota(A(U)\chi_\iota(\partial_x U))$ is locally lipschitz in a neighborhood of U^0 in H^s, for all $s \geq t_0 > 1/2$. The existence/uniqueness of a maximal solution $U^\iota \in C([0,T^\iota); H^s)$ (with $T^\iota > 0$) of (13) with initial condition U^0 ad satisfying (12) is thus a direct consequence of Cauchy-Lipschitz theorem for ODE's in Banach spces.

Step 2. Choice of a d'un symmetrizer. A natural choice is :

$$S(U) = \begin{pmatrix} b(\zeta)^{-1} & 0 \\ 0 & c(U)^{-1} \end{pmatrix}.$$

Step 3. Energy estimates. One get them by multiplying the equation for $\widetilde{U} = \Lambda^s U^\iota$ (where $\Lambda^s = (I-\Delta)^{\frac{s}{2}}$ by $S(U^\iota,)$ using commutators estimates.

Step 4. Convergence of U^ι to a solution U by a standard compactness method (see *eg* [23], Chapter 16). The proof of the blow-up condition is also classical.

Step 5. The conservation of energy results from the hamiltonia structure of (9). In fact, setting

$$H(\zeta, v) = \frac{1}{2}\int_{\mathbb{R}} [(1-\gamma)\zeta^2 + v^2 f(\zeta)]dx,$$

one can write (9) on the hamiltonian form (which corresponds to (5.24) in [9])

$$\partial_t U + \mathfrak{J}\nabla H(U) = 0, \tag{14}$$

where $\mathfrak{J}$ is the skew-adjoint operator $\mathfrak{J} = \partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

□

We refer to [13] for more precise blow-up conditions. In particular one gets the following result (which has been observed in the numerical simulations of [13]) :

Corollary 2.1 *Under the hypothesis of Theorem 2.1, if the maximal existence time T_{max} is finite and if $\gamma > 0$ then:*

— $U = (\zeta, \mathbf{v})$ *is uniformly bounded on* $[0, T_{max}) \times \mathbb{R}$

— $\lim_{t \to T_{max}} |\partial_x U(t,\cdot)|_\infty = \infty$

It is possible that the height of one of the layers vanishes when $t \to T_{max}$. In this case, a supplementary information can be given on the blow-up of $\partial_x U(t,\cdot)$:

— *If* $\lim_{t\to T_{max}} \inf_{\mathbb{R}}(1-\zeta(t,\cdot))=0$ *then* $\lim_{t\to T_{max}} \sup_{\mathbb{R}} \partial_x \mathbf{v}(t,\cdot)=\infty.$

— *If* $\lim_{t\to T_{max}} \inf_{\mathbb{R}}(1+\delta\zeta(t\cdot))=0$ *then* $\lim_{t\to T_{max}} \inf_{\mathbb{R}} \partial_x \mathbf{v}(t,\cdot)=-\infty.$

Remark 2.1 It is of course possible to have a shock on the velocity without vanishing of the height of one of the layers. This scenario is also obseved in the numerical simulations.

Remark 2.2 We refer also to [13] for the proof that, under the hypothesis of Theorem 2.1, one has always $T_{max}<\infty$ for *most of* initial data $U^0\neq(0,0)$ with compact support. This results from the fact that the domain where the system is genuinely nonlinear is a "big" subset of the domain of strict hyperbolicity.

2.2. **The two-dimensional case**

We first summarize in a lemma useful properties of the operator $\mathfrak{R}[\zeta]$ (see [13] for proofs).

Lemma 2.1 *Let* $\gamma\in[0,1)$, $\delta>0$ *and* $t_0>1$. *Suppose also* $\zeta\in H^s(\mathbb{R}^2)$, *with* $s\geq t_0+1$, *and satisfies*

$$\inf_{\mathbb{R}}(1-|\zeta|_\infty)>0 \qquad \textit{and} \qquad \inf_{\mathbb{R}}(1-\delta|\zeta|_\infty)>0.$$

Then, for all $\mathbf{v}\in L^2(\mathbb{R}^2)^2$, *one has*

$$\nabla\cdot\mathfrak{R}[\zeta]\mathbf{v}=\delta\frac{\mathfrak{S}[\zeta]\mathbf{v}}{\delta h_1+\gamma h_2}\cdot\nabla\zeta+\frac{h_2}{\delta h_1+\gamma h_2}\nabla\cdot\mathbf{v}$$

and, for $j=1,2$,

$$\partial_j\big(\mathfrak{R}[\zeta]\mathbf{v}\big)=\delta\mathfrak{R}[\zeta]\big(\frac{\mathfrak{S}[\zeta]\mathbf{v}}{h_2}\partial_j\zeta\big)+\mathfrak{R}[\zeta]\partial_j\mathbf{v}.$$

Moreover, for all $\mathbf{v}\in L^2(\mathbb{R}^2)^2$,

$$|\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2})-\frac{1}{\delta h_1+\gamma h_2}\Pi\mathbf{v}|_2\leq C\big(\frac{1}{\gamma+\delta-\delta(1-\gamma)|\zeta|_\infty},\delta(1-\gamma)|\zeta|_{H^{t_0+1}}\big)|\Pi\mathbf{v}|_{H^{-1}}.$$

Remark 2.3 The first part of the lemma shows how the divergence and derivation operators act on $\mathfrak{R}[\zeta]$.

The second part shows that $\mathfrak{R}[\zeta]$ acts on gradient vector fields as a local operator modulo a more regular term. We recall that in one dimension, $\mathfrak{R}[\zeta](\frac{\mathbf{v}}{h_2})=\frac{1}{\delta h_1+\gamma h_2}\mathbf{v}$.

It is much more delicate when $d=2$ to put (6) on a quasilinear form because of the nonlocal term $\mathfrak{R}[\zeta]\mathbf{v}$. One can nevertheless write (6) on the form :

$$\partial_t U+A^j[U]\partial_j U=0, \qquad U=(\zeta,\mathbf{v})^T, \tag{15}$$

where

$$A^j[U] = \begin{pmatrix} a^j(U) & \mathbf{b}^j(U)^T \\ \mathbf{c}^j[U] & D^j[U] \end{pmatrix}, \qquad (j = 1, 2),$$

and

$$a^j(U) = (\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j - \gamma(\mathfrak{S}[\zeta]\mathbf{v})_j \frac{h_2}{\delta h_1 + \gamma h_2}, \tag{16}$$

$$\mathbf{b}^j(U) = \frac{h_1 h_2}{\delta h_1 + \gamma h_2}\mathbf{e}^j, \tag{17}$$

$$\mathbf{c}^j[U]\bullet = \mathbf{e}^j - \gamma\Big[\mathbf{e}^j + \delta(\mathfrak{S}[\zeta]\mathbf{v})_j \mathfrak{R}[\zeta]\big(\frac{\mathfrak{S}[\zeta]\mathbf{v}}{h_2}\bullet\big)\Big], \tag{18}$$

$$D^j[U]\bullet = (\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})_j \mathrm{Id}_{2\times 2} - \gamma(\mathfrak{S}[\zeta]\mathbf{v})_j\mathfrak{R}[\zeta]\bullet, \tag{19}$$

where

$$\mathfrak{S}[\zeta]\mathbf{v} = \mathbf{v} + (1-\gamma)\mathfrak{R}[\zeta]\mathbf{v} \tag{20}$$

One can then prove, using in particular Lemma 2.1 :

Proposition 2.1 (The case $d = 2$) *Let $T > 0$, $t_0 > 1$ et $s \geq t_0 + 1$. Let also $U = (\zeta, v) \in C([0,T]; H^s(\mathbb{R}^2)^3)$ be such that for all $t \in [0,T]$,*

$$(1 - |\zeta(t,\cdot)|_\infty) > 0 \quad \text{and} \quad (1 - \delta|\zeta(t,\cdot)|_\infty) > 0 \quad \text{et} \quad \text{curl } \mathbf{v}(t,\cdot) = 0.$$

Then U solves (6) if and only if U solves (15).

Remark 2.4 The system (15) is not *stricto sensu* a quasilinear system since $\mathbf{c}^j[U]$ (resp. $D^j[U]$) is not a $\mathbb{R}^2$ function (resp. a 2×2 matrices -valued function) but a linear operator which is defined on the space of $\mathbb{R}^2$ valued functions (resp. 2×2 matrices -valued functions). However, those operators are of order zero and, as will be seen later, (15) can be roughly treated as a qasilinear system.

On the other hand, one proves that a solution of (15) which is initially curl-free remains curl-free on its maximal existence time:

Proposition 2.2 *Let $T > 0$, $t_0 > 1$ et $s \geq t_0 + 1$. Let also $U = (\zeta, \mathbf{v}) \in C([0,T]; H^s(\mathbb{R}^2)^3)$ a solution of* (15) *such that curl $\mathbf{v}(0,.) = 0$. Then curl $\mathbf{v}(t,.) = 0$ for all $t \in [0,T]$.*

Proof. Using the relation curl $\mathfrak{R}[\zeta]\mathbf{v} = 0$, one checks that the variable $\mathbf{v}$ in system (6) satisfies the equation

$$\partial_t \mathbf{v} + \nabla F - (\text{curl } \mathbf{v})(\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})^\perp = 0, \tag{21}$$

where

$$F = (1 - \gamma\zeta) + \frac{1}{2}\Big(\big|\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2\Big).$$

Let $\omega = \text{curl } \mathbf{v}$. One shows then that ω satisfies

$$\partial_t \omega + \text{curl } [\omega(\mathbf{v} - \gamma\mathfrak{R}[\zeta]\mathbf{v})^T] = 0 \tag{22}$$

Multiplying by ω and integrating on $\mathbb{R}^2$ using thefact that $\nabla(\mathbf{v}-\gamma\mathfrak{R}[\zeta]\mathbf{v}) \in L^\infty([0,T]\times\mathbb{R}^2)$, one gets after integration by part

$$\frac{d}{dt}|\omega(t,.)|_2^2 \leq C\int_0^t |\omega(s,.)|_2^2 ds,$$

and one concludes by Gronwall's lemma. □

We turn now to the study of the Cauchy of the local Cauchy problem for the two-dimensional system SW/SW (6).

The following conditions generalize the hyperbolicity conditions of the one-dimensional case.

$$\begin{cases} 1-|\zeta|_\infty > 0, \\ 1-\delta|\zeta|_\infty > 0, \\ 1-\gamma-\gamma\delta\dfrac{|\mathfrak{S}[\zeta]\mathbf{v}|_\infty^2}{\gamma+\delta-\delta(1-\gamma)|\zeta|_\infty} > 0, \end{cases} \tag{23}$$

with $\mathfrak{S}[\zeta]\mathbf{v}$ as in (20).

The main result is the following.

Theorem 2.2 *Let $\delta > 0$ and $\gamma \in [0,1)$. Let also $t_0 > 1$, $s \geq t_0 + 1$ and $U^0 = (\zeta^0, \mathbf{v}^0)^T \in H^s(\mathbb{R}^2)^3$ such that (23) are satified and* curl $\mathbf{v}^0 = 0$. *Then there exist $T_{max} > 0$ and a unique maximal solution $U = (\zeta, \mathbf{v})^T \in C([0, T_{max}); H^s(\mathbb{R}^2)^3)$ of (6) with initial data U^0. Moreover, if $T_{max} < \infty$, then at least one of the following conditions holds :*

(i) $\lim_{t\to T_{max}} |U(t)|_{H^{t_0+1}} = \infty$

(ii) One of the three conditions of (23) is violated when $t \to T_{max}$.

Proof. The proof is modelled on that of quasilinear symmetrizable hyperbolic systems (see the one-dimensional case). We indicate the main steps, insisting on the specific difficulties of the dimension two (see [13] for details).

Step 1. Regularized equations. This step is standard, one obtains an approximate system in finite dimension by truncating the high frequencies as in one dimension.

Step 2. Choice of a symetrizer. One looks for $S[U]$ under the form

$$S[U] = \begin{pmatrix} s_1(U) & 0 \\ 0 & S_2[U] \end{pmatrix}, \tag{24}$$

with $s_1(\cdot) : H^s(\mathbb{R}^2)^3 \mapsto H^s(\mathbb{R}^2)$ and $S_2[U]$ linear operator from $L^2(\mathbb{R}^2)^2$ into itself. Defining $C[U]$ by

$$\forall \widetilde{\mathbf{v}} = (\widetilde{\mathbf{v}}_1, \widetilde{\mathbf{v}}_2)^T \in L^2(\mathbb{R}^2)^2, \qquad C[U]\widetilde{\mathbf{v}} = \mathbf{c}_1[U]\widetilde{\mathbf{v}}_1 + \mathbf{c}_2[U]\widetilde{\mathbf{v}}_2,$$

a natural generalization of the (local) case of the dimension one would consist in taking $s_1(U) = b(U)^{-1}$ and $S_2[U] = C[U]^{-1}$; unfortunately, such a choice is not possible since the operator $C[U]$ is not self-adjoint. It turns out however that $C[U]$

is self-adjoint (modulo a regularizing term) on the restriction of$L^2(\mathbb{R}^2)^2$ to gradient vector fields, as shows the lemma below.

We must first define the operator $C_1[U]$ by

$$C_1[U] = (1-\gamma)\mathrm{Id} + \frac{1}{2}\delta\gamma \begin{pmatrix} c_1[U] + c_1[U]^* & 0 \\ 0 & c_1[U] + c_1[U]^* \end{pmatrix}, \tag{25}$$

with $c_1[U] : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ given by

$$c_1[U] = \frac{1}{\delta h_1 + \gamma h_2}(2\mathfrak{S}_1\mathfrak{S}_2\Pi(\mathbf{e}^2\cdot)_1 + \mathfrak{S}_1^2\Pi(\mathbf{e}^1\cdot)_1 + \mathfrak{S}_2^2\Pi(\mathbf{e}^2\cdot)_2), \tag{26}$$

and where $\mathfrak{S}_j = (\mathfrak{S}[\zeta]\mathbf{v})_j$.

Let us state the

Lemma 2.2 *Let $t_0 > 1$ and $U = (\zeta, \mathbf{v}) \in H^{t_0+1}(\mathbb{R}^2)^3$ such that (**??**) is satisfied. One defines also $C_1[U]$ as in (25) and one sets $C_2[U] = C[U] - C_1[U]$. For all $\widetilde{\zeta} \in L^2(\mathbb{R}^2)$, one has*

$$|C_2[U]\nabla\widetilde{\zeta}|_2 \leq \mathfrak{c}(U)|\widetilde{\zeta}|_2.$$

One chooses now the coefficients $s_1[U]$ and $S_2[U]$ of the symmetrizer $S[U]$ given by (24) as follows

$$s_1(U) = b(U)^{-1}, \tag{27}$$

$$S_2[U] = C_1[U]^{-1}. \tag{28}$$

One then checks that $C_1[U]$ is invertible in $\mathcal{L}(L^2(\mathbb{R}^2)^2; L^2(\mathbb{R}^2))$.

The operator $S[U]$ would be a good "symmetrizer" if $S[U]A^j[U]$ $(j = 1, 2)$ were symmetric, which unfortunately is not the case. However one can prove that the operator $\Pi S[U]A^j[U]\Pi$, where Π denotes as previously the L^2 orthogonal projection on gradient vector fields, is symmetric at principal order.

Step 3. Energy estimates. They are delicate. One uses various properties of the coefficients of matrices $A^j[U]$ and comutator estimates (see [18], Theorem 6).

Step 4. Convergence of approximate solutions. One obtains first the convergence to the unique solution of (15) which turns out to be also the unique solution of (6) since one has supposed that curl $\mathbf{v}_0 = 0$.

Step 5. Blow-up condition. It results from a standard continuation argument.

□

Remark 2.5 We do not know if (6) possesses a conserved energy or an hamiltonian structure. The existence of solutions with finite lifespan, though highly expected ie also unknown.

3. Final remarks

We conclude this article by some concluding remarks and related problems.

3.1. Numerical simulations

As previously mentionned, one can find in [13] numerical simulations under periodic conditions in x and y, and using a pseudo-spectral method. This is a natural choice for the computation of the nonlocal operator $\mathfrak{R}[\zeta]$ since all terms in its Neumann series (7) is a concatenation of Fourier multipliers (through the operator Π) with ζ.

3.2. Extensions : free upper surface; surface tension effects

A natural extension, (very relevant in applications to ocean dynamics problems) of the previous results is to consider a free upper surface instead of a rigid lid and/or a non flat bottom (nontrivial bathymetry). Asymptotic models for internal waves with a non trivial bathymetry have been derived in [12] and [11].

The main effect of surface tension in the two-layer system is to prevent the formation of Kelvin-Helmholtz instabilities. For instance, in the related problem of horizontal shear flows (see [7] and the Introduction), if we denote by σ the surface tension coefficient, the flat interface for horizontal shear flows with constant horizontal velocities U_1 and U_2 does not develop instabilities for perturbations in the direction of streaming having wave vectors $\mathbf{k}$ such that (compare to (4)):

$$\frac{\rho_1\rho_2}{(\rho_1+\rho_2)^2}(U_1-U_2)^2 < g|\mathbf{k}|\{\frac{\alpha_2-\alpha_1}{|\mathbf{k}|}+\frac{|\mathbf{k}|\sigma}{g(\rho_1+\rho_2}\}. \tag{29}$$

In particular, no Kelvin-Helmholtz instabilities are present provided the surface tension is large enough to insure that

$$(U_1-U_2)^2 < \frac{2}{\alpha_1\alpha_2}\sqrt{\frac{\sigma g(\alpha_2-\alpha_1)}{\rho_1+\rho_2}}. \tag{30}$$

Coming back to the two-layer system (1), one checks that the surface tension adds a term $-\frac{\sigma}{\rho_2}\nabla K(\widetilde{\zeta})$ to the LHS of the second equation of (1), where $K(\widetilde{\zeta}) = (\frac{\nabla\widetilde{\zeta}}{\sqrt{1+|\nabla\widetilde{\zeta}|^2}})$.

In fact, in oceanographic applications, the surface tension effects are very weak and can be ignored when deriving the aforementionned asymptotic models (they appear as a lower order effect). In situations where they are small but of the order of the "small" parameters involved in the asymptotic expansions (the $\varepsilon's$ or the $\mu's$) one has to add a cubic "capillary" term to the equation for $\mathbf{v}$ (see [11] where the various regimes are systematically investigated). For instance, in the SW/SW regime, (6) has to be replaced by

$$\begin{cases} \partial_t\zeta + \nabla\cdot\big(h_1\mathfrak{R}[\zeta]\mathbf{v}\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \frac{1}{2}\nabla\Big(\big|\mathbf{v}-\gamma\mathfrak{R}[\zeta]\mathbf{v}\big|^2 - \gamma\big|\mathfrak{R}[\zeta]\mathbf{v}\big|^2\Big) - \varepsilon\sqrt{\mu}\nu\Delta\nabla\zeta = 0, \end{cases} \tag{31}$$

where $\nu = \frac{\sigma}{\rho_2\lambda^2}$.

3.3. The case of an upper free surface

As was previously mentioned, V. Duchêne [12] has derived asymptotic models in this situation specially in the SW/SW regime.

3.4. Rigorous justifiation of the SW/SW system

We have not so far fully justify the SW/SW system. The notion of consistency we have used is not a dynamical one. A good analogy is that of the finite difference approximation of the Cauchy problem for a given PDE. In this case consistency is a purely algebraic relation between the symbols of both the PDE and the numerical scheme while the stability deals with the dynamics of the dynamical system generated by the scheme. Provided the Cauchy problem is well posed, Lax equivalence theorem states that convergence is equivalent to consistency plus stability.

Similarly, to complete the justification of the asymptotic models, one should establish that the full system (possibly with surface tension added) has solutions in relevant time scales and that the various asymptotic models have solutions (also in relevant time scales). Finally one should prove the stability analysis of the asymptotic models (that is, an estimation of the remainders which comprise the difference between the full two-layer system and the models).

Such a program has been for a large part achieved in the case of surface waves by combining the results of B. Alvarez-Samaniego and D. Lannes with [1] and [4]. In the context of internal waves, the first result in that direction is by Ohi and Iguchi [22] who proved a local well-posedness result for the 1D problem with surface tension, the existence time shrinking to zero as the surface tension parameter tends to zero. They nevertheless derived rigorously the Benjamin-Ono equation in this situation.

On the other hand, in the recent work [19] we already alluded to, D. Lannes has shown the well-posedness of the full two-layer system with (possibly small) surface tension, allowing to justify some asymptotic models in "large" time scales. In particular he derives a *stability* criterion which generalizes the classical Taylor condition for surface waves. This stability criterion (or a *practical* version of it) ensures the existence of a "stable" solution of the two-layer system, that is a solution that exists on a time scale which does not vanish as the surface tension parameter tends to zero, and which is uniformly bounded from below with respect to the physical parameters ϵ and μ. This leads in particular to the *full justification* of the SW/SW system.

3.5. The rotational case

In [6] D. Bresch and M. Renardy have derived and study a two-dimensional SW/SW system without assuming the irrotationality of the flow.

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