BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 58, 2010 - 2015 (29–42)

A general commutativity condition in topological and bornological algebras

M. Oudadess

Received 25/09/2013

Abstract

It is given a simple condition which implies commutativity in general topological algebras, real or complex, unital or not. It is shown that it is equivalent to other earlier conditions considered in different contexts. This is , in particular, the case for the Le Page inequality in complex unital Banach algebras. The condition mentioned above works also in bornological algebras; actually in bornological vector spaces with a multiplication. Another condition is given in connection with inner derivations; another one in terms of inner automorphisms. Our conditions work, as well, in non associative algebras.

1. Introduction. C. Le Page [6] has initiated the study of commutativity in the frame of complex unital Banach algebras. He showed that such an algebra is commutative whenever it satisfies, the now known as the Le Page inequality,

$$\text{(LP)} \quad \exists \alpha > 0 : \|xy\| \leq \alpha \|yx\|, \forall x, y \in E.$$

The use of the exponential function and Liouville's theorem for bounded holomorphic functions allows a very striking short proof. However, this is also an obstruction for it does not apply in the real case; take e.g., the quaternion field $\mathbb{H}$. It does not also work in the non unital case; take any anti-commutative algebra [cf. 16].

Y. Tsertos has stated [19] the analogue of (LP), in the complex non unital case; using the circle operation $x \circ y = x + y + xy$ and the set $G^q(E)$ of quasi-invertible elements of E. In[16] we have displayed an inequality in the real unital case. Y. Tsertos and myself have put [8] the appropriate inequality in the non unital real one. In the last mentioned results, no use of function theory is made; but one has to treat separately the unital and the non unital case.

The aim here is to exhibit a natural and simple condition ($\mathbb{B}$) which works in any situation (real or complex, unital or not). The latter seems to be the adequate one, for it extends to general topological algebras (Proposition 3.1). Moreover, the topology appears to be not necessary. Our condition works, as well, with bornological structures. It is a matter of bornological vector spaces which can be endowed with a multiplication possessing a boundedness property (Proposition 5.1). The result

Keywords: Topological algebra, commutativity, Le Page, bornological vector space, bornological algebra, boundedness, monoids.

Mathematics Subject Classification (2010): 46J05, 46K05.

can still be obtained for a 'kind of bilinear mappings' on monoids, with values in a separated bornological vector space (Proposition 5.5). One hopes that the latter could have concrete applications in domains with very soft structures.

In Section 5, it is also given a condition implying commutativity in bornological vector spaces with multiplication, via derivations. More precisely, inner derivations must transform straight lines into bounded subsets (Proposition 5.8).

In the unital complex Banach case, so many good properties are available; which hide the nature of the problem. The question is purely algebraic, that is the commutativity of the multiplication. In the proof of Le Page, the exponential function is used and so completeness. The continuity of multiplication is strongly used. Moreover, Liouville's theorem is a peculiar result of complex variable. Now, the condition ($\mathbb{B}$), which is anyway necessary, showed to be very flexible and dispenses with the use of completeness and function theory (real or complex). It must be the minimum one can ask for.

The following conditions are essential in the sequel.

$$(\mathbb{B}) \quad \{xy - yx : x \in E, y \in E\} \text{ is bounded.}$$

$$(\mathbb{B}G) \quad \{xy - yx : x \in G(E), y \in E\} \text{ is bounded.}$$

2. Preliminaries. In a unital algebra E (real or complex) the set of invertible elements is denoted by $G(E)$, and its complement by $\mathrm{Sing}(E)$, i.e., the set of singular elements of E. For a complex algebra, the spectrum of an element x is $Sp_E(x) = \{z \in \mathbb{C} : x - ze \notin G(E)\}$. If the algebra is real, $Sp_E(x)$ stands for the spectrum of x in the complexification $E_{\mathbb{C}}$ of E. The spectral radius of x is $\rho(x) = \sup\{|z| : z \in Sp_E(x)\}$.

A topological algebra is an algebra E over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) endowed with a topological vector space topology τ for which multiplication is separately continuous. If the map $(x, y) \longmapsto xy$ is continuous (in both variables), then E is said to be with continuous multiplication. We say that a unital topological algebra is a Q-algebra if the set $G(E)$ of its invertible elements is open.

Recall that, given a non-unital algebra E (real or complex), the circle operation, on E, is defined by $x \circ y = x + y + xy$. An element x is said to be quasi-invertible (q-invertible) if there is an $x^{'}$ such that $x \circ x^{'} = 0$ and $x^{'} \circ x = 0$. The set of q-invertible elements is denoted by $G^q(E)$. A non-unital topological algebra is a Q-algebra if the set $G^q(E)$ is open.

Let (E, τ) be a locally convex algebra (*l.c.a.*), with a separately continuous multiplication, whose topology τ is given by a family $(p_\lambda)_\lambda$ of seminorms. The algebra (E, τ) is said to be locally A-convex (*l-A-c.a.*; cf. [12]) if, for every x and every λ, there is $M(x, \lambda) > 0$ such that

$$\max\left[p_\lambda(xy), p_\lambda(yx)\right] \leq M(x, \lambda) p_\lambda(y); \forall y \in E.$$

In the case of a single space norm, $(E, \|.\|)$ is called an A-normed algebra. If $M(x, \lambda) = M(x)$ depends only on x, we say that (E, τ) is a locally uniformly A-convex algebra (*l.u-A-c.a.*; cf. [12]). If it happens that, for every λ,

$$p_\lambda(xy) \leq p_\lambda(x) p_\lambda(y); \forall x, y \in E,$$

then (E, τ) is named a locally m-convex algebra (*l.m.c.a.*; cf. [7]; see also [3],). Recall also that a *l.c.a.* has a continuous multiplication if, for every λ, there is $\lambda^{'}$ such that

$$p_\lambda(xy) \leq p_{\lambda'}(x) p_{\lambda'}(y); \forall x, y \in E.$$

The bounded structure (bornology) of a locally convex space (*l.c.s.*) (E,τ) is the collection $\mathbb{B}\tau$ of all the subsets B of E which are bounded in the sense of von Neumann-Kolmogorov, that is B is absorbed by every neighborhood of the origin. If $\tau_{\|.\|}$ is the topology induced by a norm $\|.\|$, we write $B\tau_{\|.\|}$. We say that a *l.c.s.* (E,τ) is Mackey complete if its bounded structure $\mathbb{B}\tau$ admits a fundamental system $\mathcal{B}_0$ of Banach discs that is, for every B in $\mathcal{B}_0$, the vector space generated by B is a Banach space when endowed with the gauge $\|.\|_B$ of B. All notions concerning bornology can be found in [4].

If the topology of a *l.c.s.* (E,τ) is given by a family $(p_\lambda)_{\lambda\in\Lambda}$ of seminorms, with Λ a directed set, we will often, for simplicity, write only $(p_\lambda)_\lambda$, especially when there is no risk of confusion.

Let $(E,\mathcal{B})$ be a vector space endowed with a bornology. We say that $(E,\mathcal{B})$ is a bornological vector space (*b.v.s.*) if the operations $(x,y)\mapsto x+y$ and $(\lambda,x)\mapsto\lambda x$ are bounded. A bornological algebra is an algebra endowed with a vector space bornology such that multiplication is also bounded (cf. [5]).

Let $(E,\mathcal{B})$ be a *b.v.s.*. A subset U of E is said to be Mackey open (M-open) if, for every $x\in U$, the subset $U-x=\{u-x:u\in U\}$ is bornivorous; that is absorbs every bounded subset. A bornological algebra is said to be a Mackey Q-algebra (M-Q-algebra) if the group $G(E)$ is M-open.

An element x of E is said to be bounded [1] (i-bounded in the sense of S. Warner ([20]) if there is an $\alpha>0$ such that $\{(\alpha x)^n : n=1,2,...\}$ is bounded. A subset B of E is said to be m-bounded ([1]; see also [2] and [5]) or regular (i-bounded in the sense of S. Warner ([20]) if it is absorbed by an idempotent bounded subset of E. A *l.c.a.* (E,τ) is said to be pseudo-Banach if it is pseudo-complete and every $B\in\mathbb{B}\tau$ is regular (cf. [2]; see also [5]). It is then a bornological inductive limit of Banach algebras, as it has been explicitly stated in [5].

3. Normed algebras. We reproduce the proof, of Le Page, mentioned in the introduction. It is short. Moreover a step, of it, suggests a new natural commutativity condition which will show to be somehow universal; for it applies in very general contexts.

Proposition 3.1. ([6], Théorème 5.8). Let $(E,\|.\|)$ be a complex unital Banach algebra. If there is an $\alpha>0$ such that

$$\text{(LP)} \qquad \|ab\|\leq\alpha\,\|ba\|\,,\forall a\in E,\forall b\in E,$$

then it is commutative.

Proof. For every x and y in E, consider the holomorphic function $f:\lambda\longrightarrow f(\lambda)=\exp(\lambda x)y\exp(-\lambda x)$, with $\lambda\in\mathbb{C}$. Using (LP), one obtains the boundedness of f. So $\exp(\lambda x)y\exp(-\lambda x)=y$. Hence $\exp(\lambda x)y=y\exp(\lambda x)$, for every λ. Whence $xy=yx$. □

In the proof above, one may consider directly the holomorphic function $g:\lambda\longrightarrow g(\lambda)=\exp(\lambda x)y-y\exp(\lambda x)$ and put hypotheses to make it bounded. This remark suggested the following.

Proposition 3.2. Let $(E,\|.\|)$ be a unital Q-normed algebra, real or complex. If it satisfies

$$(\mathbb{B}G) \quad \{xy-yx : x\in G\,(E)\,,y\in E\} \text{ is bounded,}$$

then it is commutative.

Proof. Take an x in $G(E)$. By hypothesis, there is a $k>0$ such that $\|xy-yx\|\leq k$, for every y in E. But then, one has $\|xy-yx\|\leq n^{-1}k$, for every y in E and $n\in\mathbb{N}^*$. Whence $xy-yx=0$. If x is not in $G(E)$, take $e-(\|x\|+1)^{-1}x$. It is invertible in the completion $\widehat{E}$ of E. But E is inverse closed in $\widehat{E}$, since it is a Q algebra. $\square$

Remark 3.3. The existence of a unit is inessential in the previous proposition. Indeed, the condition $(\mathbb{B})$ extends to the unitization E_1 of E for

$$(x+\alpha)(y+\beta)-(y+\beta)(x+\alpha)=xy-yx.$$

One also observes that $x\circ y-y\circ x=xy-yx$.

Remark 3.4. The fact that $(E,\|.\|)$ is only normed (not complete) is explained by the fact that $(\mathbb{BG})$ extends also to the completion.

Remark 3.5. It does not matter if the algebra is real or complex, for $(\mathbb{B})$ is satisfied by the complexification of E. Indeed

$$(a,b)(c,d)-(c,d)(a,b)=(ac-ca+db-bd,ad-da+bc-cb).$$

Example 3.6. An anti-commutative algebra $(E,\|.\|)$, that is $xy=-yx$, for every x and y in E, does not satisfy $(\mathbb{B})$. Indeed $\|x(ny)-(ny)x\|=2n\|xy\|$.

Example 3.7. The quaternion field $\mathbb{H}$ is not an anti-commutative algebra. It does not satisfy $(\mathbb{B})$. Indeed $|i(nj)-(nj)i|=2n$.

Example 3.8. In the quaternion field $\mathbb{H}$, one has $|xy|=|yx|$. The latter is not satisfied in its complexification $\mathbb{H}_{\mathbb{C}}$, for it should be commutative. But one can see it directly, revealing that the obstruction lies in the multiplication by complexes. Indeed, one has

$$(i,j)(1,k)=(i-jk)(ik+j)=(0,0) \text{ and } (1,k)(i,j)=(1-kj,j+ik)=(2i,2j).$$

One can also see that $\left|x^2\right|=|x|^2$ is not preserved. Let $i^{'}=(0,1)$. One has $i^{'2}=-1$. So $i^{'}+1=(1,1)$ and $\left(i^{'}+1\right)^2=2i^{'}$.

Remark 3.9. If we soften the condition $(\mathbb{BG})$ to

$$\{x(\exp\alpha y)-(\exp\alpha y)x : x\in E,\alpha\in\mathbb{R}\} \text{ bounded},$$

then the result remains true, that is the commutativity of the algebra. Indeed, one has

$$\|x(\exp\alpha y)-(\exp\alpha y)x\|\leq\frac{1}{n}M, \text{ for every } n\in\mathbb{N}^*.$$

Whence

$$x(\exp\alpha y)-(\exp\alpha y)x=0, \text{ for every } \alpha\in\mathbb{R}.$$

So differentiating, for example, one gets $xy=yx$.

Remark 3.10. By the previous remark, the subset

$$\{x(\exp\alpha y)-(\exp\alpha y)x : x\in\mathbb{H},\alpha\in\mathbb{R}\}$$

can not be bounded in the quaternion field $\mathbb{H}$. Actually, even the subset

$$\{x(\exp \alpha y) - (\exp \alpha y)x : \alpha \in \mathbb{R}\}$$

can not be bounded, for given non commuting x and y. Indeed, one has

$$x(\exp \alpha y) - (\exp \alpha y)x = \alpha(xy - yx) + \frac{1}{2}\alpha^2(xy^2 - y^2x) + \ldots + \frac{1}{n!}\alpha^n(xy^n - y^nx) + \ldots.$$

Now, take $x = i$ and $y = j$. One has $xy^n - y^nx = ij^n - j^ni$. So if $n = 2p$ is even, then

$$xy^n - y^nx = i(-1)^n - (-1)^n i = 0.$$

If $n = 2p + 1$, then

$$xy^n - y^nx = i(-1)^n j - j(-1)^n i.$$

Thus, if p is even, then

$$xy^n - y^nx = ij - ji = 2k.$$

If p is odd, then

$$xy^n - y^nx = -ij + ji = -2k.$$

So

$$x(\exp \alpha y) - (\exp \alpha y)x = \alpha(xy - yx).$$

Whence the claim

Some conditions of the complex case imply also commutativity in the real one, but others not. This phenomenon may be enlightened by the fact that when the field $\mathbb{C}$ is involved many conditions are actually equivalent. So the haze around the Le Page inequality is cleared up.

Proposition 3.11. Let $(E, \|.\|)$ be a unital complex Banach algebra. Then the following assertions are equivalent.
(i) $\|ab\| \leq \alpha \|ba\|, \forall a \in E, \forall b \in E$.
(ii) $\|ab\| \leq \alpha \|ba\|, \forall a \in G(E), \forall b \in E$.
(iii) $\forall c \in G(E) : \|c + ab\| \leq \alpha \|c + ba\|, \forall a \in G(E), \forall b \in E$.
(iv) $\exists k > 0 : \|ab - ba\| \leq k, \forall a \in G(E), \forall b \in E$.
(v) E is commutative.

Remark 3.12. Actually, not all the strength of $(\mathbb{B})$ is used. Consider the inner derivation $\delta_x : E \longrightarrow E$, for some x, that is $y \mapsto xy - yx$ and put $\mathbb{N}y = \{ny : n \in \mathbb{N}\}$. What we need is just $\delta_x(\mathbb{N}y)$ to be bounded. Notice however that the boundedness of δ_x is not sufficient; take e.g., a non commutative Banach algebra.

4. Topological algebras. The condition $(\mathbb{B})$ is so nice that it applies to multiplications for which no continuity properties are assumed. We consider topological vector spaces on which a multiplication can be defined. We will say that they are with multiplication.

Proposition 4.1. Let (E, τ) be a separated *t.v.s.* with a multiplication $(x, y) \mapsto xy$. If for x in E, the subset $\{xy - yx : y \in E\}$ is bounded then x is in the center of E.

Proof. Let V be any neighborhood of zero. By hypothesis, there is an $\alpha > 0$ such that

$$xy - yx \in \alpha V; \forall y \in E.$$

So also

$$xy - yx \in \frac{1}{n}\alpha V; \forall y \in E, \forall n \in \mathbb{N}^*.$$

Now, V can be supposed balanced. So, taking $n > \alpha$, one has

$$xy - yx \in V; \forall y \in E.$$

But this is true for every V. Hence $xy - yx = 0$, since (E, τ) is separated. If x is not in $G(E)$, there is by hypothesis a $\lambda \in \mathbb{C}$ and a $z \in G(E)$ such that $x = \lambda e - z$. One then uses the preceding. □

In topological algebras, the condition $(\mathbb{B})$ can be relaxed under an additional hypothesis.

Proposition 4.2. Let (E, τ) be a separated topological Q-algebra. If for every x in $G(E)$, the subset $\{xy - yx : y \in E\}$ is bounded then E is commutative.

Proof. As in Proposition 4.1, one shows that every x in $G(E)$ is in the center of E. Now, by hypothesis, $G(E)$ is open. Hence $e - G(E)$ is a neighborhood of zero. So, for every $y \in E$, there is α such that $\alpha y \in e - G(E)$. Then $e - \alpha y$ is in the center. □

Remark 4.3. To illustrate Proposition 4.2, let us say that it applies to p-Banach algebras [21], *l.m.c.a.′s* (cf. [7], [3]), A-normed algebras and more generally *l.u-A-c.a.′s* or *l-A-c.a.′s* (cf. [10]), etc.

Remark 4.4. The Grothendieck completion $\widehat{E}$ of a topological algebra (E, τ) is not necessarily an algebra. In the affirmative $\widehat{E}$ is not necessarily of the same type as E (cf. [10]). So, the absence of any completion condition in the above proposition is of importance.

One can give a statement concerning bilinear maps.

Proposition 4.5. Let (E, τ) be a separated *t.v.s.*and a map $f : E \times E \longrightarrow E$. If moreover $B = \{f(x, y) - f(y, x) : x, y \in E\}$ is bounded then f is symmetric; that is $f(x, y) = f(y, x)$, for all $x, y \in E$.

Remark 4.6. Actually, we can say more (cf. Proposition 5.5)

5. Non topological algebras. Our condition $(\mathbb{B})$ involves directly the commutators $xy - yx$, and boundedness. One is then tempted by a bornological setting, without appealing to topology.

Proposition 5.1. Let $(E, \mathcal{B})$ be a separated *b.v.s.* which can be endowed with a multiplication $(x, y) \mapsto xy$. If for x in E, the subset $\{xy - yx : y \in E\}$ is bounded then x is in the center of E.

Proof. By hypothesis, there is a bounded set B and an $\alpha > 0$ such that

$$xy - yx \in \alpha B; \forall y \in E.$$

So also

$$xy - yx \in \frac{1}{n}\alpha B; \forall y \in E, \forall n \in \mathbb{N}^*.$$

Now, since $\left(\frac{1}{n}\alpha\right)_n$ tends to zero, the constant sequence $(xy - yx)$ is Mackey convergent to zero. But $(E, \mathcal{B})$ is separated, hence $xy - yx = 0$.

The analogue of Proposition 4.2 is the following. □

Proposition 5.2. Let $(E, \mathcal{B})$ be a separated bornological algebra which is also an M-Q-algebra. If for every x in $G(E)$, the subset $\{xy - yx : y \in E\}$ is bounded then E is commutative.

Proof. As in Proposition 5.1, one shows that $xy - yx = 0$, every x in $G(E)$ and every $y \in E$. Now let x be a non invertible element. By hypothesis, $G(E) - e$ is bornivorous. So there is an α such that $\alpha x \in G(E) - e$. One then argues with $e + \alpha x$. □

The previous proposition applies, in particular, to the more telling pseudo-Banach algebras ([2], see also [5]).

Corollary 5.2. Let $(E, \mathcal{B})$ be a unital pseudo-Banach algebra. If for every x in $G(E)$, the subset $B(x) = \{xy - yx : y \in E\}$ is bounded then E is commutative.

Proof. The algebra is a bornological inductive limit of Banach algebras E_i, $i \in I$. Since the set of indices is (upwards) directed, take a Banach algebra E_j such that $x \in E_j$, and $B(x) \subset E_j$ and bounded there. The situation is then reduced to the Banach case. □

Remark 5.3. The previous statement is actually valid for a bornological inductive limit of Q-normed algebras.

Now, here is an example the bornology of which is not the von Neumann-Kolmogorov bounded structure of a topological vector space.

Example 5.4. Let $E = C(K)$ be an infinite dimensional Banach algebra of continuous complex valued functions on a compact space. Denote by E_c (resp. F) the vector space E endowed with the bounded structure of compact (resp. idempotent compact) discs. It is claimed ([5], Remarque 2, p.37) that the identity map $Id : E_c \to F$ is not bounded. The algebra F is pseudo-Banach while E_c is described only as a bornological convex algebra every element of which is bounded. We have shown [10] that it is actually a pseudo-Banach. The bounded structure considered can not be the one of a topological vector space.

We have announced, in the introduction, that our condition $(\mathbb{B})$ is still efficient in a very flexible context. Indeed, it can be generalized to the simple structure of monoids, yet to functions which are not bilinear as the multiplication is.

Proposition 5.5. Let E be a monoid, $(F, \mathcal{B})$ a separated $b.v.s.$ and a map $f : E \times E \longrightarrow F$ such that $f(nx, y) = n\ f(x, y)$ and $f(x, ny) = nf(x, y)$, for every $n \in \mathbb{N}^*$. If moreover $B = \{f(x,y) - f(y,x) : x, y \in E\}$ is bounded then f is symmetric; that is $f(x,y) = f(y,x)$, for all $x, y \in E$.

Proof. As in Proposition 5.1. □

Remark 5.6. In the previous proposition one can, of course consider a *t.v.s.* (E, τ), instead of $(F, \mathcal{B})$.

Remark 5.7. It is worthwhile to notice that no topology, nor a bornology is assumed in the monoid E.

Remark 3.11 leads to a weakening of $(\mathbb{B})$. The following statement seems to be more telling than that remark.

Proposition 5.8. Let $(E, \mathcal{B})$ be a separated *b.v.s.* (real or complex) with multiplication (admitting a unit or not). The following assertions are equivalent.
(i)$(E, \mathcal{B})$ satisfies $(\mathbb{B})$.
(ii) $(E, \mathcal{B})$ has the $(\mathbb{B}L)$ property; that is every inner derivation δ_x transforms straight lines into bounded subsets.
(iii) E is commutative.

Proof. (i) $\Longrightarrow$(ii) Obvious. (ii) $\Longrightarrow$(iii) By hypothesis,

$$\{\lambda(xy - yx) : \lambda \in \mathbb{K};\ \mathbb{K} = \mathbb{R},\ \mathbb{C}\}$$

is bounded. But it is also a straight line. So it is reduced to zero, since $(E, \mathcal{B})$ is separated. (iii) $\Longrightarrow$(i) Obvious. □

Remark 5.9. Different formulations can be given to other results in the paper, using the condition $(\mathbb{B}L)$.

Remark 5.10. The difference $xy - yx$ measures, let us say, the non commutativity of x and y. We want it to be zero. So it has first to be small. Condition $(\mathbb{B}L)$ ensures the requirement.

6. Locally A-convex algebras. In this class of algebras (in particular locally m-convex or uniformly A-convex ones) one can omit the Q-property and ask only for the condition $(\mathbb{B}G)$, modulo some completeness; the latter allowing, as in the Banach case, the use of the exponential function and Liouville's theorem. We first recall results which will be needed (cf. [12]).

Proposition 6.1. Let (E, τ) be a unital M-complete *l.m.c.a.* Then
(i) (E, τ) is a bornological inductive limit of Fréchet *l.m.c.a.*'s (E_i, τ_i).
(ii) If every element of (E, τ) is bounded, then it is an M-Q-algebra.

So if an algebra, as in (ii), satisfies $(\mathbb{B}G)$ then it is commutative (Proposition 5.2). Without the condition in (ii), one has to argue differently.

Proposition 6.2. Let (E, τ) be a unital M-complete *l.m.c.a.* If it satisfies $(\mathbb{B}G)$, then it is commutative.

Proof. Let (E_i, τ_i) be Fréchet *l.m.c.a.*'s whose bornological inductive limit is $(E, \mathbb{B}\tau)$. For $x \in G(E)$ and $y \in E$, let j be such that both x and y are in E_j; this is possible since the set I of indices is (upwards) directed. Now, by hypothesis, the subset $\{x \exp(\lambda y) - \exp(\lambda y)\, x : \lambda \in \mathbb{C}\}$ is bounded in (E, τ). It is then contained and bounded in some E_l. The function $g : \mathbb{C} \longrightarrow E_l$, defined by $g(\lambda) = x \exp(\lambda y) - \exp(\lambda y)\, x$, is holomorphic and bounded. By Liouville's theorem, it is constant, since (cf. [18], Theorem 3.32, p.81). $\square$

The previous result allows an extension to the locally A-convex case. For convenience, we recall some basic facts ([12]; see also [16] and [17]). If $(E, (p_\lambda)_\lambda)$ is a unital *l-A-c.a.*, then it can be endowed with a stronger m-convex topology $M(\tau)$, where τ is the topology of E. It is determined by the family $(q_\lambda)_\lambda$ of seminorms given by

$$q_\lambda(x) = \sup\{p_\lambda(xu) : p_\lambda(u) \leq 1\}.$$

If $(E, (p_\lambda)_\lambda)$ is uniformly A-convex, then there is also an algebra norm $\|.\|_0$ on E, stronger than $M(\tau)$, given by

$$\|x\|_0 = \sup\{q_\lambda(x) : \lambda\}.$$

Proposition 6.3. Let (E, τ) be a unital M-complete locally A-convex algebra. If it satisfies $(\mathbb{B}G)$, then it is commutative.

Proof. Let $M(\tau)$ be the locally m-convex topology associated to τ [14]. Since (E, τ) is M-complete, (E, τ) and $(E, M(\tau))$ have the same bornology. So $(E, M(\tau))$ is a unital M-complete *l.m.c.a.* satisfying also $(\mathbb{B}G)$. To conclude, just apply Proposition 6.2. $\square$

Remark 6.4. If (E, τ) in the preceding proposition is a unital M-complete locally uniformly A-convex algebra, then there is a Banach algebra norm $\|.\|_o$ on E such that $\mathbb{B}\tau = \mathbb{B}\tau_{\|.\|_o}$ (cf. [17] or [16]).

The condition $(\mathbb{B}G)$ is still of value in an even more general setting.

A *l.c.a.* is said to be locally uniformly convex (*l.u.c.a.*) if,

$$(\forall x)(\forall \lambda)(\exists M(x) > 0)(\exists \lambda') : p_\lambda(xy) \leq M(x) p_{\lambda'}(y), \forall y$$

with $M(x)$ depending only on x (not on λ) and λ' depending only on λ (not on x). The class of such algebras is different from that of locally uniformly A-convex ones (See [13]). Good properties of the latter are, in general, lost such as the boundedness of every element. However, we still have some analogous results. A unital *l.u.c.a.* (E, τ) can be endowed with a norm the topology of which is stronger than τ ([13], (i) of Proposition 1.1). But here the norm is not always an algebra norm; It may not even be an A-norm. However, it is a Banach algebra norm in the unital M-complete case ([13], Proposition 2.1). So one can state the following.

Proposition 6.5. Let (E, τ) be a unital M-complete *l.u.c.a.* If it satisfies $(\mathbb{B}G)$, then it is commutative.

7. Commutativity and inner automorphisms. Recall that an automorphism $\alpha : E \longrightarrow E$ is said to be inner if there is a c in $G(E)$ such that $\alpha(a) = \alpha_c(a) = c^{-1}ac$. Clearly, E is commutative if and only if every element a is a common fixed point for all inner automorphisms. It seems that a condition to have the latter property must be hard. But this is exactly what happens, in Le Page's proof, with particular ones.

He first obtains $\exp(-\lambda x)\, y \exp(\lambda x) = y$. Notice also that it is not satisfied in the quaternion field. Indeed, considering δ_i, one has $\delta_i(j) = -j$. Moreover, the existence of a common divisor is not sufficient; take e.g., any unital non commutative algebra.

Now, in the same vein of ideas in the preceding sections, we turn our attention to the notion of boundedness. The preceding suggest the consideration of the subset $\left\{c^{-1}ac - a : a \in E\right\}$. If one assumes the boundedness of the latter, then he can argue as in Proposition 4.1 and Proposition 5.1. But as in the previous sections this condition can be weakened. We give only a few results. The analogous statements of the others are easy to guess.

Proposition 7.1. Let $(E, \mathcal{B})$ be a separated *b.v.s* with multiplication (real or complex) admitting a unit and such that, for every c in $G(E)$, the map $a \mapsto c^{-1}ac - a$ transforms straight lines into bounded subsets. Then E is commutative.

Proof. Argue as in Proposition 5.8. □

Remark 7.2. Without any surprise, the condition in the previous proposition is thus equivalent to the condition $(\mathbb{B})$. But one can see this equivalence directly. Indeed, one has $c^{-1}ac - a = c^{-1}ac - c^{-1}ca = c^{-1}(ac - ca) = c^{-1}\delta_c$.

Remark 7.2. The difference $a - c^{-1}ac$ measures the impact of the transformation of a by δ_c. So it has to be small, before being zero in order a be a fixed point of δ_c. This is assured by the boundedness condition.

The analogues of Proposition 4.2 and Proposition 5.2 are respectively the following.

Proposition 7.3. Let (E, τ) be a separated topological Q-algebra. If for every x in $G(E)$, the the map $a \mapsto c^{-1}ac - a$ transforms straight lines into bounded subsets, then E is commutative.

Proposition 7.4. Let $(E, \mathcal{B})$ be a separated bornological algebra which is also an M-Q-algebra. If for every x in $G(E)$, the map $a \mapsto c^{-1}ac - a$ transforms straight lines into bounded subsets. then E is commutative.

Remark 7.5. The condition considered in this section can, of course, be added in Proposition 3.9.

8. An associativity condition. If we drop the associativity property in the definition of the multiplication, we say that the algebra is non associative. The classical example is the algebra $\mathcal{L}(E)$ of linear operators on a vector space E, endowed with the hook operation $[S, T] = ST - TS$. This is a particular case of a Lie algebra. The latter is an algebra the multiplication of which (the hook) $(x, y) \mapsto [x, y]$ satisfies the following properties

(1) $[x, x] = 0$, for every x.

(2) $[x, [y, z]] + [x, [y, z]] + [x, [y, z]] = 0$, for all x, y and z (Jacobi's identity).

No use of the second condition will be made.

Remark 8.1. One first observes that associativity is redundant in Proposition 4.1, so the latter is still true with non associative multiplications.

Let E be a unital associative separated topological algebra. Endowed with the hook operation $(x, y) \mapsto [x, y] = xy - yx$, it becomes a Lie algebra. Then $[x, y] - [y, x] = 2(xy - yx)$. Thus the commutativity condition is the same for both multiplications, though one of them is associative with a unit while the other does not have these properties. But actually, if E is commutative then the hook operation is trivial. It happens that this is true for any Lie algebra. Indeed, one has using $[x + y, y + y] = 0$ and Property (1), one gets $0 = [x + y, x + y] = 2[x, y]$. So we are led to the following statement.

Proposition 8.2. Let (E, τ) be a separated topological Lie algebra. If for every x in E, the set $\{[x, y] - [y, x] : y \in E\}$ is bounded, then E is of trivial multiplication.

Proof. Let V be any neighborhood of zero. By hypothesis, there is for every x in E, an $\alpha > 0$ such that

$$[x, y] - [y, x] \in \alpha V; \forall y \in E.$$

So also

$$[x, y] - [y, x] \in \frac{1}{n}\alpha V; \forall y \in E, \forall n \in \mathbb{N}^*.$$

Now, V can be supposed balanced. So, taking $n > \alpha$, one has

$$[x, y] - [y, x] \in V; \forall y \in E.$$

But this is true for every V. Hence $[x, y] - [y, x] = 0$, since (E, τ) is separated. Thus E is commutative.

The considerations above led us to the following somehow curious characterizations of the complex field $\mathbb{C}$ among a class of topological vector spaces. □

Proposition 8.3. Let (E, τ) be a separated *t.v.s.* such that the algebra $\mathcal{L}(E)$ of its linear operators can be endowed with a separated topological algebra structure (associative or not) with the Q-property. Then the following assertions are equivalent.

(i) For every $S \in G(\mathcal{L}(E))$, the map $T \mapsto S^{-1}TS - T$ transforms straight lines into bounded sets.

(ii) For every $S \in G(\mathcal{L}(E))$, the set $\{TS - ST : T \in E\}$ is bounded.

(iii) $\mathcal{L}(E)$ is commutative.

(iv) E is isomorphic to $\mathbb{C}$.

Hints. (i) implies (ii) for $\mathcal{L}(E)$ is then commutative (the analogue of Proposition 7.1). (ii) implies (iii) by Proposition 4.2. (iii) implies (iv) is well known. The last implication is trivial.

Now, we turn our attention to inner derivations. A derivation in a Lie algebra E is a vector space endomorphism D such that

$$D([x, y]) = [Dx, y] + [x, Dy], \text{ for all } x,\ y \text{ in } E.$$

Let δ_x be defined, on E, by $\delta_x(y) = [x, y]$. it is a derivation, called an inner derivation. We have seen that if the multiplication $[,]$ is commutative, then it is trivial. So a Lie algebra does not satisfy any condition implying that property (cf. Proposition 8.2). One can argue in a similar way and obtain the following.

Proposition 8.4. Let (E, τ) be a separated *t.v.s.* with a Lie multiplication $[,]$ (associative or not). If the inner derivations transform straight lines into bounded sets, then $[,]$ is trivial.

Proof. Let V be any balanced neighborhood of zero. By hypothesis, there is for every x in E, an $\alpha > 0$ such that

$$[x, y] \in \alpha V; \forall y \in E.$$

So also

$$[x, y] \in \frac{1}{n}\alpha V; \forall y \in E, \forall n \in \mathbb{N}^*.$$

So, taking $n > \alpha$, one has

$$[x, y] \in V; \forall y \in E.$$

But this is true for every V. Hence $[x, y] = 0$, since (E, τ) is separated. Thus E is commutative. □

By the approach adopted here one can also control associativity. A Lie algebra may be with an associative or a non associative multiplication.

Example 8.5. Let E be an anti-commutative algebra , that is $xy = -yx$, for every x and y in E. Endowed with the hook operation $(x, y) \mapsto [x, y] = xy - yx$, it becomes a Lie algebra. It is not commutative. But it is associative. Indeed, One has $[x, y] = 2xy$. Hence

$$[[x, y], z] = 2x [y, z] = 4xyz \text{ and } [x, [y, z]] = 2 [x, y] z = 4xyz.$$

Example 8.6. Take a non commutative algebra E admitting elements x and y such that $xyx \neq 0$; for example an algebra without divisors of zero. Then

$$[[x, x], y] = [x, xy - yx] = -2xyx \text{ and } [[x, x], y] = [0, y] = 0.$$

Hence the Lie algebra associated to E is not associative.

Example 8.7. Let E be a finite dimensional vector space and $A(E)$ the algebra of alternated multilinear forms on E, with the exterior product $\wedge$ as multiplication.When dim$E$ is larger than 1, the algebra $A(E)$ need not be commutative nor anti-commutative, due to the classical formula $g \wedge f = (-1)^{qr} f \wedge g$. However it is associated and unital. Also, one has $f \wedge f = (-1)^{qr} f \wedge f$; so it is not a Lie algebra. Now, consider the associated hook operation. $[f, g] = f \wedge f - g \wedge f$. Endowed with $[,]$ it becomes a Lie algebra. It is not associative. Indeed, take $f = g$ and h such that $q = r$ is even, and rs is odd. Then

$$[f, [f, g]] = [[f, f], g] \text{ if and only if } f \wedge f \wedge h = -f \wedge f \wedge h.$$

So it is sufficient to consider particular f and h which do not satisfy the last equality.

It seems worthwhile to have an associativity criterion. Here is one in a very large setting.

Proposition 8.8. Let (E, τ) be a separated *t.v.s.* with a multiplication $(x, y) \mapsto xy$. If for x in E, the subset $\{x\,(yz) - (xy)\,z : y, z \in E\}$ is bounded then the multiplication is associative.

Proof. As in Proposition 4.1. □

Proposition 8.9. Let (E, τ) be a separated topological Q-algebra. If for every x in $G\,(E)$, the subset $\{x\,(yz) - (xy)\,z : y, z \in E\}$ is bounded then the E is associative and commutative.

Proof. The associativity is obtained as in Proposition 4.2. Now taking $z = e$ the unit of E, one gets the property $(\mathbb{B}G)$ whence the commutativity again as in Proposition 4.2. □

Remark 8.10. One easily states the bornological analogues of Proposition 8.4 and Proposition 8.5.

Remark 8.11. According to Professor A Mallios, the transformation of straight lines into bounded sets could have a physical meaning. He considers that non commutativity is not inherent to the phenomena, but only to our mathematical apparatus. Commutativity should be sufficient. Moreover, the projective space is used to describe states; so straight lines are basically involved. And we do have a condition concerning them and implying commutativity.

Acknowledgements. Very warm thanks are offered to Prof. A. Mallios. He has first encouraged me to continue working on commutativity conditions, insisting on the fact that it is an interesting subject. He also has been generous in time. We had so many fruitful discussions. On the other hand, warm thanks are offered to the referee for a careful checking of the manuscript. He/she made valuable suggestions concerning the presentation and the writing. He/she also has been so kind, pointing out so many misprints.

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◇ M. Oudadess
c/o A. El Kinani
Ecole Normale Superieure
Avenue Oued Akrach
B. P. 10405, Rabat (Maroc)
oudadessm@yahoo.fr