BULLETIN OF THE
HELLENIC MATHEMATICAL SOCIETY
Volume 59, 2016 (91-98)

Some notes on the periodic distributions

Nikola Reckoski, Vasko Reckovski, Vesna Manova-Erakovikj

ABSTRACT. In this article, we give some notes concerning the convolution of the periodic distributions and the analytic representation of their Fourier transforms.

1. Introduction

A distribution $T \in D'$ is called l −periodic distribution, where $l > 0$, if it satisfies

$$(s_l T)(t) = T(t - l) = T(t),$$

where s_l is the translation operator.

It is clear that if T has a period l, then T, also, has a period kl, where k is an arbitrary integer.

Since

$$suppT = suppT(t - kl) = suppT + kl,$$

we see that the periodic distributions have unbounded supports, from which it follows that they can not be convolved.

The subspace of all l periodic distributions in D' is denoted by D_l'.

We say that the function φ is a periodic test function if is infinitely differentiable and periodic. The space of all such functions is denoted by P_l. The convergence in the space P_l is defined as follows:

A sequence $(\psi_n)_{n\in N}$ converges in the space P_l to the function ψ, if the sequence $(\psi_n^{(k)})$ converges uniformly to the function $\psi^{(k)}$, for arbitrary k.

Any test function $\varphi \in D$ generates a unique test function $\psi \in P_l$ through the expansion

$$\psi(t) = \sum_{n=-\infty}^{\infty} \varphi\,(t - nl)$$

Actually, over any bounded t interval there are only a finite number of nonzero terms in this series since φ has bounded support. Thus, we may differentiate the series term by term, and get that

$$\psi^{(k)}(t) = \sum_{n=-\infty}^{\infty} \varphi^{(k)}(t - nl), \qquad k = 1,2,\ldots$$

It is important to note that, if (φ_n) converges to the function φ in D, then the correspondent sequence of the periodic functions (ψ_n) converges to the function ψ in P_l.

Another type of functions that we will make use of, are the so-called unitary functions.

A function $\alpha(t) \in D$ is said to be a unitary function with respect to l if it satisfies

$$\sum_{n=-\infty}^{\infty} \alpha(t - nl) = 1 \qquad (1)$$

for all $t \in R$.

The set of all functions which satisfy (1) is denoted by U_l. Here, we present a construction of a unitary function.

Let $\beta \in C^{\infty}$ such that $\beta(t) = 0$ if $t \leq 0$, and $\beta(t) = 1$ if $t \geq l$ and $0 \leq \beta(t) \leq 1$, for $t \in R$. Then

$$\alpha(t) = \beta(t)\big(1 - \beta(t - l)\big) = \begin{cases} 0, & for\ t \leq 0 \\ 1, & for\ t = l \\ 0, & for\ 2l \leq t \end{cases}$$

is such that $0 \leq \alpha(t) \leq 1$ and $\alpha \in U_l$.

From

$$\sum_{n=-\infty}^{\infty} \alpha(t - nl) = 1,$$

it is clear that

$$\sum_{n=-\infty}^{\infty} \alpha^{(k)}(t - nl) = 0,$$

for $t \in R$, $k \in N$, ([4], p.246).

Clearly, if $\psi \in P_l$ and $\alpha \in U_l$, then $\alpha\psi \in D$. Also, if (ψ_n) converges in P_l to ψ, then $\alpha\psi_n$ converges to $\alpha\psi$ in D.

The space of all linear functional, defined on the space P_l and which are continuous with respect to the convergence is denoted by $P_l^{'}$.

For instance, if $T \in D_l^{'}$, then, by $A_l T$, we denote the functional on P_l defined by the formula

$$A_l T\psi = < T, \alpha\psi > = T(\alpha\psi).$$

It is easy to see that the functional $S = A_l T$ is linear on P_l, furthermore, if the sequence (ψ_n) converges to the function ψ in P_l, then the sequence $(\alpha\psi_n)$ converges to the function $\alpha\psi$ in D, and since $T \in D'$ we have that

$$S\psi_n = T(\alpha\psi_n) \to T(\alpha\psi) = S\psi,$$

Hence $S \in P_l'$.

It is known that the definition of S is independent of the choice of α of U_l ([4],p.248).

Now, we consider A_l as a mapping from D_l' into P_l'. It is linear and if the sequence of distributions (T_n) converges to the distribution T, then the sequence $(A_l T_n)$ converges to $A_l T$ in P_l' sense. In [4], it is proved that A_l is a continuous isomorphism from D_l' into P_l'. Hence, for every $\in P_l'$, there exists a unique $T \in D_l'$ such that $S = A_l T$ and, because of this, we write $T \cdot \psi$ in place of $A_l T\psi$ i.e.

$$T \cdot \psi = < T, \alpha\psi >,$$

where α is of U_l and ψ belong to P_l.

In the theory of distributions, it is well known that, if $T \in D'$ has a compact support, then

$$\sum_{n=-\infty}^{\infty} T(t - nl) \in D'$$

is a periodic distribution with period l.

For example, if $T = \delta$, the Dirac (delta) distribution, then the distribution

$$\delta_l = \frac{1}{l}\sum_{n=-\infty}^{\infty} \delta(t - nl) \in D_l'$$

belongs to D' and, for $l = 2\pi$, we have that

$$\delta_{2\pi} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \delta(t - 2n\pi).$$

Now, let T_1 and T_2 be two distributions in D_l' and let the function ψ belongs to P_l. We define a functional

$$S\psi = < T_1(t) \otimes T_2(\tau), \alpha(t)\beta(\tau)\psi(t+\tau) >,$$

where α and β are in P_l and $T_1 \otimes T_2$ is the tensor product of T_1 and T_2. Since the function $\alpha(t)\beta(\tau)\psi(t+\tau)$ is a test function of two variables (t,τ), one can easily verify that $S \in P_l'$. Since A_l is an isomorphism, there exists a unique $T \in D_l'$ such that $A_l T = S$. This distribution T is denoted by $T = T_1 \Delta T_2$ and T is called the convolution of T_1 and T_2.

From the properties of the tensor product, it follows that the commutative and the associative laws, i.e.

$$T_1 \Delta T_2 = T_2 \Delta T_1,$$
$$T_1 \Delta (T_2 \Delta T_3) = (T_1 \Delta T_2) \Delta T_3$$

hold.

Also, the following properties

1. $T\Delta\delta_l^{(k)} = T^{(k)}$,
2. $T\Delta\delta_l = T$,
3. $(T_1 \Delta T_2)^{(k)} = T_1^{(k)} \Delta T_2 = T_1 \Delta T_2^{(k)}$,
4. $s_h(T_1 \Delta T_2) = s_h T_1 \Delta T_2 = T_1 \Delta s_h T_2$

hold as well.

In particular,

$$T\Delta[\psi] = [T\Delta\psi] = [T(\tau)\psi(t-\tau)],$$

is true for $\psi \in P_l$. The function $T(\tau)\psi(t-\tau) \in P_l$ ([5], p.253).

Our first note is motivated from the function $T(\tau)\psi(t-\tau)$.

2. Convolution of a distribution $T \in D_l'$ with a function $\psi \in P_l$.

We know that the convolution of a distribution T of D' with a function $\varphi \in D$ is defined with

$$(T * \varphi)(t) = < T(\tau), \varphi(t-\tau) >$$

and the same definition holds if T has compact support and $\varphi \in C^{\infty}$.

Analogously as above, we define

$$(A\psi)(t) = (T\Delta\psi)(t) = T(\tau) \cdot \psi(t-\tau).$$

The function $(A\psi)(t)$ is called a convolution of the distribution $A_l T \in P_l'$ with $\psi \in P_l$.

Let us now consider A as a transformation of P_l into P_l.

It is evident that A is linear. Furthermore, since

$$(A\psi)(t) = < T(\tau), \alpha(\tau)\psi(t-\tau) > = < \alpha(\tau)T(\tau), \psi(t-\tau) >$$
$$= ((\alpha T) * \psi)(t)$$

(αT is with compact support and $\psi \in C^{\infty}$) we conclude that $(A\psi) \in P_l$. This is in fact another proof for the properties of the function $A\psi$.

Since

$$s_h(A\psi)(t) = (A\psi)(t-h) > = < T(\tau), \alpha(\tau)\psi(t-\tau-h) > =$$
$$< T(\tau), \alpha(\tau)(s_h\psi)(t-\tau) > = T(s_h\psi)(t-\tau) = A(s_h\psi)(t)$$

we have that the mapping A is a translation invariant.

Now let the sequence $(\psi_\nu(t))$ converges to the function ψ in P_l sense. Then $((\alpha T) * \psi_\nu)$ converges to $A\psi$ and, hence, A is continuous with respect to the convergence in P_l.

The converse. We will prove that, if the mapping $A: P_l \to P_l$ has the above properties, then there exists a unique periodic distribution T of $D_l^{'}$ such that

$$(A\psi)(t) = (T\Delta\psi)(t).$$

Proof. The proof is similar as the proofs in $D^{'}$ and in D, ([6], pg.173). We define the mapping S in the following way

$$S = (A\psi)^{\vee}(0). \qquad (2)$$

It is clear that the mapping S is a linear functional which is continuous with respect to the convergence of the sequence in P_l. Namely, if the sequence (ψ_ν) converges to the function ψ in P_l as $\nu \to \infty$, then the sequence $(\psi_\nu)^{\vee}(0)$, also, converges to the function $(\psi)^{\vee}(0)$, hence $S \in P_l^{'}$. Also, $A\psi\psi(t) = s_{-t}(A\psi\psi)(0) = S[(s_{-t}\psi\psi)^{\vee}]$, and the isomorphism $A_l: D_l^{'} \to P_l^{'}$ guarantee that there exists a unique T of $D_l^{'}$ such that

$$S[(s_{-t}\psi\psi)^{\vee}] = A_l T (s_{-t}\psi\psi)^{\vee}.$$

Consequently, we obtain

$$A\psi\psi(t) =< T(\tau), \alpha(\tau)(s_{-t}\psi\psi)^{\vee}(\tau) >=$$
$$< T(\tau), \alpha(\tau)\psi\psi(t-\tau) >= T(\tau) \cdot \psi\psi(t-\tau) = (T\Delta\psi)(t).$$

For example,

$$pf\frac{1}{sint}\Delta[1] = 0.$$

3. Tempered functions for the periodic distributions

In this section, we will determine the tempered functions for the periodic distributions. It is known that for any tempered distribution $U \in S^{'}$, it exist a continuous function f such that $f(t) = O(|t|^{\alpha})$ and $U = [f]^{(m)}$, for some $m = 0,1,2, \ldots$.

The proof for the existence of such function is not easy, ([1] p. 171), but for T of $D_l^{'}$, it is easier to determine this function by using the Fourier series for T. Indeed, let $T \in D_l^{'}$, then

$$T = [b_0] + \sum_{\nu=-\infty}^{\infty} b_\nu [e^{i\omega\nu t}],$$

where $\omega=\frac{2\pi}{l}$ and $b_\nu = \frac{1}{l} T e^{-i\omega\nu t}$ are the Fourier coefficients for T, and, for $\nu \neq 0$, there exists $a > 0$ and an integer k such that $|b_\nu| \le a|\nu|^k$.

Let us consider the function

$$g(t) = \sum_{\nu=-\infty}^{\infty} b_\nu \frac{e^{i\omega\nu t}}{(i\omega\nu)^{k+2}}$$

(b_0 is not in the sum). In accordance with the estimate of b_ν, where $\nu \neq 0$, it follows that $g(t)$ is a continuous, periodic function and $g(t) = O(|t|^0)$. Since

$$T = [b_0] + \sum_{\nu=-\infty}^{\infty} b_\nu [e^{i\omega\nu t}],$$

we write

$$T = \left(\frac{b_0}{(k+2)!} t^{k+2}\right)^{(k+2)}.$$

Consequently, the desired function is

$$f(t) = \frac{b_0}{(k+2)!} t^{k+2} + g(t)$$

or, in the distributional sense, it is

$$T = [f]^{(k+2)}.$$

4. Analytic representation for the Fourier transform in D_l'

Here we consider the space $D_{2\pi}'$. Let $T \in D_{2\pi}'$. Then there exists a continuous function

$$g(t) = \sum_{\nu=-\infty}^{\infty} d_\nu e^{i\nu t}$$

such that $g(t) = O(|t|^0)$ and $T = [d_0] + [g(t)]^{(k+2)}$ for some integer k, and

$$T - [d_0] = [g(t)]^{(k+2)}$$

for $k > -2$.

The function $(-iz)^{k+2}\hat{F}(g,z)$, $z = x + iy$, where

$$\hat{F}(g,z) = \begin{cases} \int_0^\infty g(t)e^{itz}dt, & for\ y > 0 \\ -\int_{-\infty}^0 g(t)e^{itz}dt, & for\ y < 0 \end{cases}$$

is analytic representation for the Fourier transform $F(T - [d_0])$, ([1], p.128). Since

$$F(T - [d_0]) = F(T) - F([d_0]) = F(T) - 2\pi d_0 \delta,$$

we have that

$$F(T) = F(T - [d_0]) + 2\pi d_0 \delta$$

And, from here, we conclude that the analytic representation of the Fourier transform $F(T)$ is the function

$$(-iz)^{k+2}\hat{F}(g,z) - d_0 2\pi \frac{1}{2\pi iz} = (-iz)^{k+2}\hat{F}(g,z) - \frac{d_0}{iz}$$

for $Imz \neq 0$.

Example. Let $T = pf\frac{1}{sint}$. This distribution is $2\pi -$ periodic and its Fourier series is

$$T = 2\sum_{m=0}^{\infty}[\sin(2m+1)\,t]$$

and belongs to D'. The function

$$f(t) = \sum_{m=0}^{\infty}\frac{-\sin(2m+1)\,t}{(2m+1)^2}$$

is $2\pi -$ periodic and continuous function and, since

$$|f(t)| \leq \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2},$$

we have that $f(t) = O(|t|^0)$ and it is a regular distribution of S'.

$$[f]^{(2)} = \sum_{m=0}^{\infty}[\sin(2m+1)\,t] = T$$

Consequently, the analytic representation of the Fourier transform $F(T)$ is the function

$$H(z) = (-iz)^2\begin{cases}\int_0^{\infty} g(t)e^{itz}dt, & for\ y > 0\\ -\int_{-\infty}^{0} g(t)e^{itz}dt, & for\ y < 0\end{cases}$$

where $z = x + iy$, and hence
$\lim_{y\to 0+}\int_{-\infty}^{\infty}[H(x+iy) - H(x-iy)]\varphi(x)dx =< F(T),\varphi >$.

References

[1]. H. Bremerman, Raspredelenija, kompleksnije permenenije I preobrazovanija Furje, Izdatelstvo ''Mir'' Moskva 1968.

[2]. Beltrami, E.J. and M.R. Wohlers: Distributions and the boundary values of analytic functions. Academic Press, New York 1966.

[3]. R. Carmichael, D. Mitrovic: Distributions and analytic functions, New York, 1989.

[4]. L. Jantcher: Distributionen, Walter de Gruyter Berlin. New York 1971.

[5]. N. Reckoski: One proof for the analytic representation of distributions, Matematicki Bilten 28 (LIV) 2004 (19-30) Skopje.

[6]. W. Rudin: Functional Analysis, M_c Graw-Hill, Inc.

[7]. A.H. Zemanian: Distribution Theory and Transform Analysis, M_c Graw-Hill Company New York 1965.

◊ Nikola Reckoski,
Faculty of Tourism and Hospitality, University St. Kliment Ohridski, Bitola, Republic of Macedonia -retired

◊ Vasko Reckovski,
Faculty of Tourism and Hospitality, University St. Kliment Ohridski, Bitola, Republic of Macedonia, e-mail: vaskorecko@yahoo.com

◊ Vesna Manova Erkovikj,
Ss. Cyril and Methodius University, Faculty of Mathematics and Natural Sciences, Arhimedova bb, Gazi baba, 1000, Skopje, Republic of Macedonia, e-mail: vesname@pmf.ukim.mk