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Fredholm property of matrix Wiener-Hopf plus and minus Hankel operators with piecewise almost periodic symbols

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Abstract

We will present conditions for the Fredholm property of Wiener-Hopf plus and minus Hankel operators with different Fourier matrix symbols in the C^*-algebra of piecewise almost periodic elements. In addition, under such conditions, we will obtain a formula for the sum of the Fredholm indices of these Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators. These results are useful for the treatment of different problems from Mathematical-Physics (e.g., in wedge diffraction problems).

Keywords: Fredholm property, Fredholm index, Wiener-Hopf operator, Hankel operator, piecewise almost periodic matrix-valued function.

1. Introduction

The analysis of solvability of different types of mathematical-physics problems can be equivalently reformulated by corresponding problems of analyzing the invertibility and Fredholm property of Wiener-Hopf plus and minus Hankel operators with piecewise almost periodic Fourier symbols. This is particularly evident in several examples of wave diffraction problems where an operator theoretical machinery can be applied in order to produce the indicated equivalent formulation (see, e.g., [4, 6, 7, 10, 11, 14]). Having this motivation in mind, the present paper is devoted to the Fredholm analysis of matrix Wiener-Hopf plus/minus Hankel operators of the form

$$W_{\Phi_1} \pm H_{\Phi_2} \; : \; [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \qquad (N \in \mathbb{N}), \tag{1}$$

with W_{Φ_1} and H_{Φ_2} being matrix Wiener-Hopf and Hankel operators defined by $W_{\Phi_1} = r_+\mathcal{F}^{-1}\Phi_1 \cdot \mathcal{F}$ and $H_{\Phi_2} = r_+\mathcal{F}^{-1}\Phi_2 \cdot \mathcal{F}J$, respectively. As about additional notation, for a Banach algebra B, we denote by $B^{N\times N}$ the Banach algebra of all $N \times N$ matrices with entries in B, and B^N will denote the Banach space of all N dimensional vectors with entries in B. We use $[L^2_+(\mathbb{R})]^N$ to denote the subspace of $[L^2(\mathbb{R})]^N$

formed by all the matrix functions supported on the closure of $\mathbb{R}_+ = (0, +\infty)$, r_+ represents the operator of restriction from $[L^2_+(\mathbb{R})]^N$ into $[L^2(\mathbb{R}_+)]^N$, $\mathcal{F}$ denotes the Fourier transformation, J is the reflection operator given by the rule $J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$, and $\Phi_1,\ \Phi_2 \in [L^\infty(\mathbb{R})]^{N\times N}$ are the Fourier matrix symbols.

The main purpose of the present work is to obtain conditions which will characterize the situation when $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are at the same time Fredholm operators, and to present a formula for the sum of their Fredholm indices. It is clear that it is possible to occur that $W_{\Phi_1} \pm H_{\Phi_2}$ will not be Fredholm at the same time. However, the method which we will be using does not provide independent Fredholm information for these two types of operators. Anyway, simultaneous Fredholm information about Wiener-Hopf plus Hankel and Wiener-Hopf minus Hankel operators is already very useful in the applications mentioned above (as it can be seen in [4] in a particular case). Our results will be obtained for different matrices Φ_1 and Φ_2 in the class of piecewise almost periodic elements. In particular, the present results generalize several previous results (cf. the results in [2, 3, 5, 9, 12, 13]).

2. Auxiliary notions and results

In view of defining the piecewise almost periodic functions, we will first consider the algebra of almost periodic functions.

The smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ ($\lambda \in \mathbb{R}$), where $e_\lambda(x) = e^{i\lambda x}$, $x \in \mathbb{R}$, is denoted by AP and called the algebra of *almost periodic functions* $AP := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \in \mathbb{R}\}$. We will be also using $AP_+ := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \geq 0\}$ and $AP_- := alg_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \leq 0\}$. All AP functions have a well-known mean value which existence may be formulated as in the next proposition.

Proposition 2.1 *(cf. [1], Proposition 2.22.) Let $A \subset (0, \infty)$ be an unbounded set and let $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ be a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$. If $\varphi \in AP$, then the limit*

$$M(\varphi) := \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \varphi(x)\, dx$$

exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$.

For any $\varphi \in AP$, the number $M(\varphi)$ given by Proposition 2.1 is called the *Bohr mean value* or simply the *mean value* of φ. In the matrix case the *mean value* is defined entry-wise.

Let $C(\dot{\mathbb{R}})$ (with $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) denote the set of all (bounded and) continuous functions φ on the real line for which the two limits $\varphi(-\infty) := \lim_{x\to-\infty} \varphi(x)$, $\varphi(+\infty) := \lim_{x\to+\infty} \varphi(x)$ exist and coincide. The common value of these two limits will be denoted by $\varphi(\infty)$. In addition, consider the C^*-algebra of all bounded piecewise continuous functions on $\dot{\mathbb{R}}$ denoted by PC or $PC(\dot{\mathbb{R}})$ as being the algebra of all function $\varphi \in L^\infty(\mathbb{R})$ for which the one-sided limits $\varphi(x_0 - 0) = \lim_{x \to x_0 - 0} \varphi(x)$, $\varphi(x_0 + 0) = \lim_{x \to x_0 + 0} \varphi(x)$ exist for each $x_0 \in \dot{\mathbb{R}}$. We will be also using $C(\overline{\mathbb{R}}) := C(\mathbb{R}) \cap PC(\dot{\mathbb{R}})$, where $C(\mathbb{R})$ is the usual set of continuous functions on the real line. Furthermore, PC_0 will represent the sub-class of PC of all piecewise continuous functions φ for which $\varphi(\pm\infty) = 0$.

As above mentioned, we will deal with Fourier symbols from the C^*-algebra of piecewise almost periodic elements which is defined as follows.

Definition 2.1 The C^*-algebra PAP of all *piecewise almost periodic functions* on $\mathbb{R}$ is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and PC:

$$PAP = alg_{L^\infty(\mathbb{R})}\{AP, PC\}.$$

We will use the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. The following proposition is the matrix version of a corresponding result for the scalar case (cf. [1, Proposition 3.15]).

Proposition 2.2 **(a)** *If $\Phi \in PAP^{N\times N}$, then there are uniquely determined functions $\Theta_\ell,\ \Theta_r \in AP^{N\times N}$ and $\Phi_0 \in PC_0^{N\times N}$ such that*

$$\Phi = (1-u)\Theta_\ell + u\Theta_r + \Phi_0, \tag{2}$$

where $u \in C(\overline{\mathbb{R}})$, $u(-\infty) = 0$ and $u(+\infty) = 1$.

(b) *If $\Phi \in \mathcal{G}PAP^{N\times N}$, then there exist an invertible semi-almost periodic element $\Theta \in \mathcal{G}SAP^{N\times N}$ and an invertible piecewise continuous element $\Xi \in \mathcal{G}PC^{N\times N}$ (such that $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$), which allow the construction of a factorization*

$$\Phi = \Theta\Xi, \tag{3}$$

and $W_\Phi = W_\Theta W_\Xi + K_1 = W_\Xi W_\Theta + K_2$ with compact operators $K_1,\ K_2$. The almost periodic representatives of Θ are the functions Θ_ℓ and Θ_r of part (a).

Let us now recall the so-called right and left AP factorizations.

Definition 2.2 A matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ is said to admit a *right AP factorization* if it can be represented in the form

$$\Phi(x) = \Phi_-(x)D(x)\Phi_+(x) \tag{4}$$

for all $x \in \mathbb{R}$, with

$$\Phi_- \in \mathcal{G}AP_-^{N\times N},\ \ \Phi_+ \in \mathcal{G}AP_+^{N\times N}, \tag{5}$$

and D is a diagonal matrix of the form $D(x) = \mathrm{diag}[e^{i\lambda_1 x}, \ldots, e^{i\lambda_N x}]$, $\lambda_j \in \mathbb{R}$. The numbers λ_j are called the *right AP indices* of the factorization. A right AP factorization with $D = I_{N\times N}$ is referred to be a canonical right AP factorization.

It is said that a matrix function $\Phi \in \mathcal{G}AP^{N\times N}$ admits a *left AP factorization* if instead of (4) we have $\Phi(x) = \Phi_+(x)D(x)\Phi_-(x)$ for all $x \in \mathbb{R}$ and $\Phi_\pm$ and D having the same property as above.

Note that from the above definition it follows that if an invertible almost periodic matrix function Φ admits a right AP factorization, then $\widetilde{\Phi}$ admits a left AP factorization, and also Φ^{-1} admits a left AP factorization.

The vector containing the right AP indices will be denoted by $k(\Phi)$, i.e., in the above case $k(\Phi) := (\lambda_1, \ldots, \lambda_N)$. If we consider the case with equal *right AP indices* $(k(\Phi) := (\lambda_1, \lambda_1, \ldots, \lambda_1))$, then the matrix $\mathbf{d}(\Phi) := M(\Phi_-)M(\Phi_+)$ is independent of the particular choice of the right AP factorization. In this case, this matrix $\mathbf{d}(\Phi)$ is called the *geometric mean* of Φ.

In order to relate operators and to transfer certain operator properties between the related operators, we will be also using the notion of equivalence after extension for bounded linear operators.

Definition 2.3 Consider two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, acting between Banach spaces. We say that T is *equivalent after extension to S* if there are Banach spaces Z_1 and Z_2 and invertible bounded linear operators E and F such that $\mathrm{diag}[T, I_{Z_1}] = E\ \mathrm{diag}[S, I_{Z_2}]\ F$, where I_{Z_1}, I_{Z_2} represent the identity operators in Z_1 and Z_2, respectively. This relation between T and S will be denoted by $T \overset{*}{\sim} S$.

It is clear that if T is equivalent after extension with S, then T and S have the same Fredholm regularity properties (i.e., the properties that directly depend on the kernel and on the image of the operator). It is known (cf., e.g., [8]) that such kind of operator relation holds for a diagonal operator constructed with our Wiener-Hopf plus and minus Hankel operators and a corresponding pure Wiener-Hopf operator. Namely, considering $\Phi_1, \Phi_2 \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$, then

$$\mathfrak{D}_{\Phi_{1,2}} := \mathrm{diag}\,[W_{\Phi_1} + H_{\Phi_2}, W_{\Phi_1} - H_{\Phi_2}] : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N} \tag{6}$$

is equivalent after extension to the Wiener-Hopf operator

$$W_\Psi : [L^2_+(\mathbb{R})]^{2N} \to [L^2(\mathbb{R}_+)]^{2N}$$

with Fourier symbol

$$\Psi = \begin{bmatrix} \Phi_1 - \Phi_2\widetilde{\Phi_1^{-1}}\widetilde{\Phi_2} & -\Phi_2\widetilde{\Phi_1^{-1}} \\ \widetilde{\Phi_1^{-1}}\widetilde{\Phi_2} & \widetilde{\Phi_1^{-1}} \end{bmatrix}. \tag{7}$$

3. The Fredholm property

In the present section we will work out characterizations for the Fredholm property of $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$, with possible different symbols Φ_1, $\Phi_2 \in PAP^{N\times N}$.

3.1. **General Case**

We start by recalling a Fredholm characterization for Wiener-Hopf operators with PAP matrix Fourier symbols having lateral almost periodic representatives admitting right AP factorizations.

Theorem 3.1 *(cf., e.g., [1, Theorem 3.16]) Let $\Phi \in PAP^{N\times N}$. If $\Phi \notin \mathcal{G}[PAP]^{N\times N}$, then W_Φ is not semi-Fredholm. Assume now that $\Phi \in \mathcal{G}PAP^{N\times N}$ and Φ_ℓ and Φ_r admit a right AP factorization. Then the Wiener-Hopf operator W_Φ is Fredholm if and only if:*

(i) *The almost periodic representatives Φ_ℓ and Φ_r admit canonical right AP factorizations, i.e. with $k(\Phi_\ell) = k(\Phi_r) = (0,\ldots,0)$;*

(ii) $\mathrm{sp}(\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)) \cap (-\infty, 0] = \emptyset$;

(iii) $\mathrm{sp}(\Phi^{-1}(x-0)\Phi(x+0)) \cap (-\infty, 0] = \emptyset$, *for all $x \in \mathbb{R}$.*

Using the knowledge of Proposition 2.2, if follows that if $\Psi \in \mathcal{G}PAP^{2N\times 2N}$ then this matrix function admits the following representation

$$\Psi = (1-u)\Psi_\ell + u\Psi_r + \Psi_0, \tag{8}$$

where $\Psi_{\ell,r} \in \mathcal{G}AP^{2N\times 2N}$ are defined for the particular Ψ in (7) by

$$\Psi_\ell = \begin{bmatrix} \Phi_{1\ell} - \Phi_{2\ell}\widetilde{\Phi_{1r}^{-1}}\widetilde{\Phi_{2r}} & -\Phi_{2\ell}\widetilde{\Phi_{1r}^{-1}} \\ \widetilde{\Phi_{1r}^{-1}}\widetilde{\Phi_{2r}} & \widetilde{\Phi_{1r}^{-1}} \end{bmatrix} \tag{9}$$

and

$$\Psi_r = \begin{bmatrix} \Phi_{1r} - \Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} & -\Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}} \\ \widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} & \widetilde{\Phi_{1\ell}^{-1}} \end{bmatrix} \tag{10}$$

(with $\Phi_{1\ell}$, Φ_{1r} and $\Phi_{2\ell}$, Φ_{2r} being the local representatives at $\mp\infty$ of Φ_1 and Φ_2, respectively), $u \in C(\overline{\mathbb{R}})$, $u(-\infty) = 0$, $u(+\infty) = 1$, $\Psi_0 \in [PC_0]^{2N\times 2N}$.

From (9) it follows that

$$\widetilde{\Psi_\ell^{-1}} = \begin{bmatrix} \widetilde{\Phi_{1\ell}^{-1}} & \widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} \\ -\Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}} & \Phi_{1r} - \Phi_{2r}\widetilde{\Phi_{1\ell}^{-1}}\widetilde{\Phi_{2\ell}} \end{bmatrix}. \tag{11}$$

Therefore, we obtain that

$$\Psi_r = \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix} \widetilde{\Psi_\ell^{-1}} \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}. \tag{12}$$

These representations, and the above relation between the operator $\mathfrak{D}_{\Phi_{1,2}}$ and the pure Wiener-Hopf operator, lead to the following characterization in the case when Ψ_ℓ admits a right AP factorization.

Theorem 3.2 *Let $\Psi \in \mathcal{G}PAP^{2N\times 2N}$, and assume that Ψ_ℓ admits a right AP factorization. In this case, the Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are both Fredholm if and only if the following three conditions are satisfied:*

(j) *Ψ_ℓ admits a canonical right AP factorization, i.e., $k(\Psi_\ell) = (0, \dots, 0)$;*

(jj) $\mathrm{sp}[H\mathbf{d}(\Psi_\ell)] \cap i\mathbb{R} = \emptyset$, *where* $H := \begin{bmatrix} 0 & I_N \\ I_N & 0 \end{bmatrix}$;

(jjj) $\mathrm{sp}[H\Psi(-x+0)H\Psi(x+0)] \cap (-\infty, 0] = \emptyset$, *for all* $x \in \mathbb{R}$.

Proof. Assume that $\Psi \in \mathcal{G}PAP^{2N\times 2N}$ with Ψ_ℓ admitting a right AP factorization. If the Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are both Fredholm operators then W_Ψ is also Fredholm due to the equivalence after extension relation. Employing Theorem 3.1 we obtain that Ψ_ℓ and Ψ_r admit canonical right AP factorizations,

$$\mathrm{sp}(\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)) \cap (-\infty, 0] = \emptyset, \tag{13}$$

and

$$sp(\Psi^{-1}(x-0)\Psi(x+0)) \cap (-\infty, 0] = \emptyset. \tag{14}$$

A canonical right AP factorization of Ψ_ℓ can be normalized into

$$\Psi_\ell = \theta_- \Lambda \theta_+, \tag{15}$$

where $\theta_\pm$ have the same factorization properties as the original lateral factors of the canonical factorization but with $M(\theta_\pm) = I$, and where $\Lambda := \mathbf{d}(\Psi_\ell)$. From (12) and (15) we derive that $\Psi_r = H\widetilde{\Psi_\ell^{-1}}H = H\widetilde{\theta_+^{-1}}\Lambda^{-1}\widetilde{\theta_-^{-1}}H$ which shows that

$$\mathbf{d}(\Psi_r) = H\Lambda^{-1}H \tag{16}$$

and therefore

$$\mathbf{d}^{-1}(\Psi_r) = H\Lambda H. \tag{17}$$

In this way, we conclude that $\mathrm{sp}[\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)] = \mathrm{sp}[H\Lambda H\Lambda] = \mathrm{sp}[(H\Lambda)^2]$. Thus, (13) turns out to be equivalent to $\mathrm{sp}[(H\Lambda)^2] \cap (-\infty, 0] = \emptyset$ which leads to $\mathrm{sp}[H\Lambda] \cap i\mathbb{R} = \emptyset$ and therefore, the proposition (jj) is satisfied.

Observing now that $\Psi^{-1} = H\widetilde{\Psi}H$, we conclude that

$$\begin{aligned} \mathrm{sp}[H\widetilde{\Psi}H(x-0)\Psi(x+0)] &= \mathrm{sp}[H\widetilde{\Psi}(x-0)H\Psi(x+0)] \\ &= \mathrm{sp}[H\Psi(-x+0)H\Psi(x+0)]. \end{aligned}$$

It follows that (14) is equivalent to $\mathrm{sp}[H\Psi(-x+0)H\Psi(x+0)] \cap (-\infty, 0] = \emptyset$.

Let us now assume that propositions (j)-(jjj) hold and prove that $W_{\Phi_1} \pm H_{\Phi_2}$ are both Fredholm operators. The left and right representatives of Ψ are given by (9) and (10). Since Ψ_ℓ admits a canonical right AP factorization then Ψ_ℓ^{-1} admits a canonical left AP factorization and $\widetilde{\Psi_\ell^{-1}}$ admits a canonical right AP factorization. Therefore, $H\widetilde{\Psi_\ell^{-1}}H = \Psi_r$ admits a canonical right AP factorization. These two canonical right AP factorizations and condition (jj) imply that $\mathrm{sp}(\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)) \cap (-\infty, 0] = \emptyset$. Condition (jjj) allows us to conclude that $\mathrm{sp}[\Psi^{-1}(x-0)\Psi(x+0)] \cap (-\infty, 0] = \emptyset$. All these facts together with Theorem 3.1 give us that W_Ψ is a Fredholm operator. Using the equivalence after extension relation, we obtain that the Wiener-Hopf plus and minus Hankel operators $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are both Fredholm operators. □

3.2. **The case of $\Phi_1 = \pm\Phi_2$**

Let $\Phi_2 \in \mathcal{G}PAP^{N\times N}$ and consider $\Phi_1 = \pm\Phi_2$. In this case, Ψ takes the form

$$\Psi = \begin{bmatrix} 0 & \mp \Phi_2\widetilde{\Phi_2^{-1}} \\ \pm I_N & \pm \widetilde{\Phi_2^{-1}} \end{bmatrix} = \pm \begin{bmatrix} 0 & -\Phi_2\widetilde{\Phi_2^{-1}} \\ I_N & \widetilde{\Phi_2^{-1}} \end{bmatrix}$$

and

$$W_\Psi = \pm\, r_+\mathcal{F}^{-1} \begin{bmatrix} \Phi_2\widetilde{\Phi_2^{-1}} & 0 \\ 0 & I_N \end{bmatrix} \mathcal{F}\ell_0 r_+\mathcal{F}^{-1} \begin{bmatrix} 0 & -I_N \\ I_N & \widetilde{\Phi_2^{-1}} \end{bmatrix} \mathcal{F},$$

which shows an explicit equivalence after extension relation between W_Ψ and $W_{\Phi_2\widetilde{\Phi_2^{-1}}}$ and therefore,

$$\mathfrak{D}_{\Phi_{1,2}} \overset{*}{\sim} W_{\Phi_2\widetilde{\Phi_2^{-1}}}. \tag{18}$$

From Theorem 2.2, if $\Phi_2 \in PAP^{N\times N}$, then this matrix function admits the following representation

$$\Phi_2 = (1-u)\Phi_{2\ell} + u\Phi_{2r} + \Phi_{20} \tag{19}$$

(with $\Phi_{20} \in [PC_0]^{N\times N}$) and

$$\Phi_2\widetilde{\Phi_2^{-1}} = [(1-u)\Phi_{2\ell} + u\Phi_{2r} + \Phi_{20}][(1-\widetilde{u})\widetilde{\Phi_{2\ell}} + \widetilde{u}\widetilde{\Phi_{2r}} + \widetilde{\Phi_{20}}]^{-1}.$$

Therefore, $(\Phi_2\widetilde{\Phi_2^{-1}})_\ell = \Phi_{2\ell}\widetilde{\Phi_{2r}^{-1}}$, $(\Phi_2\widetilde{\Phi_2^{-1}})_r = \Phi_{2r}\widetilde{\Phi_{2\ell}^{-1}}$. Through a similar proof of that one of the Theorem 3.1, we obtain the next result.

Corollary 3.1 *Let $\Phi_1, \Phi_2 \in \mathcal{G}PAP^{N\times N}$ such that $\Phi_1 = \pm\Phi_2$ and assume that $\Phi_{2\ell}\widetilde{\Phi_{2r}^{-1}}$ admits a right AP factorization. In this case, the Wiener-Hops plus and minus Hankel operators $W_{\pm\Phi_2} + H_{\Phi_2}$ and $W_{\pm\Phi_2} - H_{\Phi_2}$ are both Fredholm operators if and only if the following three conditions are satisfied:*

(i) $\Phi_{2\ell}\widetilde{\Phi_{2r}^{-1}}$ *admits a canonical right AP factorization;*

(ii) $\mathrm{sp}[\mathbf{d}(\Phi_{2\ell}\widetilde{\Phi_{2r}^{-1}})] \cap i\mathbb{R} = \emptyset$*;*

(iii) $\mathrm{sp}[\Phi_2(-x+0)\Phi_2^{-1}(x-0)\Phi_2(x+0)\Phi_2^{-1}(-x-0)] \cap (-\infty, 0] = \emptyset$*, for all* $x \in \mathbb{R}$*.*

4. Index formula

In the present section we will be concentrated in obtaining a Fredholm index formula for $\mathfrak{D}_{\Phi_{1,2}}$, i.e., for the sum of Wiener-Hopf plus/minus Hankel operators $W_{\Phi_1} \pm H_{\Phi_2}$ with different Fourier symbols $\Phi_1, \Phi_2 \in \mathcal{G}PAP^{N\times N}$ such that Ψ_ℓ admits a right AP factorization. For that purpose, taking into account that $PAP = SAP + PC_0$, we will first recall some known properties of Wiener-Hopf plus Hankel operators with symbols in SAP and with symbols in PC. Within this context, let us assume that $W_{\Phi_1} + H_{\Phi_2}$ and $W_{\Phi_1} - H_{\Phi_2}$ are Fredholm operators.

Let $\mathcal{G}SAP_{0,0}$ denote the set of all functions $\varphi \in \mathcal{G}SAP$ for which $k(\varphi_\ell) = k(\varphi_r) = 0$. To define the Cauchy index of $\varphi \in \mathcal{G}SAP_{0,0}$ we need the next lemma.

Lemma 4.1 *[1, Lemma 3.12] Let* $A \subset (0, \infty)$ *be an unbounded set and let* $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ *be a family of intervals such that* $x_\alpha \geq 0$ *and* $|I_\alpha| = y_\alpha - x_\alpha \to \infty$*, as* $\alpha \to \infty$*. If* $\varphi \in \mathcal{G}SAP_{0,0}$ *and* $\arg\varphi$ *is any continuous argument of* φ*, then the limit*

$$\frac{1}{2\pi} \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} ((\arg\varphi)(x) - (\arg\varphi)(-x))dx \tag{20}$$

exists, is finite, and is independent of the particular choices of $\{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ *and* $\arg\varphi$*.*

The limit (20) is denoted by $\mathrm{ind}\varphi$ and is usually called the *Cauchy index* of φ.

The following theorem provides a formula for the Fredholm index of matrix Wiener-Hopf operators with SAP Fourier symbols.

Theorem 4.1 *[1, Theorem 10.12] Let* $\Phi \in SAP^{N\times N}$*. If the almost periodic representatives* Φ_ℓ, Φ_r *admit right AP factorizations, and if* W_Φ *is a Fredholm operator, then*

$$\mathrm{Ind}\, W_\Phi = -\mathrm{ind}[\det\Phi] - \sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\ \xi_k\right\}\right) \tag{21}$$

where $\xi_1, \ldots, \xi_N \in \mathbb{C}\backslash(-\infty, 0]$ *are the eigenvalues of the matrix* $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$ *and* $\{\cdot\}$ *stands for the fractional part of a real number. Additionally, when choosing* $\arg\ \xi_k$ *in* $(-\pi, \pi)$*, we have* $\mathrm{Ind}\, W_\Phi = -\mathrm{ind}[\det\Phi] - \frac{1}{2\pi}\sum_{k=1}^{N} \arg\ \ \xi_k$*.*

Let us now consider $\Phi \in PC^{N\times N}$ and recall its auxiliary extension $\Phi^{\#}$ defined by

$$\Phi^{\#}(x, \mu) := (1-\mu)\Phi(x-0) + \mu\Phi(x+0),\ (x, \mu) \in \dot{\mathbb{R}} \times [0, 1].$$

The function $\det \Phi^{\#} : \dot{\mathbb{R}} \times [0,1] \to \mathbb{C}$ will play an important role in the Fredholm index study. Note that set $C := \{\det \Phi^{\#}(x,\mu) \in \mathbb{C} : x \in \mathbb{R}, \mu \in [0,1]\}$ is a closed continuous curve obtained from Φ by joining $\det \Phi^{\#}(x-0)$ to $\det \Phi^{\#}(x+0)$ through a line segment at the discontinuity points of $\det \Phi$. If $0 \notin C$, then the *winding number* of C with respect to the origin (denoted in this case by $\arg[\det \Phi^{\#}]$) is defined as the counter-clockwise circuits around the origin performed by the image of $\det \Phi^{\#}$. Suppose $\det \Phi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$ which implies that $\Phi(x-0)$ and $\Phi(x+0)$ are invertible for all $x \in \dot{\mathbb{R}}$. Moreover, assume that the set

$$\Delta_\Phi := \{x \in \dot{\mathbb{R}} : \Phi(x-0) \neq \Phi(x+0)\}$$

is finite. For a connected component ℓ of $\dot{\mathbb{R}} \backslash \Delta_\Phi$ we define $\mathrm{ind}_\ell \Phi$ as $(2\pi)^{-1}$ times the increment of any continuous argument of $\det \Phi$ on ℓ. Taking into account the possible jump at infinity, the winding number of C can be given by:

$$\arg[\det \Phi^{\#}] = \mathrm{ind}[\det \Phi^{\#}] + \sum_{k=1}^{N} \Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(\infty)\Big\}\Big), \qquad (22)$$

where

$$\mathrm{ind}[\det \Phi^{\#}] = \sum_{\ell} \mathrm{ind}_\ell[\det \Phi] + \sum_{x \in \Delta_\Phi} \sum_{k=1}^{N} \Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x)\Big\}\Big), \qquad (23)$$

with $\xi_1(x), \ldots, \xi_N(x)$ being the eigenvalues of $\Phi^{-1}(x-0)\Phi(x+0)$ for $x \in \Delta_\Phi$ and $\{\cdot\}$ denoting the fractional part of a real number.

Theorem 4.2 *(cf. e.g. [1, Theorem 5.10]) Let $\Phi \in PC^{N\times N}$. If W_Φ is a Fredholm operator and Φ has at most finitely many jumps, then*

$$\mathrm{Ind} W_\Phi = -\arg(\det \Phi^{\#})$$

where $\arg(\det \Phi^{\#})$ *is given by* (22) – (23). *Choosing the arguments in* $(-\pi, \pi)$, *we also have*

$$\mathrm{Ind} W_\Phi = -\mathrm{ind}[\det\ \Phi^{\#}] - \frac{1}{2\pi} \sum_{k=1}^{N} \arg \xi_k(\infty).$$

Now, we can conclude a formula for the Fredholm index of matrix Wiener-Hopf operators with PAP Fourier symbols.

Let $\Phi \in PAP^{N\times N}$. Then $\Phi = \Theta\Xi$ (with $\Theta \in \mathcal{G}SAP^{N\times N}$, $\Xi \in \mathcal{G}PC^{N\times N}$ and $\Xi(\pm\infty) = I_{N\times N}$) such that

$$W_\Phi = W_\Theta W_\Xi + K \qquad (24)$$

with K being a compact operator (cf. Proposition 2.2). Assume that W_Φ is a Fredholm operator. From (24) we derive that

$$\mathrm{Ind} W_\Phi = \mathrm{Ind} W_\Theta + \mathrm{Ind} W_\Xi.$$

Using now formulas (21), (22) and (23) and taking into account that Ξ does not have a jump at infinity, we can conclude the following theorem:

Theorem 4.3 *(cf. [2, Proposition 6.3]) Let $\Phi \in \mathcal{G}PAP^{N\times N}$. If the almost periodic representatives Φ_ℓ, Φ_r admit right AP factorizations, and if W_Φ is a Fredholm operator, then*

$$\begin{aligned}\mathrm{Ind}\, W_\Phi &= -\sum_\ell \mathrm{ind}_\ell[\det\ \Xi] - \mathrm{ind}[\det\Theta] - \sum_{x\in\Delta_\Phi}\sum_{k=1}^{N}\Big(\frac{1}{2}-\Big\{\frac{1}{2}-\frac{1}{2\pi}\arg\ \xi_k(x)\Big\}\Big)\\ &\quad -\sum_{k=1}^{N}\Big(\frac{1}{2}-\Big\{\frac{1}{2}-\frac{1}{2\pi}\arg\ \eta_k\Big\}\Big), \qquad (25)\end{aligned}$$

where $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi^{-1}(x-0)\Phi(x+0)$ and η_k are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r)\,\mathbf{d}(\Phi_\ell)$. Choosing both arguments in $(-\pi,\pi)$, (25) *simplifies into*

$$\mathrm{Ind} W_\Phi = -\sum_\ell \mathrm{ind}_\ell[\det\ \Xi] - \mathrm{ind}[\det\Theta] - \frac{1}{2\pi}\sum_{x\in\Delta_\Phi}\sum_{k=1}^{N}\arg\xi_k(x) - \frac{1}{2\pi}\sum_{k=1}^{N}\arg\eta_k. \qquad (26)$$

We will now be concerned in finding an index formula for $\mathfrak{D}_{\Phi_{1,2}}$ (cf. (6)). It directly follows from the definition of the operator $\mathfrak{D}_{\Phi_{1,2}}$ that

$$\mathrm{Ind}\ \mathfrak{D}_{\Phi_{1,2}} = \mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}].$$

Using the equivalence after extension relation mentioned in Section 2, we conclude that $\mathrm{Ind}\ \mathfrak{D}_{\Phi_{1,2}} = \mathrm{Ind}\ W_\Psi$ and consequently, $\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] = \mathrm{Ind}\ W_\Psi$. Following (25) we obtain that

$$\begin{aligned}\mathrm{Ind} W_\Psi &= -\sum_\ell \mathrm{ind}_\ell[\det\Xi] - \mathrm{ind}[\det\Theta] - \sum_{x\in\Delta_\Psi}\sum_{k=1}^{2N}\Big(\frac{1}{2}-\Big\{\frac{1}{2}-\frac{1}{2\pi}\arg\xi_k(x)\Big\}\Big)\\ &\quad -\sum_{k=1}^{2N}\Big(\frac{1}{2}-\Big\{\frac{1}{2}-\frac{1}{2\pi}\arg\zeta_k\Big\}\Big), \qquad (27)\end{aligned}$$

where $\Psi = \Theta\Xi$ is a corresponding factorization in sense of (2) for the invertible PAP matrix function Ψ, $\xi_k(x)$ are the eigenvalues of the matrix function $H\Psi(-x+0)H\Psi(x+0)$ and $\zeta_k \in \mathbb{C}\backslash(-\infty,0]$ are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell) = (H\mathbf{d}(\Psi_\ell))^2$, and $\Delta_\Psi = \{x \in \dot{\mathbb{R}} : \Psi(x-0) \neq \Psi(x+0)\}$.

Moreover, (27) can be rewritten as

$$\begin{aligned}\mathrm{Ind} W_\Psi &= -\sum_\ell \mathrm{ind}_\ell[\det\Xi] - \mathrm{ind}[\det\Theta] - \sum_{x\in\Delta_\Psi}\sum_{k=1}^{2N}\Big(\frac{1}{2}-\Big\{\frac{1}{2}-\frac{1}{2\pi}\arg\xi_k(x)\Big\}\Big)\\ &\quad -\sum_{k=1}^{2N}\Big(\frac{1}{2}-\Big\{\frac{1}{2}-\frac{1}{\pi}\arg\eta_k\Big\}\Big)\end{aligned}$$

where $\eta_k \in \mathbb{C}\backslash i\mathbb{R}$ are the eigenvalues of the matrix $H\mathbf{d}(\Psi_\ell)$.

Thus, we have just concluded the following corollary:

Corollary 4.1 *Let $\Psi \in \mathcal{G}PAP^{2N\times 2N}$ and assume that Ψ_ℓ admits a right AP factorization. If $W_{\Phi_1} \pm H_{\Phi_2}$ are Fredholm operators, then*

$$\begin{aligned}\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] &= -\sum_\ell \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] \\ - \sum_{x\in\Delta_\Psi}\sum_{k=1}^{2N}\Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi}\arg\,\xi_k(x)\Big\}\Big) &- \sum_{k=1}^{2N}\Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{\pi}\arg\,\eta_k\Big\}\Big) \qquad (28)\end{aligned}$$

where $\Psi = \Theta\Xi$ is a corresponding factorization in the sense of (3) for the invertible matrix-valued PAP function Ψ, $\xi_k(x)$ are the eigenvalues of the matrix function $H\Psi(-x+0)H\Psi(x+0)$, $\eta_k \in \mathbb{C}\backslash i\mathbb{R}$ are the eigenvalues of the matrix $H\mathbf{d}(\Psi_\ell)$.

Moreover, formula (28) simplifies into the following one:

$$\begin{aligned}\mathrm{Ind}[W_{\Phi_1} + H_{\Phi_2}] + \mathrm{Ind}[W_{\Phi_1} - H_{\Phi_2}] &= -\sum_\ell \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] \\ &-\frac{1}{2\pi}\sum_{x\in\Delta_\Psi}\sum_{k=1}^{2N}\arg\xi_k(x) - \frac{1}{\pi}\sum_{k=1}^{2N}\arg\beta(\eta_k) \qquad (29)\end{aligned}$$

when choosing $\arg\,\xi_k(x) \in (-\pi,\pi)$ *and*

$$\beta(\eta_k) = \begin{cases} \arg(\eta_k) & \textit{if } \Re e\ \eta_k > 0 \\ \arg(-\eta_k) & \textit{if } \Re e\ \eta_k < 0 \end{cases}$$

with the argument in both cases being taken in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Let us now consider a useful particular case. For the particular case of $\Phi_1 = \pm\Phi_2$, we obtain the following corollary:

Corollary 4.2 *Let $\Phi_1,\ \Phi_2 \in \mathcal{G}PAP^{N\times N}$ such that $\Phi_1 = \pm\Phi_2$ and assume that $\Phi_{2\ell}\widetilde{\Phi_{2r}^{-1}}$ admits a right AP factorization. If $W_{\pm\Psi_2} \pm H_{\Phi_2}$ are Fredholm operators, then*

$$\begin{aligned}\mathrm{Ind}[W_{\pm\Phi_2} + H_{\Phi_2}] + \mathrm{Ind}[W_{\pm\Phi_2} - H_{\Phi_2}] &= -\sum_\ell \mathrm{ind}_\ell[\det \Pi] - \mathrm{ind}[\det \Omega] \\ - \sum_{x\in\Delta_{\Phi_2\widetilde{\Phi_2^{-1}}}}\sum_{k=1}^{N}\Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi}\arg\,\tau_k(x)\Big\}\Big) &- \sum_{k=1}^{N}\Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{\pi}\arg\,\delta_k\Big\}\Big) \qquad (30)\end{aligned}$$

where $\Phi_2\widetilde{\Phi_2^{-1}} = \Omega\Pi$ is a corresponding factorization in the sense of (3) for the invertible matrix-valued PAP function Ψ, $\tau_k(x)$ are the eigenvalues of the matrix function

$\Phi_2(-x+0)\widetilde{\Phi_2^{-1}}(x-0)\Phi_2(x+0)\widetilde{\Phi_2^{-1}}(-x-0)$, $\delta_k \in \mathbb{C}\backslash i\mathbb{R}$ *are the eigenvalues of the matrix* $\mathbf{d}(\Phi_{2\ell}\widetilde{\Phi_{2r}^{-1}})$ *and* $\Delta_{\Phi_2\widetilde{\Phi_2^{-1}}} := \{x \in \dot{\mathbb{R}} : \Phi_2\widetilde{\Phi_2^{-1}}(x-0) \neq \Phi_2\widetilde{\Phi_2^{-1}}(x+0)\}$.

Moreover, formula (30) *simplifies into the following one :*

$$\begin{aligned}\operatorname{Ind}[W_{\pm\Phi_2} + H_{\Phi_2}] + \operatorname{Ind}[W_{\pm\Phi_2} - H_{\Phi_2}] &= -\sum_{\ell} \operatorname{ind}_{\ell}[\det \Pi] - \operatorname{ind}[\det \Omega] \\ &\quad - \frac{1}{2\pi} \sum_{x \in \Delta_{\Phi_2\widetilde{\Phi_2^{-1}}}} \sum_{k=1}^{N} \arg\, \tau_k(x) - \frac{1}{\pi}\sum_{k=1}^{N} \arg\, \beta(\delta_k)\end{aligned}$$

when choosing $\arg\, \tau_k(x) \in (-\pi, \pi)$ *and*

$$\beta(\delta_k) = \begin{cases} \arg(\delta_k) & \text{if } \Re e\ \delta_k > 0 \\ \arg(-\delta_k) & \text{if } \Re e\ \delta_k < 0 \end{cases}$$

with the argument in both cases being taken in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

5. Example

In the present section we will exemplify the above theory. Let us take $\Phi_1 = -\Phi_2$ with

$$\begin{aligned}\Phi_2(x) &= (1-u(x)) \begin{bmatrix} 0 & 4ie^{4ix} \\ ie^{-4ix} & 0 \end{bmatrix} \begin{bmatrix} g(x) & 0 \\ 0 & g(x) \end{bmatrix} \\ &\quad + u(x) \begin{bmatrix} 0 & 4ie^{-4ix} \\ ie^{4ix} & 0 \end{bmatrix} \begin{bmatrix} g(x) & 0 \\ 0 & g(x) \end{bmatrix} + \begin{bmatrix} e^{-|x|} & 0 \\ 0 & e^{-|x|} \end{bmatrix} \begin{bmatrix} g(x) & 0 \\ 0 & g(x) \end{bmatrix}\end{aligned}$$

where u is the real-valued function defined by

$$u(x) = \frac{1}{2} + \frac{1}{2}\tanh(x) \tag{31}$$

and

$$g(x) = \begin{cases} e^{2\frac{i\pi}{x-1}} & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases} \tag{32}$$

It is clear that Φ_2 admits a factorization

$$\Phi_2 = AB \tag{33}$$

in the sense of (3) with

$$A = (1-u(x)) \begin{bmatrix} 0 & 4ie^{4ix} \\ ie^{-4ix} & 0 \end{bmatrix} + u(x) \begin{bmatrix} 0 & 4ie^{-4ix} \\ ie^{4ix} & 0 \end{bmatrix} + \begin{bmatrix} e^{-|x|} & 0 \\ 0 & e^{-|x|} \end{bmatrix}$$

and $B = \operatorname{diag}[g(x), g(x)]$. Observing that A and B are invertible matrix functions, we conclude that Φ_2 is invertible.

From (33) we obtain that

$$\Phi_2 \widetilde{\Phi_2^{-1}} = AB\widetilde{B^{-1}}\widetilde{A^{-1}} \tag{34}$$

and

$$\Phi_{2\ell} \widetilde{\Phi_{2r}^{-1}} = I_{2\times 2}. \tag{35}$$

Therefore, $\Phi_{2\ell} \widetilde{\Phi_{2r}^{-1}}$ admits a right AP factorization and $\mathbf{d}(\Phi_{2\ell} \widetilde{\Phi_{2r}^{-1}}) = I_{2\times 2}$. Hence, the eigenvalues of this matrix are equal to $1 \notin i\mathbb{R}$.

We will now verify condition (lll) of Corollary 3.1. To this purpose, let us first observe that

$$\operatorname{sp}(\Phi_2(-x+0)\Phi_2^{-1}(x-0)\Phi_2(x+0)\Phi_2^{-1}(-x-0)) \cap (-\infty, 0] = \emptyset, \quad \text{for all} \quad x \in \mathbb{R}$$

is equivalent to

$$\operatorname{sp}(B(-x+0)B^{-1}(x-0)B(x+0)B^{-1}(-x-0)) \cap (-\infty, 0] = \emptyset, \quad \text{for all} \quad x \in \mathbb{R}.$$

Additionally, from the definition of B it follows

$$B(-x+0)B^{-1}(x-0)B(x+0)B^{-1}(-x-0) = I_{2\times 2}.$$

Consequently,

$$\operatorname{sp}(\Phi_2(-x+0)\Phi_2^{-1}(x-0)\Phi_2(x+0)\Phi_2^{-1}(-x-0)) = \{1\} \notin (-\infty, 0].$$

These are sufficient conditions for these operators $W_{-\Phi_2} \pm W_{\Phi_2}$ to have the Fredholm property (cf. Corollary 3.1).

Let us now calculate the sum of their Fredholm indices. Since B is a diagonal matrix, by (34) we can present the following factorization

$$\Phi_2 \widetilde{\Phi_2^{-1}} = A\widetilde{A^{-1}}B\widetilde{B^{-1}}. \tag{36}$$

Let $\Theta = A\widetilde{A^{-1}}$ and $\Xi = B\widetilde{B^{-1}}$. We will now calculate the determinants of Θ and Ξ.

$$\det \Theta = \det A\widetilde{A^{-1}} = \det A \det \widetilde{A^{-1}} = \det A \det A^{-1} = 1.$$

and therefore, we obtain that $\operatorname{Ind} \det \Theta = 0$. Furthermore,

$$\det \Xi = \det B\widetilde{B^{-1}} = \det B \det \widetilde{B^{-1}} = g^2(x)\widetilde{g^{-2}(x)}$$

with

$$g^2(x)\widetilde{g^{-2}(x)} = \begin{cases} e^{4\frac{i\pi}{x-1}} & \text{if } x < 0 \\ e^{-4\frac{i\pi}{x-1}} & \text{if } x \geq 0 \end{cases}$$

Therefore, $\sum_\ell \operatorname{ind}_\ell[\det \Xi] = 0$. In addition, observing that the eigenvalues of the matrix functions $\mathbf{d}(\Phi_{2\ell} \widetilde{\Phi_{2r}^{-1}})$ and $\Phi_2(-x+0)\ \Phi_2^{-1}(x-0)\ \Phi_2(x+0)\ \Phi_2^{-1}(-x-0)$ are both real, we have that their arguments are zero.

Altogether, we finally obtain $\operatorname{Ind}[W_{-\Phi_2} + H_{\Phi_2}] + \operatorname{Ind}[W_{-\Phi_2} - H_{\Phi_2}] = 0$.

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