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Moore-Penrose invertibility of singular integral operators with Carleman shift

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Abstract

We will take profit of explicit operator relations and a certain algebraization of the stability to obtain the Moore-Penrose inverse of singular integral operators with shift, having piecewise continuous functions as coefficients. This is considered for two different shifts: the reflection operator on the complex unit circle, and a weighted Carleman shift (the so-called flip operator).

Keywords: Moore-Penrose inverse, singular integral operator, shift, Fredholm property.

1. Introduction

Let $PC(\mathbb{T})$ stand for the space of all essentially bounded piecewise continuous functions on the unit circle $\mathbb{T} := \{t \in \mathbb{C} \ : \ |t| = 1\}$, i.e., functions $\varphi \in L^\infty(\mathbb{T})$ for which the one-sided limits

$$\varphi^\pm(t) := \lim_{\varepsilon \to 0^\pm} \varphi(te^{i\varepsilon})$$

exist for each $t \in \mathbb{T}$. Thus, as usual, $[PC(\mathbb{T})]^{2\times 2}$ will denote the C^*-algebra of all 2×2-matrices with entries from $PC(\mathbb{T})$. In addition, let $L^2(\mathbb{T}, w)$ be the weighted Lebesgue space over $\mathbb{T}$ equipped with the norm

$$\|f\|_{2,w} := \|wf\|_2 \, , \tag{1}$$

where $\|\cdot\|_2$ denotes the usual norm of the Hilbert space $L^2(\mathbb{T})$. We will assume that all the weights $w : \mathbb{T} \longrightarrow [0, +\infty]$ are such that $w, w^{-1} \in L^2(\mathbb{T})$, and

$$c_w := \sup_{t\in\mathbb{T}} \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_{\mathbb{T}(t,\varepsilon)} w(\tau)^2 |d\tau| \right)^{1/2} \left(\frac{1}{\varepsilon} \int_{\mathbb{T}(t,\varepsilon)} w(\tau)^{-2} |d\tau| \right)^{1/2} < \infty \, , \tag{2}$$

where

$$\mathbb{T}(t,\varepsilon) := \{\tau \in \mathbb{T} \ : \ |\tau - t| < \varepsilon\}, \qquad \varepsilon > 0.$$

The property (2) is the so-called *Hunt–Muckenhoupt–Wheeden condition*, and $A_2(\mathbb{T})$ is referred to as the set of *Hunt–Muckenhoupt–Wheeden weights*. The space $[L^2(\mathbb{T}, w)]^2$

refers to the Hilbert space of all column-vectors of length 2 with components from $L^2(\mathbb{T}, w)$.

The Cauchy singular integral operator on $\mathbb{T}$ is defined almost everywhere by

$$(S_{\mathbb{T}} f)(t) = \frac{1}{\pi i} \text{p.v.} \int_{\mathbb{T}} \frac{f(\tau)}{\tau - t} d\tau.$$

If the weight w satisfies condition (2), then $S_{\mathbb{T}} \in \mathcal{L}(L^2(\mathbb{T}, w))$. Here, $\mathcal{L}(L^2(\mathbb{T}, w))$ stands for the C^*-algebra of all bounded and linear operators acting from $L^2(\mathbb{T}, w)$ into $L^2(\mathbb{T}, w)$.

In the present work we deal with the singular integral operators

$$\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J, \tag{3}$$

with coefficients $a_0, b_0, a_1, b_1 \in PC(\mathbb{T})$, a shift operator J satisfying the Carleman condition (i.e., $J^2 = I$) which can be either the reflection operator (defined by a rotation action of π amplitude on the unit circle $\mathbb{T}$) or the flip operator (which is a weighted backward Carleman shift).

The reflection operator has the form

$$(J\varphi)(t) = \varphi(-t), \quad t \in \mathbb{T}, \tag{4}$$

in which case $\mathcal{A}$ is defined on the weighted Lebesgue space $L^2(\mathbb{T}, w)$, with weights w belonging to $A_2^e(\mathbb{T}) := \{w \in A_2(\mathbb{T}) : \; w(-t) = w(t), \; t \in \mathbb{T}\}$.

On the other hand, the flip operator is given by

$$(J\varphi)(t) = \frac{1}{t} \varphi\left(\frac{1}{t}\right), \quad t \in \mathbb{T}, \tag{5}$$

and for this case the operator $\mathcal{A}$ is assumed to be defined in $L^2(\mathbb{T})$.

Fredholm criteria for the operators $\mathcal{A}$ with shift operators as in (4) or (5) are already known (see [3, 4, 5]). The main goal of this paper is to obtain the so-called *Moore-Penrose inverse* of $\mathcal{A}$. This inverse is closely related to the *k-splitting property*, and some evidence of this will be also exposed.

To achieve our aim, we will use the following notions. We recall that two bounded linear operators $T : X_1 \longrightarrow X_2$ and $S : Y_1 \longrightarrow Y_2$ acting between Banach spaces are called *equivalent* (cf., [1], [6]) if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

$$T = E \; S \; F \tag{6}$$

holds. The notion of *equivalence after extension relation* also plays here a significant role. We say that two operators T and S are *equivalent after extension* if two additional Banach spaces Z and W exist in such a way that $T \oplus I_Z$ and $S \oplus I_W$ are equivalent operators. In the presence of the particular case $E = F^{-1}$ in (6), we will say that we have a *similarity relation* between the operators T and S. It follows from (6) that if two operators are equivalent (or equivalent after extension), then their kernels have the same dimension.

The paper is organized as follows: In section 2, we describe an equivalence relation (proved in [4]) between the operator $\mathcal{A}$ with the reflection shift operator (4) and a matrix singular integral operator $\mathcal{D}_{\mathbb{T}}$ without shift, as well as an equivalence after extension relation between the operator $\mathcal{A}$ with the flip operator J, defined in (5), and a new operator $\mathcal{D}_{\mathbb{T}}$ without flip. This last equivalence after extension relation was proved in [5]. In the final section, the so-called *projection methods*, as well as the notions of *singular values* and *stability* are considered in a general setting. These previous results and notions will be useful to relate and obtain, in explicit form, the Moore-Penrose inverse of the singular integral operator presented in (3).

2. Operator relations

In the articles [4] and [5], it was obtained a direct relation between the operator $\mathcal{A}$ and a matrix singular integral operator without additional associated operators: for the reflection shift operator (4) it is a similarity transformation $F\mathcal{A}F^{-1}$ and for the flip operator (5) it is a transformation after extension by two invertible operators $G\mathcal{A}H$. For the reader convenience, we will formulate these results below.

First we will consider the case of the reflection shift operator $(J\varphi)(t) = \varphi(-t)$, $t \in \mathbb{T}$. Let $w \in A_2^e(\mathbb{T})$ and $\mathbb{T}_+ := \{t \in \mathbb{T} \ : \ 0 < \arg t < \pi\}$. We define the following operators:

$$M : [L^2(\mathbb{T}_+, w)]^2 \longrightarrow L^2(\mathbb{T}, w)$$
$$M \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \ell_0\varphi_1 + J^{-1}\ell_0\varphi_2, \tag{7}$$

where ℓ_0 denotes the zero extension operator from $\mathbb{T}_+$ to $\mathbb{T}$ (in the corresponding spaces). Note that $M \in \mathcal{L}([L^2(\mathbb{T}_+, w)]^2, L^2(\mathbb{T}, w))$. Moreover,

$$M^{-1}\varphi = \begin{pmatrix} r_{\mathbb{T}_+}\varphi \\ r_{\mathbb{T}_+}J\varphi \end{pmatrix}, \tag{8}$$

where $r_{\mathbb{T}_+} : L^2(\mathbb{T}, w) \longrightarrow L^2(\mathbb{T}_+, w)$ denotes the restriction operator $r_{\mathbb{T}_+}\varphi = \varphi_{|\mathbb{T}_+}$. The operator M^{-1} is linear and bounded from $L^2(\mathbb{T}, w)$ onto $[L^2(\mathbb{T}_+, w)]^2$, i.e., $M^{-1} \in \mathcal{L}(L^2(\mathbb{T}, w), [L^2(\mathbb{T}_+, w)]^2)$.

We will also make use of the matrix operators

$$K^{\pm 1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix} \in \mathcal{L}([L^2(\mathbb{T}_+, w)]^2), \tag{9}$$

and

$$G^{\pm 1}(t) = \operatorname{diag}(1, t^{\pm 1}). \tag{10}$$

Finally, we will consider the operator N and its inverse N^{-1} defined by

$$N(\zeta)(t) = \zeta(t^2), \qquad N^{-1}(\zeta)(t) = \zeta(t^{1/2}), \tag{11}$$

with $N \in \mathcal{L}([L^2(\mathbb{T}, w)]^2, [L^2(\mathbb{T}_+, w)]^2)$, and $N^{-1} \in \mathcal{L}([L^2(\mathbb{T}_+, w)]^2, [L^2(\mathbb{T}, w)]^2)$.

The operators above take part in the construction of the following equivalence relation.

Theorem 2.1 ([4, Theorem 2.2]) *The initial singular integral operator with reflection* $(J\varphi)(t) = \varphi(-t)$, $t \in \mathbb{T}$,

$$\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J$$

(acting between $L^2(\mathbb{T}, w)$ *spaces) is equivalent to the matrix singular integral operator (without shift)*

$$\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}}, \qquad \mathcal{D}_{\mathbb{T}} \in \mathcal{L}[L^2(\mathbb{T}, w)]^2.$$

The operator equivalence relation between $\mathcal{A}$ *and* $\mathcal{D}_{\mathbb{T}}$ *is presented in the form of the following similarity transformation*

$$F^{-1} \mathcal{A} F = \mathcal{D}_{\mathbb{T}}, \tag{12}$$

where

$$F = MKGN \in \mathcal{L}([L^2(\mathbb{T}, w)]^2, L^2(\mathbb{T}, w)),$$
$$F^{-1} = N^{-1} G^{-1} K^{-1} M^{-1} \in \mathcal{L}(L^2(\mathbb{T}, w), [L^2(\mathbb{T}, w)]^2)$$

and the explicit form of the operators $M^{\pm 1}$, $K^{\pm 1}$, $G^{\pm 1}$, $N^{\pm 1}$ *is given in* (7), (8), (9), (10) *and* (11).

The connection between the coefficients of the operators $\mathcal{A}$ and $\mathcal{D}_{\mathbb{T}}$ is given by the formulas:

$$u_{\mathbb{T}}(t) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ t^{-1/2} & -t^{-1/2} \end{pmatrix} u_1(t^{1/2}) \begin{pmatrix} 1 & t^{1/2} \\ 1 & -t^{1/2} \end{pmatrix} \tag{13}$$

and

$$v_{\mathbb{T}}(t) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ t^{-1/2} & -t^{-1/2} \end{pmatrix} v_1(t^{1/2}) \begin{pmatrix} 1 & t^{1/2} \\ 1 & -t^{1/2} \end{pmatrix}. \tag{14}$$

where

$$u_1(t) = \begin{pmatrix} r_{\mathbb{T}_+} a_0(t) & r_{\mathbb{T}_+} a_1(t) \\ r_{\mathbb{T}_+} a_0(-t) & r_{\mathbb{T}_+} a_1(-t) \end{pmatrix},$$

and

$$v_1(t) = \begin{pmatrix} r_{\mathbb{T}_+} b_0(t) & r_{\mathbb{T}_+} b_1(t) \\ r_{\mathbb{T}_+} b_0(-t) & r_{\mathbb{T}_+} b_1(-t) \end{pmatrix}.$$

Now, we will formulate an equivalence relation for the case of the flip operator. Consider the following operators: $B : L^2(\mathbb{T}) \to L^2(\mathbb{R})$ defined by

$$(B\phi)(x) = \frac{1}{x+i}\phi\left(\frac{x-i}{x+i}\right), \quad x \in \mathbb{R} \tag{15}$$

which inverse is

$$(B^{-1}\psi)(t) = \frac{i2^{1/2}}{1-t}\psi\left(i\frac{1+t}{1-t}\right), \quad t \in \mathbb{T} \setminus \{1\}, \tag{16}$$

and the operator $M_{\mathbb{R}_+}$ given by the rule

$$M_{\mathbb{R}_+}\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \varphi(x) := \begin{cases} \varphi_1(x), & x \in \mathbb{R}_+ \\ \varphi_2(-x), & x \in \mathbb{R}_- \end{cases} \tag{17}$$

(where $\mathbb{R}_+ := (0 + \infty)$ and $\mathbb{R}_- := (-\infty, 0)$); the matrix operators

$$K^{\pm 1} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ I & -I \end{pmatrix} \in \mathcal{L}([L^2(\mathbb{R}_+)]^2), \tag{18}$$

as well the operator

$$(N_{\mathbb{R}_+}\varphi)(x) = \varphi(x^2), \qquad N_{\mathbb{R}_+} \in \mathcal{L}([L^2(\mathbb{R}_+, |x|^{-1/4})]^2, [L^2(\mathbb{R}_+)]^2), \tag{19}$$

and the operator $R_{\mathbb{R}_+}$ given by

$$R_{\mathbb{R}_+} = \begin{pmatrix} S_{\mathbb{R}_+} + U_{1,\mathbb{R}_+} & 0 \\ 0 & I_{\mathbb{R}_+} \end{pmatrix} \in \mathcal{L}([L^2(\mathbb{R}_+)]^2) \tag{20}$$

where

$$(S_{\mathbb{R}_+}f)(x) = \frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{f(u)}{u-x}du$$

and

$$(U_{1,\mathbb{R}_+}f)(x) = \frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{f(u)}{u+x}du\,, \ x \in \mathbb{R}_+.$$

Note that $S_{\mathbb{R}_+} + U_{1,\mathbb{R}_+}$ is an invertible operator and its inverse is given by $S_{\mathbb{R}_+} - U_{1,\mathbb{R}_+}$. Thus, $R_{\mathbb{R}_+}$ is also an invertible operator.

All these operators are used in the following operator equivalence relation which is given in explicit form

$$\mathcal{G}\mathcal{A}\mathcal{V} = D_{\mathbb{R}_+},$$

with operators $\mathcal{G}$ and $\mathcal{V}$ defined by

$$\begin{aligned} \mathcal{G} &= N_{\mathbb{R}_+}^{-1}K^{-1}M_{\mathbb{R}_+}^{-1}B \ \in \ \mathcal{L}(L^2(\mathbb{T}), [L^2(\mathbb{R}_+, |x|^{-1/4})]^2), \\ \mathcal{V} &= B^{-1}M_{\mathbb{R}_+}KR_{\mathbb{R}_+}N_{\mathbb{R}_+} \ \in \ \mathcal{L}([L^2(\mathbb{R}_+, |x|^{-1/4})]^2, L^2(\mathbb{T})), \end{aligned}$$

and the explicit form of the operators $B^{\pm 1}$, $M_{\mathbb{R}_+}^{\pm 1}$, $K^{\pm 1}$, $N_{\mathbb{R}_+}^{\pm 1}$ and $R_{\mathbb{R}_+}$ is given in (15)–(20). The operator $D_{\mathbb{R}_+}$ has the form

$$D_{\mathbb{R}_+} = u_{\mathbb{R}_+} I_{\mathbb{R}_+} + v_{\mathbb{R}_+} S_{\mathbb{R}_+}.$$

The relation between the coefficients $u_{\mathbb{R}_+}$ and $v_{\mathbb{R}_+}$ of this operator $D_{\mathbb{R}_+}$ and the coefficients of the operator $\mathcal{A}$ is given by the formulas:

$$u_{\mathbb{R}_+}(x) = \frac{1}{2}\begin{pmatrix} (a_1(y)+b_1(y)) - (a_1(-y)+b_1(-y)) & (a_0(y)+b_0(y)) - (a_0(-y)+b_0(-y)) \\ (a_1(y)+b_1(y)) + (a_1(-y)+b_1(-y)) & (a_0(y)+b_0(y)) + (a_0(-y)+b_0(-y)) \end{pmatrix}$$

$$v_{\mathbb{R}_+}(x) = \frac{1}{2}\begin{pmatrix} (a_0(y)-b_0(y)) + (a_0(-y)-b_0(-y)) & (a_1(y)-b_1(y)) + (a_1(-y)-b_1(-y)) \\ (a_0(y)-b_0(y)) - (a_0(-y)-b_0(-y)) & (a_1(y)-b_1(y)) - (a_1(-y)-b_1(-y)) \end{pmatrix}$$

where

$$y = \frac{x^{1/2}-i}{x^{1/2}+i}, \qquad x \in \mathbb{R}_+.$$

Next, the operator $D_{\mathbb{R}_+}$ is extended by the identity into the $[L^2(\mathbb{R}, |x|^{-1/4})]^2$ space. This is in fact an equivalence after extension relation (see [1]) applied to $D_{\mathbb{R}_+}$ where the resulting operator has the form:

$$D_{\mathbb{R}} := \begin{pmatrix} D_{\mathbb{R}_+} & 0 \\ 0 & I_{[L^2(\mathbb{R}_-, |x|^{-1/4})]^2} \end{pmatrix} \in \mathcal{L}([L^2(\mathbb{R}, |x|^{-1/4})]^2).$$

In addition, the operator $D_{\mathbb{R}}$ can also be written in the form

$$D_{\mathbb{R}} = u_{\mathbb{R}} I_{\mathbb{R}} + v_{\mathbb{R}} S_{\mathbb{R}}$$

where

$$u_{\mathbb{R}} = \chi_{\mathbb{R}_-} + \ell_0 u_{\mathbb{R}_+}, \qquad v_{\mathbb{R}} = \ell_0 v_{\mathbb{R}_+},$$

with ℓ_0 being the zero extension operator, and where $\chi_{\mathbb{R}_-}$ is the characteristic function on $\mathbb{R}_-$.

Now we pass from $\mathcal{D}_{\mathbb{R}}$ to a singular integral operator $\mathcal{D}_{\mathbb{T}}$ using the isometric isomorphism

$$B_2 := \mathrm{diag}(B, B) \tag{21}$$

from $[L^2(\mathbb{R}, |x|^{-1/4})]^2$ onto $[L^2(\mathbb{T}, \gamma)]^2$ with the (Khvedelidze) weight

$$\gamma(t) = \left| i\frac{1+t}{1-t} \right|^{-1/4}.$$

In explicit form:

$$\mathcal{D}_{\mathbb{T}} := B_2^{-1} \mathcal{D}_{\mathbb{R}} B_2 = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}},$$

with

$$u_{\mathbb{T}} I_{\mathbb{T}} = B_2^{-1} u_{\mathbb{R}} B_2, \qquad v_{\mathbb{T}} I_{\mathbb{T}} = B_2^{-1} v_{\mathbb{R}} B_2. \tag{22}$$

More precisely, $u_{\mathbb{T}} = \mathrm{diag}(B_0, B_0) u_{\mathbb{R}}$, and $v_{\mathbb{T}} = \mathrm{diag}(B_0, B_0) v_{\mathbb{R}}$, where

$$(B_0 a)(t) = a\left(i \frac{1+t}{1-t}\right), \quad t \in \mathbb{T} \setminus \{1\}.$$

Theorem 2.2 ([5, Proposition 1]) *The singular integral operator*

$$\mathcal{A} = a_0 I_{\mathbb{T}} + b_0 S_{\mathbb{T}} + a_1 J + b_1 S_{\mathbb{T}} J$$

(acting on the space $L^2(\mathbb{T})$) with Carleman shift operator $(J\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right)$, $t \in \mathbb{T}$, is equivalent after extension to the matrix singular integral operator

$$\mathcal{D}_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}}, \quad \mathcal{D}_{\mathbb{T}} \in \mathcal{L}([L^2(\mathbb{T}, \gamma)]^2), \tag{23}$$

where $\gamma(t) = |i\frac{1+t}{1-t}|^{-1/4}$, and $u_{\mathbb{T}} = \mathrm{diag}(B_0, B_0) u_{\mathbb{R}}$, and $v_{\mathbb{T}} = \mathrm{diag}(B_0, B_0) v_{\mathbb{R}}$.

3. The Moore-Penrose invertibility of $\mathcal{A}$

We will start by introducing a general framework which will then applied to our operator $\mathcal{A}$.

Let F be a finite dimensional Banach space with $\dim F = m$. We recall that the k-th approximation number ($k \in \{0, 1, \ldots, m\}$) of an operator $A \in \mathcal{L}(F)$ is defined by

$$s_k(A) = \mathrm{dist}(A, \mathcal{F}_{m-k}) := \inf\{\|A - F\| \ : \ F \in \mathcal{F}_{m-k}\},$$

where $\mathcal{F}_{n-k}$ denotes the collection of all operators (or matrices from $\mathbb{C}^{n\times n}$) having the dimension of the range equal to at most $n - k$. It is clear that

$$0 \le s_1(A) \le \cdots \le s_m(A) = \|A\|_{\mathcal{L}(F)}.$$

Notice that the approximation numbers can be also defined as the singular values of a square matrix $A_n \in \mathbb{C}^{nN \times nN}$ which are the square roots of the spectral points of $A_n^* A_n$, where A_n^* means the adjoint matrix of A_n.

Definition 3.1 A sequence (A_n) of matrices $nN \times nN$ is said to have the k-splitting property if there is an integer $k \ge 0$ such that

$$\lim_{n\to\infty} s_k(A_n) = 0 \quad \text{and} \quad \liminf_{n\to\infty} s_{k+1}(A_n) > 0.$$

The number k is called the splitting number. Alternatively, we say the singular values (computed via $A_n^* A_n$) of a sequence (A_n) of $k(n) \times l(n)$ matrices A_n have the splitting property if there exist a sequence $c_n \to 0$ ($c_n \ge 0$) and a number $d > 0$ such that they are contained in $[0, c_n] \cup [d, \infty)$ for all n. Moreover, the singular values of A_n are said to meet the k-splitting property if, in addition, for all sufficiently large n, exactly k singular values of A_n lie in $[0, c_n]$.

In general terms, given a bounded linear operator A on a Banach space X ($A \in \mathcal{L}(X)$), and an element f of X, let us consider the abstract operator equation

$$A\varphi = f. \tag{24}$$

For the approximate solution of this equation, we choose to approximate closed subspaces X_n in which the approximate solutions φ_n of (24) will be sought. In practice, the X_n spaces usually have finite dimension but we will not require this assumption. We will assume that X_n are ranges of certain projection operators $L_n : X \longrightarrow X_n$ so that these projections converge strongly to the identity operator: $s-\lim_{n\to\infty} L_n = I$. This strong convergence implies that $\cup_{n=1}^{\infty} X_n$ is dense in X.

Having fixed subspaces X_n, we choose convenient linear operators $A_n : X_n \longrightarrow X_n$ and consider in the place of (24) the equations

$$A_n\varphi_n = L_n f, \quad n = 1, 2, \dots , \tag{25}$$

with their solutions sought in $X_n = \operatorname{Im} L_n$.

A sequence (A_n) of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ is an *approximation method* for $A \in \mathcal{L}(X)$ if $A_n L_n$ converges strongly to A as $n \to \infty$.

Note that even if (A_n) is an approximation method for A, we do not yet know anything about the solvability of the equations (25) and about the relations between (eventual) solutions φ_n of (25) and the (possible) solution φ of (24).

The approximation method (A_n) for A is *applicable* if there exists a number n_0 such that the equations (25) possess unique solutions φ_n for every $n \geq n_0$ and every right-hand side $f \in X$, and if these solutions converge in the norm of X to a solution of (24). An equivalent characterization of applicable approximation methods is the notion of *stability*, where a sequence (A_n) of operators $A_n \in \mathcal{L}(\operatorname{Im} L_n)$ is called *stable* if there exists a number n_0 such that the operators A_n are invertible for every $n \geq n_0$ and if the norms of their inverses are uniformly bounded:

$$\sup_{n\geq n_0} \|A_n^{-1} L_n\| < \infty.$$

Let (L_n) be a sequence of projections converging strongly to the identity $I \in \mathcal{L}(X)$. The idea of any projection method for the approximate solution of (24) is to choose a further sequence (R_n) of projections which also converge strongly to the identity and which satisfy $\operatorname{Im} R_n = \operatorname{Im} L_n$. Thus, we choose $A_n = R_n A L_n : \operatorname{Im} L_n \longrightarrow \operatorname{Im} L_n$ as the approximate operators of A. In fact, Lemma 1.5 in [7] proves that $(R_n A L_n)$ is indeed an approximate method for A.

In the most interesting case of X being an infinite dimensional Banach space, a sequence (X_n) of finite dimensional subspaces of X needs to be considered. Moreover, we assume that there is a sequence (L_n) of projections from X onto X_n with strong limit $I \in X$ as $n \to \infty$. Let $\mathcal{F}$ refer to the set of all sequences $(A_n)_{n=0}^{\infty}$ of operators

$A_n \in \mathcal{L}(\operatorname{Im} L_n)$ which are uniformly bounded: $\sup\{\|A_n L_n\| \,:\, n \geq 0\} < \infty$. The "algebraization" of $\mathcal{F}$ is given by the natural operations

$$\lambda_1(A_n) + \lambda_2(B_n) := (\lambda_1 A_n + \lambda_2 B_n), \quad (A_n)(B_n) := (A_n B_n) \tag{26}$$

and

$$\|(A_n)\|_{\mathcal{F}} := \sup\{\|A_n L_n\| \,:\, n \geq 0\}$$

which make $\mathcal{F}$ to be an initial Banach algebra with identity $(I_{|\operatorname{Im} L_n})$. The set $\mathcal{I}$ of all sequences (G_n) in $\mathcal{F}$ with $\lim_{n\to\infty} \|G_n L_n\| = 0$ is a closed two sided ideal in $\mathcal{F}$. The Kozak's Theorem (Theorem 1.5 in [7]) establish that a sequence $(A_n) \in \mathcal{F}$ is stable if and only if its coset $(A_n) + \mathcal{I}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{I}$.

If instead of a Banach space X we consider a Hilbert space $\mathcal{H}$ and L_n to be the orthogonal projections P_n from $\mathcal{H}$ onto $\mathcal{H}_n$, then $(A_n)^* = (A_n^*)$ defines an involution in $\mathcal{F}$ which makes $\mathcal{F}$ a C^*-algebra. Note that in this case the approximation numbers of an operator $A_n \in \mathcal{L}(\mathcal{H}_n)$ are just the singular values of A_n.

Let further T be a (possible infinite) index set and suppose that, for every $t \in T$, we are given an infinite dimensional Hilbert space $\mathcal{H}^t$ with identity operator I^t as well as a sequence (E_n^t) of partial isometries $E_n^t : \mathcal{H}^t \longrightarrow \mathcal{H}$ such that the initial projections P_n^t of E_n^t converge strongly to I^t as $n \to \infty$, the range projection of E_n^t is P_n and the separation condition

$$(E_n^s)^* E_n^t \longrightarrow 0 \quad \text{weakly as } n \to \infty \tag{27}$$

holds for every $s, t \in T$ with $s \neq t$. Recall that an operator $E : \mathcal{H}' \longrightarrow \mathcal{H}''$ is a partial isometry if $EE^*E = E$ and that E^*E and EE^* are orthogonal projections (which are called the initial and the range projections of E, respectively). The restriction of E to $\operatorname{Im}(E^*E)$ is an isometry from $\operatorname{Im}(E^*E)$ onto $\operatorname{Im}(EE^*) = \operatorname{Im} E$. We write E_{-n}^t instead of $(E_n^t)^*$, and set $\mathcal{H}_n := \operatorname{Im} P_n$ and $\mathcal{H}_n^t := \operatorname{Im} P_n^t$.

Let $\mathcal{F}^T$ stand for the set of all sequences $(A_n) \in \mathcal{F}$ for which the strong limits

$$s - \lim_{n\to\infty} E_{-n}^t A_n E_n^t \quad \text{and} \quad s - \lim_{n\to\infty} (E_{-n}^t A_n E_n^t)^*$$

exist for every $t \in T$, and define mappings $W^t : \mathcal{F}^T \longrightarrow \mathcal{L}(\mathcal{H}^t)$ by

$$W^t(A_n) := s - \lim_{n\to\infty} E_{-n}^t A_n E_n^t.$$

The algebra $\mathcal{F}^T$ is a C^*-subalgebra of $\mathcal{F}$ which contains the identity, and W^t are $*$-homomorphisms. Moreover, $\mathcal{F}^T$ is a *standard* algebra. This means that any sequence $(A_n) \in \mathcal{F}^T$ is stable if and only if all the operators $W^t(A_n)$ are invertible.

The separation condition (27) ensures that, for every $t \in T$ and every compact operator $K^t \in \mathcal{K}(\mathcal{H}^t)$, the sequence $(E_n^t K^t E_{-n}^t)$ belongs to the algebra $\mathcal{F}^T$, and for all $s \in T$

$$W^s(E_n^t K^t E_{-n}^t) = \begin{cases} K^t & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases} \tag{28}$$

Conversely, the above identity implies the separation condition (27). Moreover, the ideal $\mathcal{I}$ belongs to $\mathcal{F}^T$. So we can introduce the smallest closed ideal $\mathcal{J}^T$ of $\mathcal{F}^T$ which contains all sequences $(E_n^t K^t E_{-n}^t)$ with $t \in T$ and $K^t \in \mathcal{K}(\mathcal{H}^t)$, as well as all sequences $(G_n) \in \mathcal{I}$.

Corresponding to the ideal $\mathcal{J}^T$, we introduce a class of Fredholm sequences by calling a sequence $(A_n) \in \mathcal{F}^T$ Fredholm if the coset $(A_n) + \mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$. It is also known (see [7]) that if $(A_n) \in \mathcal{F}^T$ is Fredholm, then all operators $W^t(A_n)$ are Fredholm on $\mathcal{H}^t$, and the number of the non-invertible operators among the $W^t(A_n)$ is finite.

We have now the possibility to recall an important result concerning standard algebras and relating some of the just presented notions.

Theorem 3.1 (see [7]) *Let (A_n) be a sequence from the standard C^*-algebra $\mathcal{F}^T$.*

(i) *If the coset $(A_n)+\mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{F}^T/\mathcal{J}^T$, then all operators $W^t(A_n)$ are Fredholm on $\mathcal{H}^t$, the number of the non-invertible operators among the $W^t(A_n)$ is finite, and the singular values of A_n have the k-splitting property with*

$$k(A_n) = \sum_{t\in T} \dim\ker W^t(A_n).$$

(ii) *If $W^t(A_n)$ is not Fredholm for at least one $t \in T$, then for every integer $k \geq 0$*

$$s_k(A_n) \longrightarrow 0, \quad as \quad n \longrightarrow \infty.$$

An specific algebraization of the stability for the operators under study runs as follows. We start by considering the Fourier projection $P_n \in \mathcal{L}([L^2(\mathbb{T},w)]^2)$ that in terms of the Fourier coefficients of a function $\psi \in [L^2(\mathbb{T},w)]^2$ acts componentwise according to the rule

$$\psi = \sum_{k\in\mathbb{Z}} \psi_k t^k \longmapsto \sum_{k=-n}^{n} \psi_k t^k, \qquad n \in \mathbb{N}.$$

In addition, we take the Lagrange interpolation operator L_n (which is bounded in $[L^2(\mathbb{T},w)]^2$, see for instance [2]) associated to the points

$$t_j = \exp\left(\frac{2\pi i j}{2n+1}\right), \quad j = 0,1,\dots,2n.$$

That is, L_n assigns to a function ψ its Lagrange interpolation polynomial $L_n\psi \in \operatorname{Im} P_n$, uniquely determined, on each component, by the conditions $(L_n\psi)(t_j) = \psi(t_j)$, $j = 0,1,\dots,2n$. One can show that $\|P_n\psi - \psi\|_{2,w} \longrightarrow 0$ as $n \longrightarrow \infty$ for every $\psi \in [L^2(\mathbb{T},w)]^2$ and in [9] it was proved (for the scalar case) that $\|L_n\psi - \psi\|_{2,w} \longrightarrow 0$, $n \longrightarrow \infty$.

Let us now construct

$$A_n := L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n, \quad n \in \mathbb{Z}_+, \tag{29}$$

where the operator $W_n \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$ –which componentwise is the discrete version of the flip operator (5)– acts by the rule

$$W_n\psi = \sum_{k=0}^{n} \psi_{n-k}t^k + \sum_{k=-n}^{-1} \psi_{-n-k-1}t^k.$$

First, note that the operators W_n and P_n are related as follows:

$$W_n^2 = P_n, \quad W_nP_n = P_nW_n = W_n. \tag{30}$$

On the other hand, in [7, 8, 10] it was shown that:

$$L_naI_{\mathbb{T}} = L_naL_n, \quad S_{\mathbb{T}}P_n = P_nS_{\mathbb{T}}P_n, \quad W_nL_naW_n = L_n\widetilde{a}P_n \tag{31}$$

$$(L_naP_n)^* = L_n\overline{a}P_n, \quad (P_nS_{\mathbb{T}}P_n)^* = P_nS_{\mathbb{T}}P_n \tag{32}$$

where for $a \in L^\infty(\mathbb{T})$,

$$\widetilde{a}(t) = a\left(\frac{1}{t}\right), \quad t \in \mathbb{T}.$$

We denote by T_2 the index set $\{1, 2\}$ and by $\mathcal{F}^{T_2}$ the C^*-algebra of all operator sequences (A_n), with $A_n \in \mathcal{L}(\operatorname{Im} P_n)$, for which there exist operators ($*$-homomorphisms) $W^1(A_n)$, $W^2(A_n) \in \mathcal{L}([L^2(\mathbb{T}, w)]^2)$ such that

$$s - \lim_{n\to\infty} P_nA_nP_n = W^1(A_n) \quad \text{and} \quad s - \lim_{n\to\infty} W_nA_nW_n = W^2(A_n)$$

$$s - \lim_{n\to\infty} (P_nA_nP_n)^* = W^1(A_n)^* \quad \text{and} \quad s - \lim_{n\to\infty} (W_nA_nW_n)^* = W^2(A_n)^*.$$

Furthermore, let us introduce the subsets $\mathcal{J}^1$ and $\mathcal{J}^2$ of the C^*-algebra $\mathcal{F}^{T_2}$:

$$\begin{aligned} \mathcal{J}^1 &= \{(P_nKP_n) + (G_n) \ : \ K \in \mathcal{K}([L^2(\mathbb{T}, w)]^2), \ \|G_n\| \to \infty\} \\ \mathcal{J}^2 &= \{(W_nLW_n) + (G_n) \ : \ L \in \mathcal{K}([L^2(\mathbb{T}, w)]^2), \ \|G_n\| \to \infty\}. \end{aligned}$$

Again, $\mathcal{J}^{T_2}$ is the smallest closed two-sided ideal of $\mathcal{F}^{T_2}$ which contains all sequences (J_n) such that J_n belongs to one of the ideals $\mathcal{J}^t$, $t = 1, 2$.

In the case when equation (24) is solvable, in general it is not uniquely solvable (which occurs also in the case of our operator $\mathcal{A}$). In Hilbert spaces a distinguished generalized solution of (24) –the *least square solution*– can be obtained as follows: among all x in a Hilbert space $\mathcal{H}$ which minimize $\|Ax - y\|$ choose that one with minimal $\|x\|$. The Moore-Penrose inverse A^+ of A is such that the least square solution of $Ax = y$ is given by $x = A^+y$.

In more detail, an operator $A \in \mathcal{L}(\mathcal{H})$ is said to be *Moore-Penrose invertible* if there is an operator $B \in \mathcal{L}(\mathcal{H})$ such that

$$ABA = A, \quad BAB = B, \quad (AB)^* = AB, \quad (BA)^* = BA.$$

If such an operator B exists, then it is unique and we denote it by A^+. It is also well-known that an operator is Moore-Penrose invertible if and only if its range is closed (normally solvable). In addition, note that if A is invertible then A^{-1} coincides with A^+.

Let $P_M^{\mathcal{H}}$ denote the orthogonal projection onto the closed subspace $M \subset \mathcal{H}$.

The following results about Moore-Penrose invertibility are well-known.

Proposition 3.1 (see [11]) *The following statements are equivalent:*

(i) *The operator $A \in \mathcal{L}(\mathcal{H})$ is Moore-Penrose invertible.*

(ii) *The operator $A^*A + P^{\mathcal{H}}_{\ker A}$ is invertible.*

(iii) *The operator $AA^* + P^{\mathcal{H}}_{\ker A^*}$ is invertible.*

Moreover, if one of the above conditions is fulfilled then

$$A^+ = (A^*A + P^{\mathcal{H}}_{\ker A})^{-1}A^* = A^*(AA^* + P^{\mathcal{H}}_{\ker A^*})^{-1}.$$

Moore-Penrose invertibility can be defined for elements in a C^*-algebra.

Proposition 3.2 (see [7, 11]) (i) *An element A of a C^*-algebra with identity is Moore-Penrose invertible if and only if the element AA^* is invertible or if 0 is an isolated point of the spectrum (denoted by sp) of A^*A. If this condition is fulfilled, then $\|A^+\| = \min\{\text{sp}(AA^* \setminus \{0\})\}$.*

(ii) *C^*-subalgebras of C^*-algebras with identity are inverse closed with respect to Moore-Penrose invertibility.*

A sequence of operators (A_n) satisfying $A_nP_n \longrightarrow A$ and $A_n^*P_n \longrightarrow A^*$ is said to be Moore-Penrose stable if

$$\sup_{n \geq 1} \|A_n^+\| < \infty.$$

Recall that A_n^+ exists for all n because $\dim \operatorname{Im} P_n < \infty$. Theorem 2.12 in [7] states that if (A_n) is Moore-Penrose stable, then A is Moore-Penrose invertible and $A_n^+ \longrightarrow A^+$, strongly as $n \longrightarrow \infty$.

We will apply these results to the C^*-algebra $\mathcal{F}^{T_2}$ given previously, and to some C^*-subalgebras of it. In particular, we are going to study the Moore-Penrose stability of the Fredholm sequence (A_n) defined in (29).

Proposition 3.3 (Cf. Proposition 6.9 in [7]) *Let* $a, b \in [PC(\mathbb{T})]^{2\times 2}$ *and suppose that* $(L_n(aI_{\mathbb{T}} + bS_{\mathbb{T}})P_n)$ *is a Fredholm sequence (equivalently, suppose* $aI_{\mathbb{T}} + bS_{\mathbb{T}}$ *and* $\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}}$ *to be Fredholm operators). If* $\ker(aI_{\mathbb{T}} + bS_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$ *and* $\ker(\widetilde{a}I_{\mathbb{T}} + \widetilde{b}S_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$ *for a certain* n_0*, then*

$$P^{\operatorname{Im} P_n}_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)} = P^{[L^2(\mathbb{T},w)]^2}_{\ker(aI_{\mathbb{T}}+bS_{\mathbb{T}})} + W_n P^{[L^2(\mathbb{T},w)]^2}_{\ker(\widetilde{a}I_{\mathbb{T}}+\widetilde{b}S_{\mathbb{T}})} W_n$$

for all sufficiently large n.

From this result, the connection between the k-splitting property and the Moore-Penrose stability is clear:

$$\dim\ker A_n = \dim\ker W^1(A_n) + \dim\ker W^2(A_n).$$

The above proposition implies that (A_n) is a Moore-Penrose stable sequence, and from Proposition 6.5 in [7] we have that the sequence $A_n^* A_n + P^{\operatorname{Im} P_n}_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)}$ is stable and the sequence

$$B_n := \left(A_n A_n^* + P^{\operatorname{Im} P_n}_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)}\right)^{-1} A_n^*, \quad \text{for all sufficiently large } n,$$

is the Moore-Penrose inverse A_n.

Finally, we are now in conditions to provide the explicit Moore-Penrose inverse of the operator $\mathcal{A}$ defined on (3) with the Carleman shift operator J as in (4) or in (5).

Theorem 3.2 *Let us suppose* $\mathcal{A}$ *to be Fredholm. Moreover, assume that for a certain* n_0*,* $\ker(\mathcal{D}_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$ *and* $\ker(\widetilde{\mathcal{D}}_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$*, where the operator* $\mathcal{D}_{\mathbb{T}}$ *is given as in Theorem 2.1 in the case of* J *to be the shift operator* (4) *and as in Theorem 2.2 for* J *in* (5) *with, in each case,* $\widetilde{\mathcal{D}}_{\mathbb{T}} = \widetilde{u}_{\mathbb{T}} I_{\mathbb{T}} + \widetilde{v}_{\mathbb{T}} S_{\mathbb{T}}$*, where for a function* $a \in [PC(\mathbb{T})]^{2\times 2}$ *we have* $\widetilde{a}(t) = a\left(\frac{1}{t}\right)$*,* $t \in \mathbb{T}$*. Then, the operator* $\mathcal{A}$ *is Moore-Penrose invertible by* $\mathcal{A}^+$*, where:*

(1) *For the shift operator* $(J\varphi)(t) = \varphi(-t)$*,*

$$\mathcal{A}^+ = MKGN\left[\left(\mathcal{D}_{\mathbb{T}}^*\mathcal{D}_{\mathbb{T}} + P^{[L^2(\mathbb{T},w)]^2}_{\ker\mathcal{D}_{\mathbb{T}}}\right)^{-1}\mathcal{D}_{\mathbb{T}}^*\right] N^{-1}G^{-1}K^{-1}M^{-1},$$

with $\mathcal{A}^+ \in \mathcal{L}(L^2(\mathbb{T}, w))$ *and* $w \in A_2^e(\mathbb{T})$*. We recall that the explicit form of the operators* $M^{\pm 1}$*,* K*,* $G^{\pm 1}, N^{\pm 1}$ *and* $\mathcal{D}_{\mathbb{T}}$ *are given in* (7)–(12)*;*

(2) *In the case of the shift operator* $(J\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right)$*, we have*

$$\begin{aligned}\mathcal{A}^+ \;=\;& B^{-1}M_{\mathbb{R}_+}KR_{\mathbb{R}_+}N_{\mathbb{R}_+}\operatorname{Rest}|_{[L^2(\mathbb{R}_+,|x|^{-1/4})]} B_2\left[\left(\mathcal{D}_{\mathbb{T}}^*\mathcal{D}_{\mathbb{T}} + P^{[L^2(\mathbb{T},\gamma)]^2}_{\ker\mathcal{D}_{\mathbb{T}}}\right)^{-1}\mathcal{D}_{\mathbb{T}}^*\right] \\ & B_2^{-1}N_{\mathbb{R}_+}^{-1}K^{-1}M_{\mathbb{R}_+}^{-1}B,\end{aligned}$$

with $\mathcal{A}^+ \in \mathcal{L}(L^2(\mathbb{T}))$ *and* γ *is the weight* $\gamma(t) = \left| i\frac{1+t}{1-t} \right|^{-1/4}$. *The explicit form of the operators* $B^{\pm 1}, M^{\pm 1}_{\mathbb{R}_+}, N^{\pm 1}_{\mathbb{R}_+}, K, R_{\mathbb{R}_+}, B_2^{\pm 1}$ *and* $\mathcal{D}_{\mathbb{T}}$ *are given in* (15)–(21) *and* (23).

Proof. Since $\mathcal{A}$ is a Fredholm operator, then $\mathcal{A}$ is a Moore-Penrose invertible operator. Also, from Theorem 2.1 (Theorem 2.2) we have that $\mathcal{A}$ is equivalent (equivalent after extension) to the operator $\mathcal{D}_{\mathbb{T}}$ with coefficients $u_{\mathbb{T}}$ and $v_{\mathbb{T}}$ depending of the shift J.

On the other hand, we know that $(A_n) = (L_n(u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}})P_n)$ converge strongly to $\mathcal{D}_{\mathbb{T}}$, as $n \longrightarrow \infty$. Also, the hypothesis that $\ker(\mathcal{D}_{\mathbb{T}}) \subseteq \operatorname{Im} P_{n_0}$ and $\ker(\widetilde{\mathcal{D}_{\mathbb{T}}}) \subseteq \operatorname{Im} P_{n_0}$ for a certain n_0, allow us to apply Proposition 3.3. Thus, we have that (A_n) is a Moore-Penrose stable sequence with Moore-Penrose inverse

$$B_n := (A_n A_n^* + P^{\operatorname{Im} P_n}_{\ker(L_n(aI_{\mathbb{T}}+bS_{\mathbb{T}})P_n)})^{-1} A_n^*.$$

Moreover, $B_n \longrightarrow \mathcal{D}_{\mathbb{T}}^+$.

The explicit form of $\mathcal{A}^+$ is obtained from B_n, as $n \longrightarrow \infty$, and from the operators given in Theorem 2.1, for the shift operator J given on (4), and Theorem 2.2 for J defined on (5) (being fundamental in this last step to have in complete explicit form the corresponding operator relations). □

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