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Solutions of the matrix equation $AXB^T = C$ over singular matrices*

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Abstract

In this note, the computation of the solutions of the matrix equation $AXB^T = C$ is presented with respect to X, where A, B and C are given in $\mathbb{C}^{m\times n}$. We consider the case that A and B are singular matrices and the method used relies on Kronecker matrix products.

Keywords: Linear Matrix Systems; Kronecker Products; Singular Matrices
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1. Introduction

Throughout the paper, $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices. For any $A \in \mathbb{C}^{m\times n}$ symbols $\mathcal{R}(A)$, $rank(A)$, $\mathcal{N}(A)$ and A^D stand for the range, the rank, the null space and the Drazin inverse of A, respectively.

In this short paper, we provide an initial discussion for the computation of the solutions of linear systems (1.1) of type:

$$AXB^T = C, \tag{1.1}$$

where $A, X, B \in \mathbb{C}^{n\times n}$ with $det(A) = det(B) = 0$.

In literature of linear matrix equations, there have been proposed several methods to estimate the unknown parameter matrix X; see for instance Tian [13], Deng et al. [5], Liao and Lei [10], Liu [11], Jokar and Mehrmann [7] and the references therein.

*This short paper is dedicated to our beloved Professor, Prof. Grigoris Kalogeropoulos, on the occasion of his 70th birthday. Prof. G. Kalogeropoulos was the former President of the Greek Mathematical Society from 2009-2013.

2. Notation and Preliminaries: Properties of the Kronecker product and Pseudo-Inverses

In this section, the main properties of the Kronecker product will be presented; see for instance Neudecker [12] and Bellman [2] for further details.

Definition 2.1 Let $A = [a_{ij}] \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{k\times l}$. The **kronecker tensor** product is defined as follows

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \in \mathbb{C}^{m\cdot k\times n\cdot l}. \tag{2.1}$$

Associate with the matrix $X = [x_{ij}] \in \mathbb{C}^{m\times n}$, we can consider the useful transformation U, such as the following column vector is derived

$$U(X) = \left[x_{11}\ x_{12} ...\ x_{1n} \vdots\ x_{21}\ x_{22} ...\ x_{2n} \vdots ... \vdots x_{m1}\ x_{m2} ...\ x_{mn} \right] \in \mathbb{C}^{m\cdot n\times 1}. \tag{2.2}$$

Moreover, we denote the "inverse" transformation $\mathcal{U}$:

$$\mathcal{U}(U(X)) = \mathcal{U}\left(\left[x_{11}\ x_{12} ...\ x_{1n} \vdots\ x_{21}\ x_{22} ...\ x_{2n} \vdots ... \vdots x_{m1}\ x_{m2} ...\ x_{mn} \right] \right) = X \in \mathbb{C}^{m\times n}. \tag{2.3}$$

This definition follows Neudecker [12], which is also slightly different from the adopted elsewhere Barnett [1], but is a somewhat more natural one since when is applied to product the ordering is maintained, i.e.

$$U(CXD) = \left(C \otimes D^T\right) U(X), \tag{2.4}$$

where $C \in \mathbb{C}^{p\times m}$ and $D \in \mathbb{C}^{n\times q}$.

Lemma 2.1 *Let $A, B \in \mathbb{C}^{n\times n}$ and $C, D \in \mathbb{C}^{m\times m}$, then*

$$(A \otimes C) \cdot (B \otimes D) = AB \otimes CD. \tag{2.5}$$

Lemma 2.2 *Let $A, B \in \mathbb{C}^{n\times n}$ and $A \otimes B \in \mathbb{C}^{n^2\times n^2}$ are non-singular matrices, then*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}. \tag{2.6}$$

Lemma 2.3 *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$, then*

$$\det(A \otimes B) = (\det(A))^n (\det(B))^m. \tag{2.7}$$

Remark 2.1 Let $A \in \mathbb{C}^{n\times n}$ with $\det A = 0$, then $\det(A \otimes I_m) = (\det(A))^n = 0$.

In this part of the paper, some necessary notation and preliminaries from the Pseudo inverses and particularly for the Drazin inverse are provided based on Ben-Israel and Greville [3], Greville [6] and Campbell and Meyer [4].

Lemma 2.4 *Let $\underset{\sim}{A}$ be a linear transformation on $\mathbb{C}^n$. There exists a non-negative integer k such that $\mathbb{C}^n = R(\underset{\sim}{A}^k) \dotplus N(\underset{\sim}{A}^k)$.*

The number k which has been introduced above is really very important.

Definition 2.2 Let $\underset{\sim}{A}$ be a linear transformation on $\mathbb{C}^n$. The smallest non-negative integer k such that $\mathbb{C}^n = R(\underset{\sim}{A}^k) \dotplus N(\underset{\sim}{A}^k)$ or equivalently, the smallest non-negative integer k such that $rank(\underset{\sim}{A}^k) = rank(\underset{\sim}{A}^{k+1})$ is called the index of $\underset{\sim}{A}$ and is denoted by $Ind(\underset{\sim}{A})$.

Theorem 2.1 *(The Canonical form representation for A and A^D)*
If $A \in \mathbb{C}^{n\times n}$ is such that $Ind(A) = k > 0$, then there exists a non-singular matrix P such that

$$A = P \begin{bmatrix} C & \mathbb{O} \\ \mathbb{O} & N \end{bmatrix} P^{-1} \tag{2.8}$$

where C is non-singular and N is nilpotent of index k. Furthermore, if P, C and N are any matrices satisfying the above conditions, then

$$A^D = P \begin{bmatrix} C^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} P^{-1}. \tag{2.9}$$

Let $B = \{v_1, v_2, ...v_r, v_{r+1}, ..., v_n\}$ be the linear transformation induced on $\mathbb{C}^n$ by A and the basis for $\mathbb{C}^n$ constructed in Lemma 2.4 respectively, so that $\{v_1, ..., v_r\}$ is a basis for $R(\underset{\sim}{A}^k)$ and $\{v_{r+1}, ..., v_n\}$ is a basis for $N(\underset{\sim}{A}^k)$. Since $R(\underset{\sim}{A}^k)$ and $N(\underset{\sim}{A}^k)$ are invariant subspaces for $\underset{\sim}{A}$ and $\underset{\sim}{A}^k(N(\underset{\sim}{A}^k))$, we have the block form for A if $P = [v_1, ..., v_n]$.

3. Main results

In this section, the main results for the linear matrix system (1.1) are presented analytically. The whole discussion extends the existing literature and provides an alternative way to solve $AXB^T = C$ with respect to X considering also singularities on matrices A and B. At this point, it should be mentioned that in a recent paper by Karageorgos et al. [8], the matrix pencil approach has been recommended as an interesting, other alternative methodology to solve linear matrix systems with singularities on the related matrices. The Kronecker approach, which will be discussed here, has been also applied very successfully in matrix differential equations; see for instance Barnett [1] and Karageorgos et al. [9]. After the next lemma, a necessary condition is given for the solvability of system (1.1).

Lemma 3.1 *The system (1.1) can be written as follows:*

$$\begin{bmatrix} I_{rs} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} (Q \otimes P)^{-1} U(X) = (Q \otimes P)^{-1} (A^D \otimes B^D) U(C), \tag{3.1}$$

where $P, Q \in \mathbb{C}^{n \times n}$ are invertible matrices and U is the transformation that has been introduced in the previous section.

Proof. First, we apply the transformation U on system (1.1) such that

$$U(AXB^T) = U(C) \Leftrightarrow (A \otimes B)U(X) = U(C) \Leftrightarrow$$

$$(A^D \otimes B^D)(A \otimes B)U(X) = (A^D \otimes B^D)U(C) \Leftrightarrow$$

$$(A^D A \otimes B^D B)U(X) = (A^D \otimes B^D)U(C). \tag{3.2}$$

However, $A^D A = P \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} P^{-1}$ and $B^D B = Q \begin{bmatrix} I_s & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} Q^{-1}$, see also the previous section and Theorem 2.1.

Then, after some algebraic calculations we take

$$A^D A \otimes B^D B = (P \otimes Q) \left(\begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \begin{bmatrix} I_s & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \right) (P^{-1} \otimes Q^{-1}) =$$

$$(P \otimes Q) \begin{bmatrix} I_r s & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} (P^{-1} \otimes Q^{-1}).$$

Thus, eq. (3.2) is reformulated to eq. (3.1). □

Remark 3.1 For the solvability of system (3.2), the following condition should be satisfied:

$$\begin{bmatrix} \mathbb{O}_{rs} & \mathbb{O}_{rs,n^2-rs} \\ \mathbb{O}_{n^2-rs,rs} & \mathbb{O}_{n^2-rs} \end{bmatrix} (P \otimes Q)^{-1} (A^D \otimes B^D) U(X) = 0. \tag{3.3}$$

The above remark is derived naturally from the analysis of equation (3.1). Here, unnecessary details are omitted as the result is rather straightforward.

Theorem 3.1 *The transformed solution $U(X)$ of the matrix X is given by*

$$U(X) = \Gamma \begin{bmatrix} \underline{\theta}_{rs} - T_{rs,n^2-rs} \underline{z}_{n^2-rs} \\ \underline{z}_{n^2-rs} \end{bmatrix} \in \mathbb{C}^n, \tag{3.4}$$

where $\Gamma \in \mathbb{C}^{n \times n}$, $T \in \mathbb{C}^{rs \times (n^2-rs)}$, $\underline{\theta}_{rs} \in \mathbb{C}^{rs}$ and the arbitrary vector $\underline{z} \in \mathbb{C}^{n^2-rs}$.

Proof. Let's assume that the matrix $(Q \otimes P)^{-1}$ can be re-written as follows

$$\begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ M_{n^2-rs,rs} & \Pi_{n^2-rs} \end{bmatrix}.$$

Obviously, the above matrix is invertible, then we take

$$\begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ \mathbb{O}_{n^2-rs,rs} & \mathbb{O}_{n^2-rs} \end{bmatrix},$$

as the following equation holds,

$$\begin{bmatrix} I_{rs} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ M_{n^2-rs,rs} & \Pi_{n^2-rs} \end{bmatrix} = \begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix},$$

and the *rank* $\begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = rs < n^2$.

Indeed, there is always a transformation for the above matrix such that

$$\begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \Gamma = \begin{bmatrix} I_{rs} & I_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}.$$

Thus, eq. (3.1) is reformulated to

$$\begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} U(X) = (Q \otimes P)^{-1}(A^D \otimes B^D)U(C).$$

Let's introduce now the following transformation,

$$U(X) = \Gamma \begin{bmatrix} \underline{y}_{rs} \\ \underline{y}_{n^2-rs} \end{bmatrix} \in \mathbb{C}^n. \tag{3.5}$$

Consequently, we have

$$\begin{bmatrix} E_{rs} & \Lambda_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \Gamma \begin{bmatrix} \underline{y}_{rs} \\ \underline{y}_{n^2-rs} \end{bmatrix} = (Q \otimes P)^{-1}(A^D \otimes B^D)U(C) \Leftrightarrow$$

$$\begin{bmatrix} I_{rs} & T_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \underline{y}_{rs} \\ \underline{y}_{n^2-rs} \end{bmatrix} = (Q \otimes P)^{-1}(A^D \otimes B^D)U(C). \tag{3.6}$$

Based on Theorem 1 [3], the $\{1\}$- inverse (i.e. the property $AA^{(1)}A = A$ is satisfied) of matrix

$$\begin{bmatrix} I_{rs} & T_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$$

is given by $\begin{bmatrix} I_{rs} & \mathbb{O} \\ \mathbb{O} & L_{n^2-rs} \end{bmatrix}$, for any $L_{n^2-rs} \in \mathbb{C}^{(n^2-rs)\times(n^2-rs)}$ matrix.

Then, the system (3.5) is solvable if and only if there is a $\{1\}$-inverse $\begin{bmatrix} I_{rs} & T_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}$ such that

$$\begin{bmatrix} I_{rs} & TL_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}(P\otimes Q)^{-1}(A^D\otimes B^D)U(C) = (P\otimes Q)^{-1}(A^D\otimes B^D)U(C). \quad (3.7)$$

Following the remark 3.1 and the derived equation (3.3), then we can assume that

$$(P \otimes Q)^{-1}(A^D \otimes B^D)U(C) = \begin{bmatrix} \underline{\theta}_{rs} \\ 0 \end{bmatrix}$$

[Otherwise, the initial system is not solvable.]

So, the system (3.6) can be written as

$$\begin{bmatrix} I_{rs} & TL_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}(P \otimes Q)^{-1}(A^D \otimes B^D)U(C) =$$

$$\begin{bmatrix} I_{rs} & TL_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix}\begin{bmatrix} \underline{\theta}_{rs} \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{\theta}_{rs} \\ 0 \end{bmatrix}.$$

Thus, the solution is given by

$$\begin{bmatrix} \underline{y}_{rs} \\ \underline{y}_{n^2-rs} \end{bmatrix} =$$

$$\begin{bmatrix} I_{rs} & \mathbb{O} \\ \mathbb{O} & L_{n^2-rs} \end{bmatrix}(P\otimes Q)^{-1}(A^D\otimes B^D)U(C)+(I_{n^2}-\begin{bmatrix} I_{rs} & \mathbb{O} \\ \mathbb{O} & L_{n^2-rs} \end{bmatrix}\begin{bmatrix} I_{rs} & T_{rs,n^2-rs} \\ \mathbb{O} & \mathbb{O} \end{bmatrix})\underline{z}$$

where, $\underline{z} \in \mathbb{C}^{n^2}$ is an arbitrary vector.

Then, after some simple algebraic calculations, it is derived that

$$\begin{bmatrix} \underline{y}_{rs} \\ \underline{y}_{n^2-rs} \end{bmatrix} = \begin{bmatrix} I_{rs} & \mathbb{O} \\ \mathbb{O} & L_{n^2-rs} \end{bmatrix}(P \otimes Q)^{-1}(A^D \otimes B^D)U(C) + \begin{bmatrix} \mathbb{O} & -T_{rs,n^2-rs} \\ \mathbb{O} & I_{n^2-rs} \end{bmatrix}\underline{z}. \quad (3.8)$$

Then, using the transformation (3.5) and previous assumptions, we take

$$U(X) = \Gamma\left(\begin{bmatrix} I_{rs} & \mathbb{O} \\ \mathbb{O} & L_{n^2-rs} \end{bmatrix}\begin{bmatrix} \underline{\theta}_{rs} \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbb{O} & -T_{rs,n^2-rs} \\ \mathbb{O} & I_{n^2-rs} \end{bmatrix}\underline{z}\right). \quad (3.9)$$

where, the eq. (3.4) is derived after some algebraic calculations made on eq. (3.9). □

The following expression is based on the "inverse" transformation $\mathcal{U}$. Thus, the solution X is derived as follows, see (3.10).

Corollary 3.1 *The solution of eq. (1.1) is given by*

$$X = \mathcal{U}\left(\Gamma\begin{bmatrix} \underline{\theta}_{rs} - T_{rs,n^2-rs}\underline{z}_{n^2-rs} \\ \underline{z}_{n^2-rs} \end{bmatrix}\right) \in \mathbb{C}^{n\times n}, \quad (3.10)$$

where Γ, T, $\underline{\theta}$ *and* $\underline{z}$ *have been defined in Theorem 3.1.*

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