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# Borg-Levinson theorem for magnetic Schrödinger operator

V. S. Serov

**Abstract**

We prove that the boundary spectral data, i.e. the Dirichlet eigenvalues and normal derivatives of the normalized eigenfunctions at the boundary uniquely determine the coefficients of the magnetic Schrödinger operator in the bounded domains.

*Keywords*: Magnetic Schrödinger operator, Borg-Levinson theorem.

## 1. Green's function

Let $\Omega$ be a bounded domain with smooth boundary in $R^n, n \geq 2$. We consider in this domain a magnetic Schrödinger operator

$$H_m = -(\nabla + i\vec{A}(x))^2 + V(x)\cdot, \quad x \in \Omega, \tag{1}$$

where the coefficients $\vec{A}(x)$ and $V(x)$ are assumed to be real-valued. We assume also that

$$\vec{A}(x) \in (L^n(\Omega))^n, \quad n \geq 3; \quad \vec{A}(x) \in (L^p(\Omega))^2, \quad 2 < p \leq \infty, \quad n = 2, \tag{2}$$

and

$$V_+(x) \in L^1(\Omega), \quad n \geq 2,$$

$$V_-(x) \in L^{\frac{n}{2}}(\Omega), \quad n \geq 3; \quad V_-(x) \in L^s(\Omega), \quad 1 < s \leq \infty, \quad n = 2, \tag{3}$$

where $V_+ = \max(V, 0)$ and $V_- = \min(V, 0)$.

It is well-known (see, for example, [13]) that under the conditions (2) and (3) for the coefficients the following Gårding's inequality holds:

$$(H_m u, u)_{L^2} \geq c_1 \|\nabla u\|^2_{L^2} - c_2 \|u\|^2_{L^2}, \tag{4}$$

where $0 < c_1 < 1, c_2 > 0$. This inequality allows us to define symmetric operator (1) by the method of quadratic forms. $H_m$ has a self-adjoint Friedrichs extension denoted by $(H_m)_F$ with domain

$$D((H_m)_F) = \{f(x) \in \overset{\circ}{W}{}_2^1(\Omega) : H_m f(x) \in L^2(\Omega)\},$$

where $\overset{\circ}{W}{}_2^1(\Omega)$ denotes the closure of the space $C_0^\infty(\Omega)$ by the norm of Sobolev space $W_2^1(\Omega)$. The spectrum of this extension is purely discrete, of finite multiplicity and has an accumulation point only at the $+\infty$:

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ... \to +\infty.$$

The corresponding orthonormal eigenfunctions $\{\varphi_k(x)\}_{k=1}^\infty$ form orthonormal basis in $L^2(\Omega)$. The Gårding's inequality (4) allows us to conclude also that there is a positive constant $\mu_0$ such that the operator $(H_m)_F + \mu_0 I$ is positive. Then the diamagnetic inequality of Barry Simon (see [12] and [2]) says that for any $t \geq 0$ and any $f \in L^2(\Omega)$

$$|e^{-t((H_m)_F+\mu_0 I)} f(x)| \leq e^{-t(-\Delta)_F}|f|(x), \quad a.e. \quad x \in \Omega, \tag{5}$$

where $(-\Delta)_F$ denotes the Friedrichs self-adjoint extension of the Laplacian in $L^2(\Omega)$ with purely discrete spectrum. The operators $e^{-t((H_m)_F+\mu_0 I)}$ and $e^{-t((-\Delta)_F}$ in (5) can be understood via J. von Neumann spectral theorem. Even more is true in that case. Namely, since both these operators have purely discrete spectrum they are integral with kernels denoted by $P(t, x, y)$ and $P_0(t, x, y)$, respectively. Since $P_0$ is a heat kernel maximum principle implies the following estimate:

$$0 \leq P_0(t, x, y) \leq \frac{1}{(\sqrt{4\pi t})^n} e^{-\frac{|x-y|^2}{4t}}.$$

Thus, the inequality (5) can be rewritten as

$$|\int_\Omega P(t, x, y) f(y)\, dy| \leq \frac{1}{(\sqrt{4\pi t})^n} \int_\Omega e^{-\frac{|x-y|^2}{4t}} |f(y)|\, dy, \quad a.e. \quad x \in \Omega, \tag{6}$$

where $f \in L^2(\Omega)$. Using then the Hardy-Littlewood maximal functions from the inequality (6) we obtain

$$|P(t, x, y)| \leq \frac{1}{(\sqrt{4\pi t})^n} e^{-\frac{|x-y|^2}{4t}}, \quad x, y \in \Omega. \tag{7}$$

For any $\lambda \geq \mu_0$, where $\mu_0$ from (5), the operator $(H_m)_F + \lambda I$ is positive and its inverse

$$((H_m)_F + \lambda I)^{-1} : L^2(\Omega) \to L^2(\Omega) \tag{8}$$

is bounded integral operator with kernel denoted by $G(x, y, \lambda)$. If we use for this integral operator the symbol $\widehat{G}(\lambda)$ then we have

$$((H_m)_F + \lambda I)\widehat{G}(\lambda) = I, \quad \widehat{G}(\lambda)((H_m)_F + \lambda I) = I, \quad G(x, y, \lambda) = \overline{G(y, x, \lambda)}. \tag{9}$$

**Definition 1.1** The kernel $G(x, y, \lambda)$ of the integral operator $\widehat{G}(\lambda)$ is called the Green's function of the operator $(H_m)_F + \lambda I$.

Our first result is

**Theorem 1.1** *Suppose that $\vec{A}(x)$ and $V(x)$ satisfy the conditions (2) and (3), respectively. Then for any $\lambda \geq \mu_0$ the Green's function of the operator $(H_m)_F + \lambda I$ satisfies the following estimates:*

$$|G(x, y, \lambda)| \leq C|x - y|^{2-n} e^{-|x-y|\sqrt{\lambda}}, \quad n \geq 3, \tag{10}$$

*and*

$$|G(x, y, \lambda)| \leq C \left(1 + |\log(|x - y|\sqrt{\lambda}|\right) e^{-|x-y|\sqrt{\lambda}}, \quad n = 2, \tag{11}$$

*where $x, y \in \Omega$ and $C > 0$ does not depend on $x, y \in \Omega$ and $\lambda$ .*

*Proof.* Due to J. von Neumann spectral theorem the Green's function $G(x, y, \lambda)$ can be calculated as a Laplace transform of $P(t, x, y)$

$$G(x, y, \lambda) = \int_0^\infty e^{-t\lambda} P(t, x, y)\, dt. \tag{12}$$

Using (7) we can easily obtain from (12) that for $\lambda \geq \mu_0$

$$|G(x, y, \lambda)| \leq (2\pi)^{-\frac{n}{2}} \left(\frac{|x - y|}{\sqrt{\lambda}}\right)^{\frac{n-2}{2}} K_{\frac{n-2}{2}}(|x - y|\sqrt{\lambda}), \tag{13}$$

where $K_\nu$ is the Macdonald function of order $\nu$. Equation (13) and asymptotic expansions of $K_\nu(z)$ for $z \to 0$ anf for $z \to +\infty$ give us estimates (10) and (11). Theorem 1.1 is proved. □

We have three immediate corollaries of Theorem 1.1.

**Corollary 1.1** *Assume that $\vec{A}$ and $V$ are as above and $\sigma > \frac{n}{4}, n \geq 2$. Then for any function $f(x) \in L^2(\Omega)$ the following inequality holds*

$$\|((H_m)_F + \lambda I)^{-\sigma} f\|_{L^\infty(\Omega)} \leq C\lambda^{\frac{n}{4}-\sigma} \|f\|_{L^2(\Omega)},$$

*where $\lambda \geq \mu_0$ with $\mu_0$ as in Theorem 1.1.*

**Corollary 1.2** *Assume that $\sigma > \frac{n}{4}, n \geq 2$. There is a constant $C > 0$ depending only on $\Omega$, such that the estimate*

$$\sum_{k=1}^{\infty} \frac{|\varphi_k(x)|^2}{(\lambda_k + \lambda)^{2\sigma}} \leq C\lambda^{\frac{n}{2}-2\sigma}$$

*holds uniformly in $x \in \Omega$ and $\lambda \geq \mu_0$.*

**Corollary 1.3** *Assume that* $\sigma > \frac{n}{4}, n \geq 2$. *Then the following series*

$$\sum_{k=1}^{\infty} \frac{1}{(\lambda_k + \mu_0)^{2\sigma}}$$

*converges.*

**Remark 1.1** It can be mentioned here that the estimates (10) and (11) of the Green's function of the magnetic Schrödinger operator (1) are obtained in Theorem 1.1 for very weak conditions of the coefficients of $H_m$. As far as we know they never appeared in the literature for such conditions of the coefficients.

## 2. Dirichlet-to-Neumann map and eigenfunctions

In this chapter we assume (a bit more than in (2) and (3)) that

$$\vec{A}(x) \in (W_p^1(\Omega))^n, \quad V(x) \in L^p(\Omega), \tag{14}$$

for some $p > \frac{n}{2}, n \geq 2$.

Let $\lambda \geq \mu_0$ with $\mu_0$ as in Chapter 1. Consider the following Dirichlet problem:

$$((H_m)_F + \lambda I)u(x) = 0, \quad x \in \Omega, \quad u(x) = f(x), \quad x \in \partial\Omega, \tag{15}$$

where the boundary function $f(x)$ satisfies the following condition:

$$f(x) \in B_{pp}^t(\partial\Omega), \quad t = \frac{p-1}{p}, \quad p > n, \quad t > \frac{n-1}{p}, \quad \frac{n}{2} < p \leq n, \tag{16}$$

where $B_{pp}^t(\partial\Omega)$ denotes Besov space on the boundary.

It is well known (see, for example, [3], Remarks on pp. 209, 241, and Corollary 9.18, p. 243) that there exists a unique solution of (15) from the spaces

$$u(x) \in W_{p,loc}^2(\Omega) \cap W_p^{t+\frac{1}{p}}(\Omega). \tag{17}$$

Thus, we may define the Dirichlet-to-Neumann map $\Lambda_{\vec{A},V+\lambda}$ as follows

$$\Lambda_{\vec{A},V+\lambda} f(x) := \frac{\partial u(x)}{\partial \nu} + i\vec{A} \cdot \nu f(x), \tag{18}$$

where $\nu$ is outward normal vector at the boundary $\partial\Omega$. Conditions (14), (16) and (17) imply that the Dirichlet-to-Neumann map (18) acts as (for fixed $\lambda$)

$$\Lambda_{\vec{A},V+\lambda} : B_{pp}^t(\partial\Omega) \to B_{pp}^{t-1}(\partial\Omega) \tag{19}$$

with $t$ and $p$ as in (16).

The following lemma plays the crucial role in this work.

**Lemma 2.1** *Assume that $\vec{A}_1, \vec{A}_2$ and $V_1, V_2$ satisfy the conditions (14) and $f$ satisfies the condition (16). In addition we assume that $\vec{A}_1(x) = \vec{A}_2(x)$ on the boundary $\partial\Omega$. Then*

$$\lim_{\lambda\to+\infty} \|\Lambda_{\vec{A}_2,V_2+\lambda} f - \Lambda_{\vec{A}_1,V_1+\lambda} f\|_{B^{\delta}_{pp}(\partial\Omega)} = 0 \tag{20}$$

*for any $0 < \delta < 1 - \frac{1}{p}$.*

*Proof.* Let $\omega(x) := u_1(x) - u_2(x)$, where $u_j(x), j = 1, 2$, solves the problem (15) with $\vec{A}_j, V_j$, respectively. We denote the corresponding magnetic Schrödinger operators by $H_{m,j}$. Then $\omega(x)$ solves the boundary value problem

$$(H_{m,1} + \lambda I)\omega(x) = (H_{m,2} - H_{m,1})u_2(x), \quad x \in \Omega,$$

$$\omega(x) = 0, \quad x \in \partial\Omega.$$

This problem can be rewritten as

$$(-\Delta + \lambda I - 2i\vec{A}_1 \cdot \nabla + Q_1)\omega(x) = 2i(\vec{A}_1 - \vec{A}_2) \cdot \nabla u_2(x) + Qu_2(x), \quad x \in \Omega,$$

$$\omega(x) = 0, \quad x \in \partial\Omega, \tag{21}$$

where the potentials $Q(x)$ and $Q_1(x)$ are defined by

$$Q(x) = i\nabla \cdot (\vec{A}_1(x) - \vec{A}_2(x)) + (|\vec{A}_2(x)|^2 - |\vec{A}_1(x)|^2) + (V_2(x) - V_1(x))$$

and

$$Q_1(x) = -i\nabla \cdot \vec{A}_1(x) + |\vec{A}_1(x)|^2 + V_1(x),$$

respectively. The conditions (14) easily imply that $Q(x)$ and $Q_1(x)$ belong to $L^p(\Omega)$ with $p$ as in (14). Denote by $\hat{G}_0(\lambda)$ the integral operator with the kernel which is the Green's function of $-\Delta + \lambda I$ in $\Omega$. Applying this operator we obtain from (21) the following integral equation

$$(I + K)\omega(x) = F(x), \tag{22}$$

where the integral operator $K$ and the function $F$ are given by

$$K := \hat{G}_0(\lambda)(-2i\vec{A}_1 \cdot \nabla + Q_1)$$

and

$$F(x) := \hat{G}_0(\lambda)(2i(\vec{A}_1 - \vec{A}_2) \cdot \nabla u_2 + Qu_2)(x).$$

We consider this equation (22) in the space of functions from Sobolev space $W^2_p(\Omega)$ which vanish at the boundary $\partial\Omega$. Due to the assumptions (14) for the coefficients $\vec{A}_j$ and $V_j, j = 1, 2$, and embedding (17) we may conclude that $F$ belongs to this space and $K$ is compact there. Since the operator $H_{m,1} + \lambda I$ is positive for $\lambda \geq \mu_0$ then the boundary value problem

$$(H_{m,1} + \lambda I)\omega(x) = 0, \quad x \in \Omega,$$

$$\omega(x) = 0, \quad x \in \partial\Omega$$

has only the trivial solution $\omega \equiv 0$. The same is true for the homogeneous equation corresponding to (22). By Fredholm's alternative the operator $I + K$ has a bounded inverse in the indicated Sobolev space and therefore the solution $\omega$ of the equation (22) satisfies

$$\|\omega\|_{W_p^2(\Omega)} \leq C\|F\|_{W_p^2(\Omega)} \leq C\left(\|(\vec{A}_1 - \vec{A}_2)\cdot\nabla u_2\|_{L^p(\Omega)} + \|Qu_2\|_{L^p(\Omega)}\right). \tag{23}$$

If $p > n$ then the conditions (16) and (17), and the inequality (23) imply that

$$\|\omega\|_{W_p^2(\Omega)} \leq C\left(\|(\vec{A}_1 - \vec{A}_2)\|_{L^\infty(\Omega)}\|\nabla u_2\|_{L^p(\Omega)} + \|Q\|_{L^p(\Omega)}\|u_2\|_{L^\infty(\Omega)}\right)$$

$$\leq C\left(\|(\vec{A}_1 - \vec{A}_2)\|_{W_p^1(\Omega)}\|u_2\|_{W_p^1(\Omega)} + \|Q\|_{L^p(\Omega)}\|u_2\|_{W_p^1(\Omega)}\right) \leq C\|u_2\|_{W_p^1(\Omega)}, \tag{24}$$

where $C$ depends only on the coefficients of the magnetic Schrödinger operators $H_{m,1}$ and $H_{m,2}$. If $\frac{n}{2} < p \leq n$ using Sobolev embedding theorem and by analogy with the previous case we obtain

$$\|\omega\|_{W_p^2(\Omega)} \leq C\left(\|(\vec{A}_1 - \vec{A}_2)\|_{L^s(\Omega)}\|\nabla u_2\|_{L^r(\Omega)} + \|Q\|_{L^p(\Omega)}\|u_2\|_{L^\infty(\Omega)}\right)$$

$$\leq C\left(\|(\vec{A}_1 - \vec{A}_2)\|_{W_p^1(\Omega)}\|u_2\|_{W_p^\alpha(\Omega)} + \|Q\|_{L^p(\Omega)}\|u_2\|_{W_p^\alpha(\Omega)}\right) \leq C\|u_2\|_{W_p^\alpha(\Omega)}, \tag{25}$$

where $s = \frac{pn}{n-p}$, $r = n$ if $p < n$ and $s < \infty$, $r > p$ if $p = n$, and $\alpha > \frac{n}{p}$.
We apply now the result from [15](see inequality (5.46), p. 183) and obtain that

$$\|(-\Delta + \lambda I)\omega\|_{L^p(\Omega)} \geq C\lambda\|\omega\|_{L^p(\Omega)}. \tag{26}$$

By combining the inequalities (23)-(26) we may get the following inequality

$$\|\omega\|_{L^p(\Omega)} \leq \frac{C}{\lambda}\|u_2\|_{W_p^{t+\frac{1}{p}}(\Omega)}, \tag{27}$$

where $t$ is as in (16). The interpolation of (24), (25) and (27) leads us to the inequality

$$\|\omega\|_{W_p^s(\Omega)} \leq \frac{C}{\lambda^{1-\frac{s}{2}}}\|u_2\|_{W_p^{t+\frac{1}{p}}(\Omega)}, \tag{28}$$

where $0 \leq s \leq 2$ and $t$ is as in (16). Thus, due to (18), (28) and the conditions of this lemma we have

$$\|\Lambda_{\vec{A}_2, V_2+\lambda}f - \Lambda_{\vec{A}_1, V_1+\lambda}f\|_{B_{pp}^\delta(\partial\Omega)} \leq \|\frac{\partial\omega}{\partial\nu}\|_{B_{pp}^\delta(\partial\Omega)}.$$

Using (28) we can estimate the latter term as follows:

$$\|\frac{\partial\omega}{\partial\nu}\|_{B_{pp}^\delta(\partial\Omega)} \leq \|\omega\|_{B_{pp}^{\delta+1}(\partial\Omega)} \leq C\|\omega\|_{W_p^{\delta+1+\frac{1}{p}}(\Omega)} \leq \frac{C}{\lambda^{\frac{1-\delta}{2}-\frac{1}{2p}}}\|u_2\|_{W_p^{t+\frac{1}{p}}(\Omega)}. \tag{29}$$

Since $\delta < 1 - \frac{1}{p}$ taking into account the boundedness of the norm of $u_2$ in $\lambda$ we may conclude from (29) that Lemma 2.1 is completely proved. □

We are in the position now to estimate the normalized eigenfunctions of the magnetic Schrödinger operator.

**Lemma 2.2** *Under the assumptions (14) for the coefficients $\vec{A}$ and $V$ the orthonormal eigenfunctions $\varphi_k(x)$ satisfy the estimate*

$$\|\varphi_k\|_{W_p^s(\Omega)} \le C(\lambda_k + \mu_0)^{\frac{s}{2}+\frac{n}{4}}, \tag{30}$$

*where $0 \le s \le 2$, $p > \frac{n}{2}$ and $\mu_0$ is as in (5).*

*Proof.* Let $\lambda_k$ be an eigenvalue and $\varphi_k(x)$ corresponding orthonormal eigenfunction. Then the inequality (5) ((6)) can be rewritten for $f = \varphi_k$ as

$$e^{-t(\lambda_k+\mu_0)}|\varphi_k(x)| \le \frac{1}{(\sqrt{4\pi t})^n} \int\limits_{\Omega} e^{-\frac{|x-y|^2}{4t}} |\varphi_k(y)|\, dy$$

$$\le \frac{1}{(\sqrt{4\pi t})^n} \left( \int_{R^n} e^{-\frac{|x-y|^2}{2t}}\, dy \right)^{\frac{1}{2}} = \left(\frac{1}{8\pi t}\right)^{\frac{n}{4}}.$$

The latter inequality immediately implies

$$\|\varphi_k\|_{L^\infty(\Omega)} \le \left(\frac{e}{2\pi n}\right)^{\frac{n}{4}} (\lambda_k + \mu_0)^{\frac{n}{4}}$$

and

$$\|\varphi_k\|_{L^p(\Omega)} \le C(\lambda_k + \mu_0)^{\frac{n}{4}}, \tag{31}$$

where $1 \le p \le \infty$ and constant $C > 0$ depends only on $n$, $p$ and $Vol(\Omega)$.

Rewriting the equation for the eigenfunctions $\varphi_k(x)$ in the form

$$(-\Delta + \mu_0 I - 2i\vec{A}\cdot\nabla - i\nabla\cdot\vec{A} + |\vec{A}|^2 + V)\varphi_k(x) = (\lambda_k + \mu_0)\varphi_k(x), \quad x \in \Omega,$$

$$\varphi_k(x) = 0, \quad x \in \partial\Omega,$$

and applying the inequality (23), we obtain for any $p > \frac{n}{2}$ that

$$\|\varphi_k\|_{W_p^2(\Omega)} \le C(\lambda_k + \mu_0)\|\varphi_k\|_{L^p(\Omega)} \le C(\lambda_k + \mu_0)^{1+\frac{n}{4}}. \tag{32}$$

Now by interpolating (31) and (32) we may obtain (30). Thus, Lemma 2.2 is proved. □

The next lemma shows us the representation for the kernel of the operator $\Lambda_{\vec{A},V+\lambda}$.

**Lemma 2.3** *For $l = n + 1$ and $f$ as in (16) we have*

$$\left(\frac{d}{d\lambda}\right)^l \left(\Lambda_{\vec{A},V+\lambda} f(x)\right) = \int\limits_{\partial\Omega} g_l(x, y, \lambda) f(y)\, d\sigma(y), \tag{33}$$

*where $g_l$ is defined by*

$$g_l(x, y, \lambda) = (-1)^{l+1} l! \sum_{k=1}^{\infty} \frac{\frac{\partial\varphi_k(x)}{\partial\nu}\frac{\partial\varphi_k(y)}{\partial\nu}}{(\lambda_k + \lambda)^{l+1}}, \tag{34}$$

*$\lambda \geq \mu_0$ and the right-hand side is convergent in $L^p(\partial\Omega \times \partial\Omega)$.*

*Proof.* Integration by parts for the problem (15) with $f$ from (16) and $\lambda \geq \mu_0$ leads to

$$u(x) = -\int_{\partial\Omega} \frac{\partial \overline{G(x, y, \lambda)}}{\partial \nu_y} f(y)\, d\sigma(y) = -\int_{\partial\Omega} \frac{\partial G(y, x, \lambda)}{\partial \nu_y} f(y)\, d\sigma(y), \tag{35}$$

where $G(x, y, \lambda)$ is the Green's function of $(H_m)_F + \lambda I$ defined in (8)-(10) and $\nu_y$ denotes the outward normal vector in $y$. In our case the Green's function is given by

$$G(x, y, \lambda) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)\varphi_k(y)}{\lambda_k + \lambda}. \tag{36}$$

Since $u$ solves the problem (15)-(16) ($u$ therefore depends on $\lambda$ also) then using J. von Neumann spectral theorem it can be easily proved by induction that

$$\left(\frac{d}{d\lambda}\right)^l u(x, \lambda) = (-1)^l l! \left((H_m)_F + \lambda I\right)^{-l} u(x, \lambda), \quad l = 1, 2, .... \tag{37}$$

The operator $\left((H_m)_F + \lambda I\right)^{-l}$ is well-defined by the spectral theorem and it is the integral operator with kernel denoted by $G_l(x, y, \lambda)$

$$G_l(x, y, \lambda) = \sum_{k=1}^{\infty} \frac{\varphi_k(x)\varphi_k(y)}{(\lambda_k + \lambda)^l}.$$

This fact allows us to represent $\left((H_m)_F + \lambda I\right)^{-l} u(x, \lambda)$ as follows

$$\left((H_m)_F + \lambda I\right)^{-l} u(x, \lambda) = \int_{\Omega} G_l(x, y, \lambda) u(y, \lambda)\, dy = \sum_{k=1}^{\infty} \frac{\varphi_k(x) u_k(\lambda)}{(\lambda_k + \lambda)^l}, \tag{38}$$

where $u_k(\lambda)$ is given by

$$u_k(\lambda) = \int_{\Omega} \varphi_k(y) u(y, \lambda)\, dy.$$

Integration by parts in the last equality gives us

$$u_k(\lambda) = -\frac{1}{\lambda_k + \lambda} \int_{\partial\Omega} \frac{\partial \varphi_k(y)}{\partial \nu} f(y)\, d\sigma(y). \tag{39}$$

Combining (38) and (39) we obtain the following equality

$$((H_m)_F + \lambda I)^{-l} u(x,\lambda) = -\int\limits_{\partial\Omega} \sum_{k=1}^{\infty} \frac{\varphi_k(x)\frac{\partial\varphi_k(y)}{\partial\nu_y}}{(\lambda_k+\lambda)^{l+1}}\, d\sigma(y) \tag{40}$$

which coincides for $l = 0$ with (35).

Since $u$ solves the boundary value problem (15)-(16) using (18) we can obtain

$$\left(\frac{d}{d\lambda}\right)^l \left(\Lambda_{\vec{A},V+\lambda} f(x)\right) = \frac{\partial}{\partial\nu}\left(\left(\frac{d}{d\lambda}\right)^l u(x,\lambda)\right).$$

Thus, the equalities (37) and (40) give us that (33) and (34) are formally obtained. This lemma will be proved if we show the convergence of the series (34) in $L^p(\partial\Omega\times\partial\Omega)$. To this end, the inequality (30) from Lemma 2.2 and Sobolev's imbedding theorem allow us to conclude that for any $0 < \delta < 1 - \frac{1}{p}$

$$\|\frac{\partial\varphi_k}{\partial\nu}\|_{L^p(\partial\Omega)} \le C\|\varphi_k\|_{B_{pp}^{\delta+1}(\partial\Omega)} \le C\|\varphi_k\|_{W_p^{\delta+1+\frac{1}{p}}(\Omega)} \le C(\lambda_k+\mu_0)^{\frac{1}{2}(\delta+1+\frac{1}{p})+\frac{n}{4}}.$$

By using this estimate and taking now $m = n+1$, we have

$$\|\int\limits_{\partial\Omega} g_l(x,y,\lambda) f(y)\, d\sigma(y)\|_{L^p(\partial\Omega)} \le C\sum_{k=1}^{\infty} \frac{\|\frac{\partial\varphi_k}{\partial\nu}\|^2_{L^p(\partial\Omega)}}{(\lambda_k+\lambda)^{n+2}} \|f\|_{L^{p'}(\partial\Omega)}$$

$$\le C\|f\|_{B_{pp}^t(\partial\Omega)} \sum_{k=1}^{\infty} \frac{1}{(\lambda_k+\mu_0)^{\frac{n}{2}+1-\frac{1}{p}-\delta}}.$$

Since $0 < \delta < 1 - \frac{1}{p}$ then due to Corollary 1.3 of Theorem 1.1 the latter series converges and therefore, Lemma 2.3 is completely proved. □

## 3. Borg-Levinson theorem

Classical one-dimensional Borg-Levinson theorem is formulated as follows: let $q$ be real valued potential in $L^\infty(0,1)$ and $y(x,\lambda)$ solves the initial value problem

$$-y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), \quad y(0,\lambda) = 0, y'(0,\lambda) = 1.$$

Define the Dirichlet eigenvalues $\lambda_k(q)$ by the condition

$$y(1,\lambda_k) = 0$$

and define the norming constants $c_k(q)$ by

$$c_k(q) = \int_0^1 y_k^2(x,\lambda_k)\, dx.$$

The result of Borg [1] and Levinson [6] is:
If for all $k = 1, 2, ...$

$$\lambda_k(q_1) = \lambda_k(q_2), \quad c_k(q_1) = c_k(q_2),$$

then $q_1 = q_2$. It can be reformulated as follows. If for all $k = 1, 2, ...$

$$\lambda_k(q_1) = \lambda_k(q_2), \quad y_k'(1, \lambda_k; q_1) = y_k'(1, \lambda_k; q_2),$$

then $q_1 = q_2$. Thus, the Dirichlet eigenvalues and normal derivatives of the eigenfunctions at the boundary uniquely determine a potential. We generalize the latter formulation of Borg-Levinson theorem for multidimensional case. The first result is:

**Theorem 3.1** *Assume that $\vec{A}_j \in W_p^1(\Omega)$ and $V_j \in L^p(\Omega)$, $j = 1, 2$, for some $p > \frac{n}{2}, n \geq 2$. Assume in addition that $\vec{A}_1(x) = \vec{A}_2(x)$ at the boundary $\partial\Omega$. Assume also that for each $k = 1, 2, ...$*

$$\lambda_k(\vec{A}_1, V_1) = \lambda_k(\vec{A}_2, V_2) \tag{41}$$

*and*

$$\frac{\partial \phi_k}{\partial \nu}(x; \vec{A}_1, V_1) = \frac{\partial \phi_k}{\partial \nu}(x; \vec{A}_2, V_2). \tag{42}$$

*Then for all $\lambda \geq \mu_0$*

$$\Lambda_{\vec{A}_1, V_1 + \lambda} = \Lambda_{\vec{A}_2, V_2 + \lambda}. \tag{43}$$

*Proof.* From the conditions (41), (42), and Lemma 2.3 it follows that for all $\lambda \geq \mu_0$

$$\left(\frac{d}{d\lambda}\right)^{n+1} \left(\Lambda_{\vec{A}_1, V_1 + \lambda} f(x) - \Lambda_{\vec{A}_2, V_2 + \lambda} f(x)\right) = 0$$

for any $f \in B_{pp}^t(\partial\Omega)$ and $p > \frac{n}{2}$. This equation reads as

$$\Lambda_{\vec{A}_1, V_1 + \lambda} - \Lambda_{\vec{A}_2, V_2 + \lambda} = \sum_{j=0}^{n} \lambda^j L_j, \tag{44}$$

where $L_j$ are bounded operators from $B_{pp}^t(\partial\Omega)$ to $L^p(\partial\Omega)$. But Lemma 2.1 implies that the polynomial in the right-hand side of (44) is zero. Hence, $\Lambda_{\vec{A}_1, V_1 + \lambda} = \Lambda_{\vec{A}_2, V_2 + \lambda}$ for all $\lambda \geq \mu_0$. Thus, Theorem 3.1 is proved. □

Using now the paper of Salo [11] (see also very resent result of Päivärinta, Salo and Uhlmann [10]) we obtain

**Theorem 3.2 (Borg-Levinson)** *If $\vec{A}_j \in W_\infty^1(\Omega)$ and $V_j \in L^\infty(\Omega)$, $j = 1, 2, n \geq 3$, and all conditions of Theorem 3.1 are satisfied, then*

$$d\vec{A}_1 = d\vec{A}_2, \quad V_1 = V_2,$$

*where the 2-form $d\vec{A}$ of the vector $\vec{A} = (a_1, a_2, ..., a_n)$ is defined by*

$$d\vec{A} = \sum_{j,k=1}^{n} \left(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k}\right) dx^k \wedge dx^j.$$

### Historical remarks

The multidimensional Borg-Levinson theorem for the Schrödinger operators ($\vec{A} = 0$) for the first time was proved by Nachman, Sylvester and Uhlmann [7] for the potentials $V \in C^\infty(\overline{\Omega})$. Their proof remains however valid if one assumes that $V \in L^\infty(\Omega)$. The proof uses the convolution type estimates of the Green's function for the Schrödinger operator in the weighted $L^2-$spaces. Finally, the problem is reduced to the fact that the Dirichlet-to-Neumann map uniquely determines such potentials [14]. The same result was obtained independently by Novikov [8]. For singular potentials $V \in L^p(\Omega), \frac{n}{2} < p \leq \infty, n \geq 2$, this theorem was proved by Päivärinta and Serov [9]. For inverse boundary spectral problems on Riemannian manifolds some related results were proved by Kachalov, Kurylev and Lassas [4] (see also [5]). As far as we know Borg-Levinson theorem for the magnetic Schrödinger operators is never met in the literature.

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◇ V. S. Serov
Department of Mathematical Sciences
University of Oulu
P.O. Box 3000, FIN-90014, Oulu, FINLAND
vserov@cc.oulu.fi