BULLETIN OF THE
HELLENIC MATHEMATICAL SOCIETY
Volume 59, 2016 (107-118)

Number Theory in Byzantium according to Codex Vindobonensis phil. Graecus 65 of the 15th cent The numbers and the numeric position in Byzantium

Maria Chalkou

Abstract

In this article we describe some elements from the chapter of Mathematics „Number Theory", as they appear in the Codex Vindobonensis phil. Gr. 65, a Byzantine Ms kept in the National Library of Austria in Vienna. This codex contains a comprehensive program of teaching Mathematics which was addressed to an audience consisting of students probably of all the grades of what is today's primary and secondary education, but also state functionaries, merchants, craftsmen of various specialities such as silversmiths and goldsmiths, builders, etc. The mathematical branch of Number Theory and the numerical position system is integrated in codex 65 in the respective chapters of the four operations and their checks.

Keywords: Codex Vindobonensis phil. Graecus 65, Number Theory, Greek Mss, Numerical Positioning System, Byzantine Mathematics, Indian digits, digits in Byzantium, Greek numbers, Zero.

Introduction

At the beginning of this article we briefly describe Codex Vindobonensis phil. Gr. 65 fols 11r —126r, which was written circa AD 1436 by an anonymous author, and was intended for teaching purposes. We record the syllabus units so that the reader will form an initial idea about the content of the Ms, and then we analyze in detail the problems related to the Number Theory, but also more generally to the numerical system of position in Byzantium. We explain the practical methodologies of the anonymous author with the proofs which are based on the modern Theories of

Mathematics, thus demonstrating the sound education of Byzantine teachers. Finally we cite the respective chapters from the transcribed text of the Ms so that the readers themselves will ascertain the correctness of the interpretations given during the study of the mathematical content of the Ms right after its transcription.

Description of the Ms

Codex Vindobonensis phil. gr. 65 is made of paper and dates back to the 15th c. AD.[1] The author and the origin of the codex[2] are unknown. The codex was obtained by Augerius von Busbeck when he was ambassador of emperor Ferdinand I in the court of sultan Suleiman II (1555—1562).[3] Fols 126v—140r contain a book of Arithmetic with solved problems, which was published in 1963 by H. Hunger and K. Vogel. The greatest part of the codex (fols 11r—126r) contains an anonymous book of Arithmetic, the preamble and first two chapters of which were published by J. L. Heilberg in 1899.[4]

It is a voluminous work of about 230 densely written A3 pages,[5] by an anonymous author, the content of which was taught before the Fall of Constantinople to a wider public coming from various social strata, as was usual at that time.

The Ms contains a large number of problems on various topics, which allows us to come to safe conclusions about the role played by Byzantine scholars in the progress of the science of Mathematics and mathematical education in general.

In addition, it offers a wealth of information about the problems the Byzantines faced in their everyday lives related to commerce, transactions,

---

[1] Its dimensions are 297/300x210/212 mm. And its issues are numbered as follows: II. 163 BI (I 160=V; +61/1; 142/1; 156/1; 157/1). 36Z. See H. HUNGER, *Katalog der griechischen Handschiften der Oesterreichishen National Bibliothek*, Teil 1 Codices Historici, Philosophici et Philologici, Wien 1961, 183.

[2] The codex consists of sheets (or the issues) which are not joined into a scroll, but they are spread between tablets of wood or some other material; it is considered the precursor of today's book. It appears in the 2nd century AD and replaces the cylinder because of its handiness. Around the same time parchment (usually the skin of a calf or sheep) replaces papyrus as writing material. The codex consists of joined issues, and each issue consists of a (variable) number of sheets folded into two. See E. MIONI, Introduzione alla paleografia greca, trans. in Greek by N. M. ΠΑΝΑΓΙΩΤΑΚΗΣ, *Εισαγωγή στην Ελληνική Παλαιογραφία,* Athens 1994, 46, 49.

[3] E. STAMATIS, *Κριτική βυζαντινού βιβλίου αριθμητικής* (Criticism of a byzantine book of arithmetic), Athens 1965, 5.

[4] H. HUNGER, Die Hochsprachliche profane Literatur der Byzantiner, trans. in Greek by T. ΚΟΛΙΑΣ, *Βυζαντινή Λογοτεχνία* 3, Athens 1994, 65.

[5] Appendix I

the manufacture of gold and silver articles, the construction of fortifications, etc.

The Ms is of considerable importance since the scientific research carried out indicated that it is in essence the Mathematical Encyclopaedia of the Byzantines, and possibly the first Mathematical Encyclopaedia.[6]

In addition, a second piece of evidence which reinforces the view that it is an important scientific work of its time is that from the scientific findings so far it follows that it is the first Greek Ms in which there appears the problem of chapter 177 (ροζ) of the construction of a square inscribed in an equilateral triangle, so that one of its sides touches one side of the equilateral triangle. Certainly, the theoretical part of the construction is not recorded in the Ms; what is only asked is to calculate the side of the square in relation to the side of the equilateral triangle.[7]

It must be noted that in the past there were some attempts to make the transcription, the study and the comments of Codex Vindobonensis phil. Graecus 65 by H. Hunger (1978) and J. L. Heiberg (1899), which resulted in publishing small parts of it despite the fact that the work was usually carried out by scientific teams. Yet this is expected because the difficulty of publishing a study of this kind lies in the fact that what is required is knowledge of Greek Paleography, Theoretical Mathematics, Methodology of Mathematics, and a very good knowledge of both Modern and 15th century Greek.

The initial study of the codex was published in 2004 by the National and Kapodistrian University of Athens (NKUA) as a Doctoral Thesis, which was accepted Summa Cum Laude by the Department of Mathematics of NKUA.

The improved and completed edition of the codex was published in 2006 by the Byzantine Research Centre of Aristotelian University of Thessaloniki and concerns fols 11r—126r. This part of the codex, which had remained unpublished until then since studies of the content of such Mss are rare, was called „Tractatus Mathematicus Vindobonensis Graecus“.[8]

The edition of the Doctoral Thesis published by the NKUA as well as the completed one published by the Byzantine Research Centre of

[6] M. CHALKOU, The Greek Mathematical Encyclopaedia of the Byzantines, *Byzantina* 27 (2007), 91—117.

[7] ANONYMOUS, *Arithmetica (Codex Vindobonensis phil. Gr. 65 of the 15th century)*, ed. M. CHALKOU, Thessaloniki 2006, 405—407.

[8] TractMathVindGr. See: ANONYMOUS, *Arithmetica (Codex Vindobonensis phil. Gr. 65 of the 15th century),* 17 (see footnote 6).

Aristotelian University of Thessaloniki are found, apart from other University Libraries, in the Library of Harvard University, too, labeled „Source“ for the History of Mathematics.[9]

The units of Tractatus Mathematicus Vindobonensis Graecus

UNIT 1. Chapters 1—39, 101, 102. Operations between real numbers.

UNIT 2. Chapters 40—56. Fractions, ratios, proportions.

UNIT 3. Chapters 57—60. Progressions.

UNIT 4. Chapters 61—94. Problems on first-degree equations.Solution through practical arithmetic.

UNIT 5. Chapters 95—100, 154, 155. Interest on loans or debts.

UNIT 6. Chapters 103—106. problems of sharing in proportional parts.

UNIT 7. Chapters 107—116. Problems on manufacture of gold and silver articles.

UNIT 8. Chapters 117—134, 239, 240. Roots of real numbers.

UNIT 9. Chapters 135—140. Solution of equations.

UNIT 10. Chapters 141—153, 156—165, 234. Systems of equations.

UNIT 11. Chapters 166—184. Plane geometry.

UNIT 12. Chapters 202—226. Areas of plane figures.

UNIT 13. Chapters 227—233, 235—238. Solid geometry.

The first unit contains the four arithmetical operations and their checks (or proofs according to the author of Ms). The symbols used for the numbers are the letters of the Greek alphabet,[10] but the calculations are done with the then new decimal Arabic numeration. The author appears not to have adjusted to the new method, and in order to represent zero he uses a symbol which resembles a calligraphic inverted h (ɥ).[11] Yet it must be stressed that using letters and not numbers did not influence the result since the numeric position was used, that is both the position of the letter and its value as a digit determined its contribution to the value of the number.[12] Thus the author insists on maintaining this old symbolism, while other

[9] Harvard University.

[10] Appendix

[11] Between the 12th and the 13th cent. AD the Byzantines completed the nine Greek alphabetic digits with the specific symbol, which denoted „ouden“. See C. B. BOYER— U. C. MERZBACH, A History of Mathematics (2nd ed.), trans. in Greek by B. ΚΟΥΣΟΥΛΑΚΟΥ, *Η Ιστορία των Μαθηματικών*, Athens 1997 (2nd ed.) 284.

[12] The Cambridge Medieval History 4: The Byzantine Empire 2, XXVIII: K. VOGEL, The Byzantine Science, trans. in Greek by N. ΣΑΟΥΛ, *Η Ιστορία της Βυζαντινής Αυτοκρατορίας* 2, Η Βυζαντινή Επιστήμη, Athens 1979, 803—833, 815.

earlier scholars, such as Maximos Planoudis (1255—1305) in Byzantium, and Fibonacci (born in 1170), who introduced the new symbolism into the West, had become familiar with the new symbolism and the numeric position.[13] Yet the new symbolism was not in general use in Byzantium; in fact, we know that it was not used by distinguished scholars such as Georgios Pachymeris (a contemporary of Planoudis), Moschopoulos, Nicholaos Ravdas, Ioannis Pediasimos, Varlaam of Seminara, Isaac Argyros (14th c. AD).[14] Most probably the author of Codex 65 did not adopt the new digits because their use caused various problems in commercial Mathematics.[15]

In the 5th chapter the author mentions the term „milliouni“, which, as it follows from the definition, means a „million“. Certainly we know that M. Planoudis was one of the first who used the term „milleton“ for a million. Yet, according to D.E. Smith, who as it seems did not know the existence of Codex 65, this term first appeared in 1478 in the Italian Ms Arithmetic of Treviso.[16]

At this point we present a table with the correspondence of Arabic and Greek numbers, as well as examples of writing numbers in the Arabic and the Greek system, so that the mathematical operations which follow will be better understood.

[13] In his work „Liber abacci“ Fibonacci uses the new digits, and Planoudis does so in his work „Ψηφιφορία κατ' Ινδούς η λεγομένη μεγάλη“ (Arithmetic after the Indian method). See H. HUNGER, Die Hochsprachliche profane Literatur der Byzantiner, trans. in Greek by T. ΚΟΛΙΑΣ, *Βυζαντινή Λογοτεχνία* 3, Athens 1994, 42, 49.

[14] K. VOGEL, Calculus and Indian Digits in Byzantium, *Neusis 5* (1996), 80.

[15] In AD 1299 the municipality of Florence issued a decree according to which writing numbers in columns and using the Indian digits were prohibited because 0 could easily be altered and become 6 or 9, a risk which was avoided with Roman digits. Also a directive issued in Antwerp in 1594 warned the merchants against using Indian digits in contracts and bills of exchange. See B. L. VAN DER WAERDEN, Science Awakening I, trans. in Greek by I. ΧΡΙΣΤΙΑΝΙΔΗΣ, *Η Αφύπνηση της Επιστήμης*, Heraklion 2000, 58.

[16] D. E. SMITH, *History of Mathematics* 1, New York 1958, 81.

Table of correspondence between Greek and Arabic numbers

G	A	G	A	G	A	G	A
α	1	ι	10	ρ	100	͵α	1000
β	2	κ	20	σ	200	͵β	2000
γ	3	λ	30	τ	300	͵γ	3000
δ	4	μ	40	υ	400	ɥ **ή u**	0
ε	5	ν	50	φ	500		
ς στίγμα	6	ξ	60	χ	600		
ζ	7	ο	70	ψ	700		
η	8	π	80	ω	800		
θ	9	ϟ κόππα	90	ϡ σαμπί	900		

Examples of writing numbers

ρκγ denotes number 123. So does αβγ.

ϡ ϟε denotes number 995. So does θθε.

͵ετξ denotes number 5360. So does εγςɥ.

We observe that the numbers in the examples are symbolized by the author in the Greek system of writing in two ways. The second way can be used because here we have the numeric position, where each letter takes its value according to its position in the number.

Some elements from Mathematical commentary on chapters 1—5. (α—ε)

The author of our Ms uses the decimal numeric position, which the Byzantines received from the Persians not directly but through the Latins 100 years before the Tractatus Mathematicus Vindobonensis Graecus was written, as the author himself mentions in chapter two. The mediation of the Latins was, as expected, due to the commercial transactions between the two peoples.

In the Ptolemy's Treatise, zero is called „ouden“ (nothing) and is symbolized with 0. In our manuscript symbols α, β, γ, δ, ε, ς, ζ, η, θ are used for numbers one to nine and zero is symbolized with a letter which resembles an inverted h (ɥ). At this point we mention that in the transcription of the Codex Vindobonensis phil. Gr. 65, the mathematical commentary and the broader study digit 0 has been symbolized with the letter u.

One thousand is symbolized by the author as ͵α, two thousand as ͵β, etc.

10,000 is called „myriades“, and 100 myriades (that is our million) a „milliouni“. Also, today’s billions are called „legeones“.

Below we mention some operations as they are described by the author, so that the reader will understand the use of the numbers and that of the numeric position as they appear in the codex 65 of the 15th century.

Execution of the operations

As a first example we consider the multiplication of

η (eight) by βα (21) (chapter 16).

η

βα

αςη

We observe that the operations do not differ from those we would execute today, that is we multiply α (one) by η (eight), and exactly below α and η we write the result, which is again η. Next we would multiply β (two) by η (eight) and we would write ας (16), in which case the final result of the operation will be αςη (168).

In the above example we can observe that the numbers η, α and η are placed on the right-hand column (3^{rd}) and denote the units. Numbers β and ς are placed on the second column and denote the tens, and the number α on the first column and denotes one hundred.

The important element here is that the result αςη denotes the number 168 according to the operations. This number could also be symbolized as ρξη since ρ denotes one hundred, ξ the number 60, and η the number eight. That is we see that when ς (six) is in the second place from the right, it denotes six tens, and when α is in the third place from the right, it denotes one hundred. Therefore each letter takes an arithmetical value according to its position in the number.

As a second example we mention the multiplication of αγ (13) by αε (15). This operation appears in the manuscript as follows:

αγ

αε

αθε

The author multiplies γ (three) by ε (five) and has ιε (15). He writes ε and carries α. As he says, he multiplies αγ by αε „crosswise“, that is he multiplies α by ε, which gives a result ε, and γ by α, which gives a result γ, and adds the results ε and γ, in which case he has a result η. He adds α, which he has carried from the first multiplication, to η, and he places θ, which is their sum, in the place of tens. Finally, he multiplies α of αγ by α of αε and writes their product, which is again α, in the place of hundreds. Thus the result of the multiplication is the number αθε (195).

These operations by the anonymous author are justified by applying the distributive law of multiplication over addition and as follows:

αγ can be written as α.10+γ, and αε as α.10+ε. Therefore the product of αγ by αε is:

αγ.αε=(α.10+γ). (α.10+ε)=
α.α.100+α.ε.10+γ.α.10+γ.ε=
α.α.100+α.ε.10+γ.α.10+α.10+ε=
α.α.100+(α.ε+γ.α+α).10+ε=
100+(5+3+1).10+5=
100+90+5=195

<u>As a third example</u> we mention the multiplication of αε (15) by αβγ (123).

ϥαε α
αβγ α
αηδε

The author multiplies γ by ε and he has αε. He writes ε and carries α. Next he multiplies α by γ, β by ε, and adds α to them. The result is αδ. He writes δ and carries α. He multiplies „ouden“ by γ, α by ε, β by α, and adds α to what he finds. The result is η. He multiplies „ouden“ by β, and α by α. The result is α. The product of the multiplication is αηδε.

According to the distributive law of multiplication over addition we have:

15.123= (10+5)(100+2.10+3)=
1000+2.100+3.10+5.100+10.10+3.5=
1000+8.100+4.10+5

<u>As a fourth example</u> we mention the multiplication of αβε (125) by βεζ (257).

αβε γ
βεζ δ
γβαβε γ
α

He multiplies ε by ζ, which gives γε. He writes ε and carries γ. He multiplies β by ζ, ε by ε, and adds γ. The result is δβ. He writes β and carries δ. He multiplies α by ζ, β by ε, again β by ε, and adds δ. The result is γα. He writes α and carries γ. He multiplies α by ε, β by β, and adds γ. The result is αβ. He writes β and carries α. He multiplies α by β, and adds α. The result is γ. The final product is γβαβε.

On the check of the operation of multiplication ch. 14, 15 (ιδ, ιε)[17]

The Greek author of the 15th century suggests for the proof of multiplication: „Ἄφελε τὰ 15 ὁσάκις χωρῶσι ἐπὶ τῶν 7· δὶς οὖν 7 γίνονται 14, περιττεύει 1 μέχρι τῶν 15.... ." According to the author this means that he asks for the remainder of the division of 15 by seven, which is the unit. Because the remainder of the division of six by seven is six, he multiplies the unit by six and places the product, which equals six, in a circle. Finally he finds the remainder of the division of 90 by seven, which is six, and compares it with the number he has placed in the circle, that is also 6. If the two results are the same, then the multiplication is trusted to be correct.

His method is probably based on the following:

If we symbolize the multiplicand with Π and the multiplier with P, then $\Pi.P \equiv \Pi.P (\text{mod} 7)$, and if

$\Pi = 7\kappa + \upsilon$, and $P = 7\lambda + \nu$ with κ, λ, υ, ν natural numbers and $\upsilon, \nu < 7$,

then $\Pi.P \equiv (7\kappa+\upsilon).(7\lambda+\nu) \ (\text{mod} 7)$,

that is $\Pi.P \equiv \upsilon.\nu + (7\kappa\lambda + \kappa\nu + \lambda\upsilon).7 (\text{mod} 7)$,

in which case $\Pi.P \equiv \upsilon.\nu (\text{mod} 7)$

This is the mathematical justification of the method with the remainders of the division by seven, which is used by the author of the 15th century.

On the check of the operation of addition ch. (101 ρα)[18]

In another problem the Byzantine author deals with the addition of final sums which result from various „logariasmous daneistikous" (loan accounts). He adds 1695+1393+3454+4565 and finds 11107.

Nowadays we do not refer to a check of additions with more than two addends. However, in Codex Vindobonensis phil. gr. 65 (fols 11r—126r) there is a check with the same method which the author applies to the operation of multiplication, which is as follows:

He divides each addend by seven, in which case from the first division he has the unit as remainder, from the second one a remainder zero, from the third one a remainder three, and from the fourth one as remainder the unit.

[17] ANONYMOUS, *Arithmetica (Codex Vindobonensis phil. Gr. 65 of the 15th century),* 83 (see footnote 6).

[18] ANONYMOUS, *Arithmetica (Codex Vindobonensis phil. Gr. 65 of the 15th century),* 212—215 (see footnote 6).

Next he divides the sum 11107 by 7 and he finds a remainder five. He adds the remainders of the divisions of the four addends by seven, and again he finds 5 (1+0+3+1). Then he writes that the addition is correct.

This procedure can be justified as follows:

We symbolize the above addends with A_1, A_2, A_3, A_4, the respective remainders of their divisions by seven with $\upsilon_1 = 1$, $\upsilon_2 = 0$, $\upsilon_3 = 3$, $\upsilon_4 = 1$, and the respective quotients of the same divisions with κ, λ, μ, ν .

Then we will have:

$A_1+A_2+A_3+A_4 \equiv A_1+A_2+A_3+A_4 (\mathrm{mod} 7)$,

In which case $7\kappa+\upsilon_1+7\lambda+\upsilon_2+7\mu+\upsilon_3+7\nu+\upsilon_4 \equiv A_1+A_2+A_3+A_4 (\mathrm{mod} 7)$,

that is $\upsilon_1+\upsilon_2+\upsilon_3+\upsilon_4+7(\kappa+\lambda+\mu+\nu) \equiv A_1+A_2+A_3+A_4 (\mathrm{mod} 7)$,

or $\upsilon_1+\upsilon_2+\upsilon_3+\upsilon_4 \equiv A_1+A_2+A_3+A_4 (\mathrm{mod} 7)$[19]

Thus we observe that apparently the problems are dealt with by using Arithmetic, while in essence the author has a good knowledge of Number Theory, elements of which he uses in his teaching.

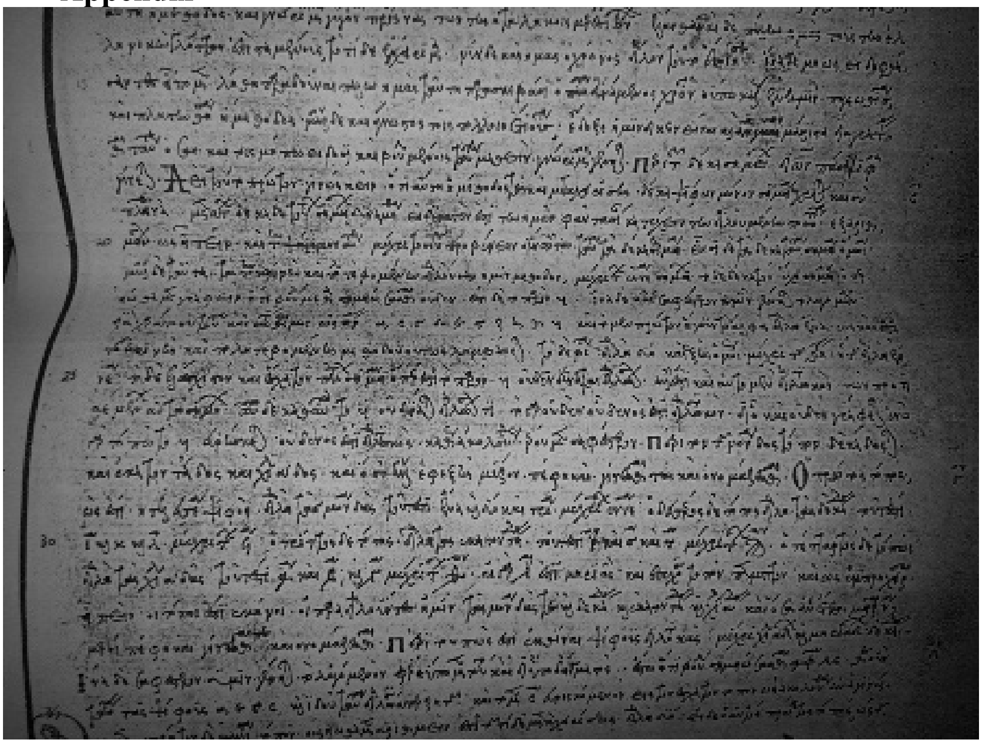
Conclusions

In this article data were presented about the origin and the content of Codex Vindobonensis phil. Gr. 65 of the 15th century. The mathematical branch on Number Theory and numeric position in Byzantium was extensively presented, especially as regards its theoretical part, on which the anonymous author seems to base the practical methods of the four operations and their tests. It was ascertained that the Byzantine author had a sound education and a scientific knowledge of the numeric position, which was used in Byzantium at the end of the 15th century. Finally we cited parts of the transcribed text, so that through an adequate number of examples the reader will be able to follow the thinking governing the methodological approach to the chapters to be taught.

Keywords: Codex Vindobonensis phil. Graecus 65, Number Theory, Greek Mss, Numerical Positioning System, Byzantine Mathematics, Indian digits, digits in Byzantium, Greek numbers, Zero.

[19] M. CHALKOU, Historical Mathematical Methods according to Tractatus Mathematicus Vindobonensis Graecus, *EUCLIDES* III, 69 (2008), 102—122. See also: ANONYMOUS, *Arithmetica (Codex Vindobonensis phil. Gr. 65 of the 15th century)*, 82, 83, 212, 213 (see footnote 6).

Appendix



Bibliography

ANONYMOUS, *Arithmetica (Codex Vindobonensis phil. Gr. 65 of the 15th century)*, ed. MARIA CHALKOU, Thessaloniki 2006.

C. B. BOYER— U. C. MERZBACH, A History of Mathematics (2nd ed.), trans. in Greek by BIBIAN ΚΟΥΣΟΥΛΑΚΟΥ, *Η Ιστορία των Μαθηματικών*, Athens 1997 (2nd ed.).

MARIA CHALKOU, Historical Mathematical Methods according to Tractatus Mathematicus Vindobonensis Graecus, *EUCLIDES* III, 69 (2008), 102-122.

MARIA CHALKOU, The Greek Mathematical Encyclopaedia of the Byzantines, *Byzantina* 27 (2007), 91-117.

H. HUNGER, *Katalog der griechischen Handschiften der Osterreichishen National Bibliothek*, Teil 1 Codices Historici, Philosophici et Philologici, Wien.

H. HUNGER, Die Hochsprachliche profane Literatur der Byzantiner, trans. in Greek by T. ΚΟΛΙΑΣ, *Βυζαντινή Λογοτεχνία* 3, Athens 1994.

E. MIONI, Introduzione alla paleografia greca, trans. in Greek by N. M. ΠΑΝΑΓΙΩΤΑΚΗΣ, *Εισαγωγή στην Ελληνική Παλαιογραφία,* Athens 1994.

D. E. SMITH, *History of Mathematics* 1, New York 1958.

E. STAMATIS, *Κριτική βυζαντινού βιβλίου αριθμητικής* (Criticism of a byzantine book of arithmetic)*,* Athens 1965.

B. L. VAN DER WAERDEN, Science Awakening I, trans. in Greek by I. ΧΡΙΣΤΙΑΝΙΔΗΣ, *Η Αφύπνηση της Επιστήμης*, Heraklion 2000.

K. VOGEL, Calculus and Indian Digits in Byzantium, *Neusis 5* (1996), 80.

The Cambridge Medieval History 4: The Byzantine Empire 2, XXVIII: K. VOGEL, The Byzantine Science, trans. in Greek by N. ΣΑΟΥΛ, *Η Ιστορία της Βυζαντινής Αυτοκρατορίας* 2, Η Βυζαντινή Επιστήμη, Athens 1979.

◊ Maria Chalkou
Ph.D. on Mathematics, National and Kapodistrian University of Athens,
State High School Advisor
Academy of Institutions and Cultures
m.chalkou@academy.edu.gr
mchalkou@gmail.com