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RIESZ MEANS VIA HEAT KERNEL BOUNDS

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ABSTRACT. Let (M, ρ, μ) be a space of doubling volume growth and L a self-adjoint positive definite operator. We assume that the corresponding heat kernel satisfies certain upper Gaussian bounds and we prove L^p estimates for Bochner-Riesz means $\sigma_{\alpha,R}(L)$, based only on these bounds.

1. INTRODUCTION

Riesz means play a prominent role in Harmonic analysis, especially in the theory of Fourier multipliers. Over the last three decades these operators have been studied extensively on Lie groups, symmetric spaces, spheres, graphs, groups and other geometric contexts. See indicatively [3, 5, 6, 9, 10, 12, 17, 21, 26, 27, 30, 31, 33, 34, 35, 36] and the references therein, the book [37] may be used as a survey. Let us focus on the result of Alexopoulos and Lohoué [3] which we are going to generalize.

Let M be a Riemannian manifold of dimension $n \in \mathbb{N}$, with positive Ricci curvature and Δ be the Laplace-Beltrami operator. The operator Δ admits a spectral resolution $\Delta = \int_0^\infty \lambda dE_\lambda$ and the Riesz means are defined as spectral multipliers

$$\sigma_{\alpha,R}(\Delta) = \int_0^\infty \Big(1 - (\lambda/R)^2\Big)_+^\alpha dE_\lambda, \quad R > 0, \ \alpha > 0,$$

where $(x)_+ := \max(x, 0)$. In [3] Alexopoulos and Lohoué proved that
(i) If $\alpha > n/2$ then the operators $\sigma_{\alpha,R}(\Delta), \ R > 0$ are uniformly bounded on L^p, for all $p \in [1, \infty]$.
(ii) If $\alpha \leq n/2$ then $\sigma_{\alpha,R}(\Delta), \ R > 0$ are uniformly bounded on L^p, for all $p \in (1, \infty)$ satisfying

$$\alpha > n \left| \frac{1}{p} - \frac{1}{2} \right|.$$

The main vehicle on the proof of Alexopoulos and Lohoué is the finite propagation speed property of the kernel of the wave operator $\cos t\sqrt{\Delta}$ [38].

In the present article we will extend the result of Alexopoulos and Lohoué on the more general setting of a doubling volume space, associated with a self-adjoint operator, whose heat kernel satisfies certain upper Gaussian bounds (see (1.3) below). Moreover our approach is based only on those heat kernel bounds, since in our context the finite propagation speed property may not hold.

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The basic ingredients in our development are the following:

I. We consider a metric measure space (M,ρ,μ) with distance $\rho(\cdot,\cdot)$ and a positive measure μ satisfying the *doubling volume condition*:

$$|B(x,2r)| \leq c_*|B(x,r)|, \text{ for all } x\in M \text{ and } r>0, \tag{1.1}$$

where $B(x,r) := \{y : \rho(x,y) < r\}$, $|B(x,r)| := \mu(B(x,r))$ and $c_* > 1$ is a constant. We set $d := \log_2 c_*$ the *homogeneous dimension* and from (1.1) we have

$$|B(x,r)| \leq c_* \left(\frac{r}{t}\right)^d |B(x,t)|, \text{ for } x\in M,\ r\geq t>0. \tag{1.2}$$

II. We also assume that L is a positive definite operator such that the associated heat kernel, i.e. the kernel of the semigroup e^{-tL}, $t>0$, satisfies the *upper Gaussian bound*: There are constants $c,b>0$ and $m>1$ such that for all $x,y\in M$, $t>0$,

$$|p_t(x,y)| \leq c\exp\left(-b\frac{\rho(x,y)^{m/(m-1)}}{t^{1/(m-1)}}\right)|B(y,t^{1/m})|^{-1}. \tag{1.3}$$

We point out that assumption II is common in the literature. For the case of a manifold M with $\mathrm{Ric}(M)\geq 0$ and for $L=\Delta$, the estimate (1.3) holds true for $m=2$. Another realization is when L is a power of the Laplacian or more generally when L is uniformly elliptic operator with measurable coefficients on $\mathbb{R}^d$ [4]. Moreover, Dungey [16] in the context of spaces of uniform polynomial growth, proved that if H is a self-adjoint positive operator, whose heat kernel satisfies (1.3) with $m=2$, then for every $\kappa\in\mathbb{N}$ the heat kernel of $L=H^\kappa$, satisfies (1.3) for $m=2\kappa$. In the recent article [29] as well as in [24] one can find more examples for our frame work, where (1.3) holds for $m>2$. See also [1, 2, 11, 13, 14, 17, 19, 22, 23, 35].

We denote by dE_λ the spectral measure of L. In view of the spectral theorem, if $m:[0,\infty)\to\mathbb{R}$ is a multiplier i.e. a bounded Borel function, we can define the spectral multiplier

$$m(L) = \int_0^\infty m(\lambda)dE_\lambda,$$

which is a bounded operator on $L^2(M)$, with $\|m(L)\|_{2\to2}\leq\|m\|_\infty$.

Let $\alpha,R>0$. The Bochner-Riesz means, or simply *Riesz means*, $\sigma_{\alpha,R}(L)$ of order α are defined by the multiplier

$$\sigma_{\alpha,R}(\lambda) := \Big(1-(\lambda/R)^2\Big)_+^\alpha.$$

Our goal in this article is to prove the following:

Theorem 1. *Let (M,ρ,μ) be a doubling volume space and L a positive operator as above. For d as in (1.2) we have the followings:*

(i) *If $\alpha>d/2$ then the operators $\sigma_{\alpha,R}(L)$, $R>0$ are uniformly bounded on L^p, for all $p\in[1,\infty]$.*

(ii) *If $\alpha\leq d/2$ then the operators $\sigma_{\alpha,R}(L)$, $R>0$ are uniformly bounded on L^p, for all $p\in(1,\infty)$ satisfying*

$$\alpha > d\left|\frac{1}{p}-\frac{1}{2}\right|.$$

Our setting covers of course the Euclidean space associated with the Laplace operator, or one of its powers. It includes the cases of Riemannian manifolds with nonnegative Ricci curvature associated with the Laplace-Beltrami operator and Lie groups of polynomial volume growth with left invariant sub-Laplacians, studied in [3]. Also homogeneous groups with homogeneous Laplacians are contained in our setting, see for example [8] for nilpotent and [31] for stratified groups, or the book [20] for general informations about homogeneous groups. The interval associated with the Jacobi operator as well as Simplex are two more examples [15].

We have only the bound (1.3) at our disposal and thus our approach will be based entirely on that. Such a strategy has been introduced by Alexopoulos [1]; see especially Lemma 3.1.

In this paper, we first decompose Riesz means $\sigma_{\alpha,R}(\lambda)$ as a sum of multipliers $m_{j,R}(\lambda)$ with compact supports, in which we control the magnitude of $1-(\lambda/R)^2$ in terms of j. Then we express the kernels, $K_{j,R}(x,y)$, of the spectral multipliers $m_{j,R}(L)$ using the heat semigroup e^{-tL} and the heat kernel $p_t(x,y)$ (see section 2). The next decisive step is to obtain pointwise estimates between e^{-tL} and $p_t(x,y)$. Then, we are able to obtain sharp L^2-bounds for the kernels $K_{j,R}(x,y)$ (see section 3). The conclusion of this approach is based on interpolation and duality.

We would like to mention here that the critical exponent of Theorem 1 is $(d-1)/2$ for the Laplace-Beltrami operator on $\mathbb{R}^d$ and on compact manifolds of dimension d. But if we consider uniformly elliptic operators on $\mathbb{R}^d$, sub-Laplacians on Lie groups, or discrete Laplacians on graphs and discrete groups, we derive again the index $d/2$ (see [3, 21, 25, 36]).

A natural question here is to find the "right" assumptions for the setting, in order to obtain sharp results for Riesz-means and sharp spectral multipliers in general. This problem has been studied during the last decade, in some extended and very interesting papers such as [7, 17, 18, 35] and still attracts the interest of the researchers in the area of global analysis.

Firstly, Duong, Ouhabaz and Sikora in [17] assumed additionally the so-called "Plancherel type estimates" (see [17, §3]) and then they derived the L^1-boundedness of $\sigma_{\alpha,R}(L)$ for every $\alpha>(d-1)/2$.

In [35] Sikora, Yan and Yao, used the "Stein-Tomas restriction type conditions" (see [35, §2.2]), for obtaining sharp bounds for spectral multipliers and Riesz means especially.

In the, just published, article [7] of Chen, Ouhabaz, Sikora and Yao, the "Restriction type estimates" and "Dispersive type estimates" are introduced (see [7, §I,II]) and, together with the finite speed propagation property for the L, sharp spectral multipliers revisited.

The above conditions are true for the Euclidean space associated with the Laplacian, compact manifolds, asymptotically conic manifolds and other settings [7, §III], [17, §7], [35, §6] (it is although still unclear what conditions could work for manifolds with positive Ricci curvature) and lead to sharp spectral multipliers.

The purpose of this article is *not* to offer some new sufficient conditions for obtaining sharp spectral multipliers. Here we aim to give a proof using heat kernel localization and only. Consequently our indices, will not be as sharp as in [17, 35].

Notation: We will denote for briefness

$$P[t]f(x) := e^{-tL}f(x) = \int_M p_t(x,y)f(y)dy,\ f\in\mathcal{C}_0^\infty(M),\ x\in M.$$

$$B_q(y) := B(y, 2^{q(m-1)/m}),\ A_q(y) := B_{q+1}(y)\setminus B_q(y),\ q\in\mathbb{R},\ y\in M.$$

Throughout this article, c will denote a constant which does not necessarily have always the same value. The dependence of the constant c on a parameter q will be stated as c_q. The sets of non-negative and positive integers will be denoted by $\mathbb{N}$, $\mathbb{N}^*$ respectively.

2. Preliminaries

2.1. An Approximation Lemma.

Lemma 1. *(see [1, 32]) For $f \in \mathcal{C}_0^k(\mathbb{R})$ and $k \in \mathbb{N}$, we set*

$$\|f\|_{\mathcal{C}^k} = \|f\|_\infty + \|f'\|_\infty + \cdots + \|f^{(k)}\|_\infty.$$

Then for every $\lambda > 0$, there is a continuous and integrable function ψ_λ and a constant $c > 0$, independent of λ and f, such that

$$\|\widehat{\psi}_\lambda\|_\infty \leq c, \;\; \operatorname{supp}(\widehat{\psi}_\lambda) \subset [-\lambda, \lambda] \;\; \text{and} \;\; \|f - f * \psi_\lambda\|_\infty \leq c\|f\|_{\mathcal{C}^k}\lambda^{-k}.$$

2.2. **Decomposition of Riesz means.** Let α, $R > 0$. The Riesz means of order α, $\sigma_{\alpha,R}(L)$, is defined by the multiplier

$$\sigma_{\alpha,R}(\lambda) = \left(1 - \frac{\lambda^2}{R^2}\right)_+^\alpha.$$

We begin by adapting [3]. Let the $\mathcal{C}^\infty$ function

$$\psi(\lambda) := \begin{cases} e^{-(1/\lambda^2)}, & \lambda > 0 \\ 0, & \lambda \leq 0 \end{cases}. \tag{2.1}$$

We then set the $\mathcal{C}^\infty$ function $\phi(\lambda) := \psi(\lambda+5/4)\psi(-\lambda-1/4)$, which is supported on $[-5/4, -1/4]$ and we put for all $j \in \mathbb{N}$,

$$\phi_{j,R}(\lambda) := \phi\left(2^{j(m-1)}\left(\frac{\lambda^2}{R^2} - 1\right)\right).$$

Let now,

$$\chi_{j,R}(\lambda) := \frac{\phi_{j,R}(\lambda)}{\sum_{\ell \geq 0} \phi_{\ell,R}(\lambda)}, \quad \text{for every } \; j \in \mathbb{N}.$$

Then $\sum_{j \geq 0} \chi_{j,R}(\lambda) = 1$ and

$$\frac{1}{4}2^{-j(m-1)} \leq 1 - (\lambda/R)^2 \leq \frac{5}{4}2^{-j(m-1)},$$

for every $\lambda \in \operatorname{supp}(\chi_{j,R})$.

We divide the Riesz means $\sigma_{\alpha,R}(\lambda)$ into the compactly supported terms

$$m_{j,R}(\lambda) := \chi_{j,R}(\lambda)\sigma_{\alpha,R}(\lambda), \; j \in \mathbb{N}.$$

From the above procedure, we get the decomposition for the multiplier

$$\sigma_{\alpha,R}(\lambda) = \sum_{j \geq 0} m_{j,R}(\lambda) \tag{2.2}$$

and therefore

$$\Lambda_{\alpha,R}(x, y) = \sum_{j \geq 0} K_{j,R}(x, y), \tag{2.3}$$

where $\Lambda_{\alpha,R}(x, y)$ and $K_{j,R}(x, y)$ are the Schwartz kernels of $\sigma_{\alpha,R}(L)$ and $m_{j,R}(L)$ respectively.

Our main aim is to decompose $K_{j,R}$ in terms of the heat kernel and heat operator. This leads us to consider the functions

$$g_{j,R}(\lambda) := m_{j,R}(-R\log\lambda)\lambda^{-1},\ j\in\mathbb{N}.$$

Then we derive that for every $j\in\mathbb{N}$

$$m_{j,R}(L) = g_{j,R}(P[1/R])P[1/R] \tag{2.4}$$

which finally offers the desired expression

$$K_{j,R}(x,y) = g_{j,R}(P[1/R])p_{1/R}(x,y). \tag{2.5}$$

2.3. **Estimation of $\|g_{j,R}\|_{\mathcal{C}^k}$, $k\in\mathbb{N}$.** We need to estimate $\|g_{j,R}\|_\infty$ and $\|g_{j,R}\|_{\mathcal{C}^k}$, for $k\in\mathbb{N}^*$. By the construction of $\chi_{j,R}(\lambda)$, we obtain inductively

$$\|\chi_{j,R}^{(\nu)}\|_\infty \le c_\nu R^{-\nu}2^{j(m-1)\nu},\ \text{for every } \nu\in\mathbb{N}. \tag{2.6}$$

By Leibniz rule and induction we get

$$\sigma_{\alpha,R}^{(2\nu)}(\lambda) = \sum_{\mu=0}^{\nu} cR^{-2\nu}\left(\frac{\lambda}{R}\right)^{2\mu}\left(1-\left(\frac{\lambda}{R}\right)^2\right)_+^{\alpha-\nu-\mu}$$

and

$$\sigma_{\alpha,R}^{(2\nu+1)}(\lambda) = \sum_{\mu=0}^{\nu} cR^{-2\nu-1}\left(\frac{\lambda}{R}\right)^{2\mu+1}\left(1-\left(\frac{\lambda}{R}\right)^2\right)_+^{\alpha-\nu-\mu-1},$$

where the constants $c\in\mathbb{R}$ above, depend on α, μ and ν.

For every $\lambda\in\operatorname{supp}(m_{j,R})$ it is easy to see that $|\lambda|\le R$ and $(1-(\lambda/R)^2)\sim 2^{-j(m-1)}$, so for every $\nu\in\mathbb{N}$

$$|\sigma_{\alpha,R}^{(\nu)}(\lambda)| \le c_\nu R^{-\nu}2^{-j(m-1)(\alpha-\nu)}. \tag{2.7}$$

Using Leibniz rule, (2.6) and (2.7) we derive for every $\nu\in\mathbb{N}$

$$\|m_{j,R}^{(\nu)}\|_\infty \le c_\nu R^{-\nu}2^{-j(m-1)(\alpha-\nu)}. \tag{2.8}$$

Let $\lambda\in\operatorname{supp}(g_{j,R})$, then we observe that $\lambda^{-1}<e$, so

$$||g_{j,R}||_\infty \le e\|m_{j,R}\|_\infty \le c2^{-\alpha(m-1)j}. \tag{2.9}$$

Now, some simple calculus show that for every $\nu\in\mathbb{N}^*$

$$\begin{aligned}\left|\left(\frac{d}{d\lambda}\right)^\nu\left[m_{j,R}(-R\log\lambda)\right]\right| &\le \sum_{\mu=1}^{\nu} c_\nu\frac{R^\mu}{\lambda^\nu}\left|m_{j,R}^{(\mu)}(-R\log\lambda)\right|\\ &\le c_\nu 2^{-j(m-1)(\alpha-\nu)},\end{aligned}$$

where for the last estimation we used (2.8).

By Leibniz rule again we conclude that

$$\|g_{j,R}\|_{\mathcal{C}^k} \le c_k 2^{-(\alpha-k)(m-1)j}\ \text{for every } k\in\mathbb{N}. \tag{2.10}$$

3. Preparation for the proof of Theorem 1

We consider $r \in \mathbb{R}$ such that $R = 2^{(m-1)r}$. A fundamental step in our approach is the following pointwise estimates between the heat operator $P[t]$ and the heat kernel p_t, inspired by [1]:

Lemma 2. *There exist constants $c > 0$ and $\delta \in (0,1)$, such that for all $q \geq -r$, $y \in M$, $x \in A_q(y)$ and $|t| \leq \delta 2^{(q+r)(m-1)/m}$*

$$\left| e^{itP[1/R]} p_{1/R}(x,y) \right| \leq \frac{ce^{-c2^{(q+r)(m-1)/m}}}{|B_{-r}(y)|}. \tag{3.1}$$

Proof. By the fact

$$e^{itP[1/R]} = \sum_{n \geq 0} \frac{(it)^n}{n!} \big(P[1/R]\big)^n = \sum_{n \geq 0} \frac{(it)^n}{n!} P[n/R],$$

we obtain

$$\begin{aligned} \left| e^{itP[1/R]} p_{1/R}(x,y) \right| &\leq \sum_{n \geq 0} \frac{|t|^n}{n!} |P[n/R] p_{1/R}(x,y)| \\ &= \sum_{n \in N_1} \frac{|t|^n}{n!} |p_{(n+1)/R}(x,y)| \\ &+ \sum_{n \in N_2} \frac{|t|^n}{n!} |p_{(n+1)/R}(x,y)| \\ &=: \Sigma_1 + \Sigma_2, \end{aligned} \tag{3.2}$$

where we denoted

$$N_1 := \{n \in \mathbb{N} : n \leq 2^{(q+r)(m-1)/m}\} \ \text{ and } \ N_2 := \{n \in \mathbb{N} : n > 2^{(q+r)(m-1)/m}\}.$$

We proceed now to estimate the sums Σ_1 and Σ_2.

Estimation of Σ_1. Using (1.3) we have for every $x \in A_q(y)$, $n \in N_1$,

$$\begin{aligned} |p_{(n+1)/R}(x,y)| &\leq c \exp\Big(\frac{-c\rho(x,y)^{m/(m-1)}}{((n+1)/R)^{1/(m-1)}} \Big) |B(y, ((n+1)/R)^{1/m})|^{-1} \\ &\leq c \frac{e^{-c2^{(q+r)(m-1)/m}}}{|B_{-r}(y)|}, \end{aligned} \tag{3.3}$$

where for the last inequality we used the observation

$$B_{-r}(y) \subseteq B(y, ((n+1)/R)^{1/m}), \ \text{while} \ n \in \mathbb{N}.$$

Now, there are constants $c > 0$, $\delta \in (0,1)$, such that for $|t| \leq \delta 2^{(q+r)(m-1)/m}$,

$$\begin{aligned} \Sigma_1 &\leq c|B_{-r}(y)|^{-1} e^{-c2^{(q+r)(m-1)/m}} e^{|t|} \\ &\leq c|B_{-r}(y)|^{-1} e^{-c2^{(q+r)(m-1)/m}} e^{\delta 2^{(q+r)(m-1)/m}} \\ &\leq c|B_{-r}(y)|^{-1} e^{-c2^{(q+r)(m-1)/m}}. \end{aligned} \tag{3.4}$$

Estimation of Σ_2. It is $\frac{1}{n!} \leq c\big(\frac{e}{n}\big)^n$. Applying (1.3) we get as above

$$\begin{aligned} \Sigma_2 &\leq c|B_{-r}(y)|^{-1} \sum_{n\in N_2} \frac{|t|^n}{n!} \\ &\leq c|B_{-r}(y)|^{-1} \sum_{n\in N_2} \Big(\frac{e\delta 2^{(q+r)(m-1)/m}}{2^{(q+r)(m-1)/m}}\Big)^n \\ &\leq c|B_{-r}(y)|^{-1} \sum_{n\in N_2} e^{-cn} \\ &\leq c|B_{-r}(y)|^{-1} e^{-c2^{(q+r)(m-1)/m}}, \qquad (3.5) \end{aligned}$$

for δ small enough. Then (3.1) follows by (3.2), (3.4) and (3.5).

□

We proceed now with L^2-estimates for the kernels $K_{j,R}$ of the multipliers $m_{j,R}$.

Lemma 3. *There is a constant $c > 0$ such that for all $q \geq -r,\ k \in \mathbb{N},\ y \in M$*

$$\text{(i)} \quad \|K_{j,R}(.,y)\|_2 \leq \frac{c2^{-\alpha(m-1)j}}{\sqrt{|B_{-r}(y)|}},$$

$$\text{(ii)} \quad \|K_{j,R}(.,y)\|_{L^2(A_q(y))} \leq \frac{c2^{-(\alpha-k)(m-1)j}2^{-k(q+r)(m-1)/m}}{\sqrt{|B_{-r}(y)|}}.$$

Proof. (i) It follows from (1.3) and (1.2) that

$$\|p_{1/R}(.,y)\|_2 \leq \frac{c}{\sqrt{|B_{-r}(y)|}}. \qquad (3.6)$$

In addition

$$\|g_{j,R}(P[1/R])\|_{2\to 2} \leq \|g_{j,R}\|_\infty,$$

and hence from (2.5) and (2.9) we arrive at

$$\|K_{j,R}(.,y)\|_2 \leq \|g_{j,R}\|_\infty \|p_{1/R}(.,y)\|_2 \leq \frac{c2^{-\alpha(m-1)j}}{\sqrt{|B_{-r}(y)|}}.$$

(ii) By Lemma 1, there exists a function $\psi_{r,q}$ satisfying

$$\|\widehat{\psi}_{r,q}\|_\infty \leq c, \qquad (3.7)$$

and

$$\|g_{j,R} - g_{j,R} * \psi_{r,q}\|_\infty \leq c\|g_{j,R}\|_{\mathcal{C}^k} 2^{-k(q+r)(m-1)/m}, \qquad (3.8)$$

with $\operatorname{supp}(\widehat{\psi}_{r,q}) \subseteq [-\delta 2^{(q+r)(m-1)/m}, \delta 2^{(q+r)(m-1)/m}]$, for the δ of Lemma 2.

Combining (3.6), (3.8) and (2.9) we derive

$$\begin{aligned} &\|(g_{j,R} - g_{j,R} * \psi_{r,q})(P[1/R])p_{1/R}(.,y)\|_{L^2(A_q(y))} \\ &\qquad \leq \|g_{j,R} - g_{j,R} * \psi_{r,q}\|_\infty \|p_{1/R}(.,y)\|_2 \\ &\qquad \leq c\|g_{j,R}\|_{\mathcal{C}^k} 2^{-k(q+r)(m-1)/m} |B_{-r}(y)|^{-1/2} \\ &\qquad \leq c2^{-(\alpha-k)(m-1)j} 2^{-k(q+r)(m-1)/m} |B_{-r}(y)|^{-1/2}. \qquad (3.9) \end{aligned}$$

For every $x \in A_q(y)$, by inverse Fourier transform, Lemma 2 implies that

$$\begin{aligned}
&|(g_{j,R} * \psi_{r,q})(P[1/R])p_{1/R}(x,y)| \\
&\leq c \int\limits_{|t|\leq \delta 2^{(q+r)(m-1)/m}} |\widehat{g_{j,R}}(t)||\widehat{\psi_{r,q}}(t)| \left| e^{itP[1/R]} p_{1/R}(x,y) \right| dt \\
&\leq c \|\widehat{g_{j,R}}\|_\infty 2^{(q+r)(m-1)/m} e^{-c2^{(q+r)(m-1)/m}} |B_{-r}(y)|^{-1} \\
&\leq c 2^{-(\alpha-k)(m-1)j} e^{-c2^{(q+r)(m-1)/m}} |B_{-r}(y)|^{-1}
\end{aligned}$$

where for the second inequality we used relations (3.7) and (3.8), and for the last inequality, relation (2.9). Hence, it follows from (1.2) that

$$\begin{aligned}
&\|(g_{j,R} * \psi_{r,q})(P[1/R])p_{1/R}(.,y)\|_{L^2(A_q(y))} \\
&\leq |A_q(y)|^{1/2} \|(g_{j,R} * \psi_{r,q})(P[1/R])p_{1/R}(.,y)\|_\infty \\
&\leq c 2^{-(\alpha-k)(m-1)j} e^{-c2^{(q+r)(m-1)/m}} |B_{-r}(y)|^{-1} |B_{q+1}(y)|^{1/2} \\
&\leq c 2^{-(\alpha-k)(m-1)j} e^{-c2^{(q+r)(m-1)/m}} 2^{d(q+r)(m-1)/2m} |B_{-r}(y)|^{-1/2} \\
&\leq c 2^{-(\alpha-k)(m-1)/j} 2^{-k(q+r)(m-1)/m} |B_{-r}(y)|^{-1/2}. \qquad (3.10)
\end{aligned}$$

The claim (ii) now is just a consequence of (3.9) and (3.10). □

4. Proof of Theorem 1

Let $j \in \mathbb{N}$. We denote briefly $B := B_{mj-r}(y)$ and we introduce the countable set $Q := \{mj - r + n,\ n \in \mathbb{N}\}$. Then $M = B \cup \bigcup_{q\in Q} A_q(y)$. From the claim (i) of Lemma 3 and (1.2) we have

$$\begin{aligned}
\|K_{j,R}(.,y)\|_{L^1(B)} &\leq |B|^{1/2} \|K_{j,R}(.,y)\|_2 \leq c2^{-\alpha(m-1)j} \left(\frac{|B|}{|B_{-r}(y)|} \right)^{1/2} \\
&\leq c 2^{-\alpha(m-1)j} 2^{((mj-r)+r)(m-1)d/2m} \\
&= c 2^{-(\alpha-(d/2))(m-1)j}. \qquad (4.1)
\end{aligned}$$

By (ii) of Lemma 3 and (1.2) we get for every $q \in Q,\ k \in \mathbb{N}$,

$$\begin{aligned}
\|K_{j,R}(.,y)\|_{L^1(A_q(y))} &\leq |A_q(y)|^{1/2} \|K_{j,R}(.,y)\|_{L^2(A_q(y))} \\
&\leq c 2^{-(\alpha-k)(m-1)j} 2^{-k(q+r)(m-1)/m} 2^{(q+r)d(m-1)/2m}. \qquad (4.2)
\end{aligned}$$

We choose now $k > d/2$. A summation over q implies that

$$\begin{aligned}
\sum_{q\in Q} \|K_{j,R}(.,y)\|_{L^1(A_q(y))} &\leq c 2^{-\alpha(m-1)j} 2^{k(m-1)j} 2^{-k(m-1)j} 2^{d(m-1)j/2} \\
&= c 2^{-(\alpha-(d/2))(m-1)j}. \qquad (4.3)
\end{aligned}$$

Combining the relations (4.1) and (4.3) we arrive at

$$\|K_{j,R}(.,y)\|_1 \leq c 2^{-(\alpha-(d/2))(m-1)j}. \qquad (4.4)$$

Proof of claim (i). For $\alpha > d/2$, summing (4.4) over j, we get according to (2.3),

$$\|\Lambda_{\alpha,R}(.,y)\|_1 \leq \sum_{j\geq 0} \|K_{j,R}(.,y)\|_1 \leq c.$$

Then $\sigma_{\alpha,R}(L)$ is bounded on L^∞. By interpolation and duality we have that $\sigma_{\alpha,R}(L)$ is bounded on L^p for all $p \in [1,\infty]$.

Proof of claim (ii). We will use interpolation.

For $\alpha \leq d/2$, relation (4.4) gives

$$\|m_{j,R}(L)\|_{1\to 1} \leq \sup_{y\in M} \|K_{j,R}(.,y)\|_1 \leq c2^{-(\alpha-(d/2))(m-1)j}. \tag{4.5}$$

On the other hand, estimation (2.9) implies that

$$\|m_{j,R}(L)\|_{2\to 2} \leq \|m_{j,R}\|_\infty \leq c2^{-\alpha(m-1)j}. \tag{4.6}$$

Let $1 < p < 2$ and we set $q := \frac{2}{p} - 1$. Then, by interpolation, we derive using (4.5) and (4.6)

$$\begin{aligned} \|m_{j,R}(L)\|_{p\to p} &\leq \Big(\|m_{j,R}(L)\|_{1\to 1}\Big)^q \Big(\|m_{j,R}(L)\|_{2\to 2}\Big)^{1-q} \\ &\leq c2^{-\left(\alpha - d\left(\frac{1}{p}-\frac{1}{2}\right)\right)(m-1)j}. \end{aligned} \tag{4.7}$$

From (2.2), (4.7) and for

$$\alpha > d\left|\frac{1}{p} - \frac{1}{2}\right|$$

we conclude that

$$\|\sigma_{\alpha,R}(L)\|_{p\to p} \leq \sum_{j\geq 0} \|m_{j,R}(L)\|_{p\to p} \leq c.$$

So $\sigma_{\alpha,R}(L)$ is bounded on L^p.

Duality guarantees the boundedness for $p > 2$. We note that the bounds are independent of R which finishes the proof of Theorem 1.

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