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A TWO-PARAMETER FINITE FIELD ERDŐS-FALCONER DISTANCE PROBLEM

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Abstract. We study the following two-parameter variant of the Erdős-Falconer distance problem. Given $E, F \subset \mathbb{F}_q^{k+l}$, $l \geq k \geq 2$, the $k+l$-dimensional vector space over the finite field with q elements, let $B_{k,l}(E,F)$ be given by

$$\{(\|x'-y'\|, \|x''-y''\|) : x=(x',x'') \in E, y=(y',y'') \in F; x',y' \in \mathbb{F}_q^k, x'',y'' \in \mathbb{F}_q^l\}.$$

We prove that if $|E||F| \geq Cq^{k+2l+1}$, then $B_{k,l}(E,F) \supseteq \mathbb{F}_q^* \times \mathbb{F}_q^*$. Furthermore this result is sharp if k is odd. For the case of $l=k=2$ and q a prime with $q \equiv 3 \mod 4$ we get that for every positive C there is c such that

$$\text{if } |E||F| > Cq^{6+\frac{2}{3}}, \text{ then } |B_{2,2}(E,F)| > cq^2.$$

1. Introduction

The Erdős-Falconer distance problem in $\mathbb{F}_q^d$ is to determine how large $E \subset \mathbb{F}_q^d$ needs to be to ensure that

$$\Delta(E) = \{||x-y|| : x,y \in E\},$$

with $||x|| = x_1^2 + x_2^2 + \cdots + x_d^2$, is the whole field $\mathbb{F}_q$, or at least a positive proportion thereof. Here and throughout, $\mathbb{F}_q$ denotes the field with q elements and $\mathbb{F}_q^d$ is the d-dimensional vector space over this field.

The distance problem in vector spaces over finite fields was introduced by Bourgain, Katz and Tao in [2]. In the form described above, it was introduced by the second listed author of this paper and Misha Rudnev ([5]), who proved that $\Delta(E) = \mathbb{F}_q$ if $|E| > 2q^{\frac{d+1}{2}}$. It was shown in [4] that this exponent is essentially sharp for general fields when d is odd. When $d=2$, it was proved in [3] that if if $E \subset \mathbb{F}_q^2$ with $|E| \geq cq^{\frac{4}{3}}$, then $|\Delta(E)| \geq C(c)q$. We do not know if improvements of the $\frac{d+1}{2}$ exponent are possible in even dimensions ≥ 4. We also do not know if improvements of the $\frac{d+1}{2}$ exponent are possible in any even dimension if we wish to conclude that $\Delta(E) = \mathbb{F}_q$, not just a positive proportion.

In this paper we introduce a two-parameter variant of the Erdős-Falconer distance problem. Given $E, F \subset \mathbb{F}_q^{k+l}$, $l \geq k \geq 2$, the $k+l$-dimensional vector space over the finite field with q elements, define $B_{k,l}(E,F)$ by

$$\{(\|x'-y'\|, \|x''-y''\|) : x=(x',x'') \in E, y=(y',y'') \in F; x',y' \in \mathbb{F}_q^k, x'',y'' \in \mathbb{F}_q^l\}.$$

This formulation introduces immediate interesting geometric complications. For example, let $k=l=2$, let

$$E = \{(x,0,0) : ||x|| = 1\} \text{ and } F = \{(0,0,y) : ||y|| = 1\}.$$

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Then $B_{2,2}(E,F)=\{(1,1)\}$. However, we are going to see that if $|E||F|$ is sufficiently large, then $B_{k,l}(E,F)=\mathbb{F}_q\times\mathbb{F}_q$. Our first result is the following.

Theorem 1.1. *Let $E,F\subseteq\mathbb{F}_q^{k+l}$, $l\geq k\geq 2$. There is a $C>0$ such that*

$$\textit{if } |E||F|>Cq^{k+2l+1} \textit{ then } B_{k,l}(E,F)\supseteq\mathbb{F}_q^*\times\mathbb{F}_q^*.$$

If k is odd, this result is best possible, up to the value of the constant C.

When k is even, we can hope to improve the exponent a bit. We are able to accomplish this in the case $k=l=2$. Our second result is the following.

Theorem 1.2. *Let q be a prime with $q\equiv 3 \mod 4$. For every positive C there is c such that for $E,F\subseteq\mathbb{F}_q^{2+2}$*

$$\textit{if } |E||F|>Cq^{6+\frac{2}{3}}, \textit{ then } |B_{2,2}(E,F)|>cq^2.$$

While this result probably is not sharp, we show the exponent cannot go below 6.

2. Proof of Theorem 1.1

We begin with a quick review of Fourier analytic preliminaries.

Let χ be a nontrivial additive character on $\mathbb{F}_q$. Given $f:\mathbb{F}_q^d\to\mathbb{C}$, define

$$\widehat{f}(m)=q^{-d}\sum_{x\in\mathbb{F}_q^d}\chi(-x\cdot m)f(x).$$

Observe that

$$f(x)=\sum_{m\in\mathbb{F}_q^d}\chi(x\cdot m)\widehat{f}(m),$$

$$\sum_{m\in\mathbb{F}_q^d}\left|\widehat{f}(m)\right|^2=q^{-d}\sum_{x\in\mathbb{F}_q^d}|f(x)|^2$$

and

$$\sum_{x\in\mathbb{F}_q^d}\chi(x\cdot m)=0 \text{ if } m\neq\vec{0} \text{ and } q^d \text{ otherwise.}$$

Lemma 2.1. *Let $S_t^{d-1}=\{x\in\mathbb{F}_q^d:\|x\|=t\}$, where $\|x\|=x_1^2+\cdots+x_d^2$. If $t\neq 0$ and $m\neq\vec{0}$, then*

$$|\widehat{S}_t^{d-1}(m)|\leq 2q^{-\frac{d+1}{2}}.$$

Lemma 2.2. *With the notation above,*

$$|S_t^{d-1}|=q^{d-1}+O(q^{d-2}).$$

For a proof of Lemma 2.1 and Lemma 2.2, see [5]. See also [8] and [6]. See [9] on a spectral graph theory viewpoint on similar phenomena.

We now move on to the proof of Theorem 1.1. Let $E(X), F(Y)$ denote the indicator functions of E, F, respectively, where $X = (x', x'')$ and $Y = (y', y'')$. For some $a, b \in \mathbb{F}_q^*$ we consider

$$\begin{aligned}
&\sum_{\|x'-y'\|=a;\|x''-y''\|=b} E(X)F(Y) \\
&= \sum_{X,Y} S_a^{k-1}(x'-y')S_b^{l-1}(x''-y'')E(X)F(Y) \\
&= \sum_{X,Y,m',m''} \widehat{S}_a^{k-1}(m')\widehat{S}_b^{l-1}(m'')\chi((x'-y')\cdot m')\chi((x''-y'')\cdot m'')E(X)F(Y) \\
&= \sum_{X,Y,m',m''} \widehat{S}_a^{k-1}(m')\widehat{S}_b^{l-1}(m'')\chi((X-Y)\cdot M)E(X)F(Y) \\
&= q^{2(k+l)} \sum_{M} \widehat{S}_a^{k-1}(m')\widehat{S}_b^{l-1}(m'')\overline{\widehat{E}(M)}\widehat{F}(M). \qquad (1)
\end{aligned}$$

We shall now break up the sum into three pieces. The first piece is the sum over $m' = m'' = \vec{0}$. The second piece is the sum over $m' \neq \vec{0}, m'' \neq \vec{0}$. The third sum is over $m' = \vec{0}, m'' \neq \vec{0}$ or $m'' = \vec{0}, m' \neq \vec{0}$.

2.1. The term $m' = \vec{0}, m'' = \vec{0}$.

Plugging this condition into (1) we obtain

$$|E||F||S_a^{k-1}||S_b^{l-1}|q^{-k-l}. \qquad (2)$$

2.2. The term $m' \neq \vec{0}, m'' \neq \vec{0}$.

Using Cauchy-Schwarz we see that

$$\left| \sum_{m' \neq \vec{0} \neq m''} \widehat{S}_a^{k-1}(m')\widehat{S}_b^{l-1}(m'')\overline{\widehat{E}(M)}\widehat{F}(M) \right|^2 \leq \sum_{m' \neq \vec{0} \neq m''} |\widehat{S}_a^{k-1}(m')\widehat{S}_b^{l-1}(m'')\widehat{F}(M)|^2 \sum_{n' \neq \vec{0} \neq n''} |\widehat{E}(N)|^2.$$

Now for the first sum we see by using Lemma 2.1 and Plancherel that it is bounded by

$$\left(2q^{-\frac{k+1}{2}}\right)^2 \left(2q^{-\frac{l+1}{2}}\right)^2 \sum_M |\widehat{F}(M)|^2 = 16q^{-(k+l+2)}q^{-k-l}|F|.$$

And again by Plancherel, we notice

$$\sum_{n' \neq \vec{0} \neq n''} |\widehat{E}(N)|^2 \leq q^{-k-l}|E|.$$

Therefore,

$$q^{2(k+l)} \left| \sum_{m' \neq \vec{0} \neq m''} \widehat{S}_a^{k-1}(m')\widehat{S}_b^{l-1}(m'')\overline{\widehat{E}(M)}\widehat{F}(M) \right| \leq 4q^{\frac{k+l}{2}-1}\sqrt{|E||F|}.$$

2.3. The term $m' \neq \vec{0}, m'' = \vec{0}$.

We obtain

$$q^{2(k+l)} \cdot q^{-l}|S_b^{l-1}| \sum_{m' \neq \vec{0}} \widehat{S}_a^{k-1}(m')\overline{\widehat{E}\left(m', \vec{0}\right)}\widehat{F}\left(m', \vec{0}\right). \qquad (3)$$

Very similarly to the previous case we see

$$\left|\sum_{m'\neq\vec{0}} \widehat{S}_a^{k-1}(m')\overline{\widehat{E}\left(m',\vec{0}\right)}\widehat{F}\left(m',\vec{0}\right)\right|^2 \leq \sum_{m'\neq\vec{0}}\left|\widehat{S}_a^{k-1}(m')\widehat{F}\left(m',\vec{0}\right)\right|^2\sum_{n'\neq\vec{0}}\left|\widehat{E}\left(n',\vec{0}\right)\right|^2$$

$$\leq 4q^{-k-1}\sum_{m'}\left|\widehat{F}\left(m',\vec{0}\right)\right|^2\sum_{n'}\left|\widehat{E}\left(n',\vec{0}\right)\right|^2$$

And furthermore we have the following

Lemma 2.3. *If $E\subset\mathbb{F}_q^{k+l}$ then*

$$\sum_{m'\in\mathbb{F}_q^k}\left|\widehat{E}\left(m',\vec{0}\right)\right|^2 \leq q^{-k-l}|E|.$$

Proof. This is a simple application of Plancherel. However, we write out the argument from scratch for the reader's convenience.

$$\sum_{m'\in\mathbb{F}_q^k}\left|\widehat{E}\left(m',\vec{0}\right)\right|^2 = \sum_{m'\in\mathbb{F}_q^k} q^{-2(k+l)}\sum_{\substack{x',y'\in\mathbb{F}_q^k\\ x'',y''\in\mathbb{F}_q^l}}\chi((x'-y')m')E(x',x'')E(y',y'')$$

$$= q^{-k-2l}\sum_{\substack{x'\in\mathbb{F}_q^k\\ x'',y''\in\mathbb{F}_q^l}} E(x',x'')\underbrace{E(x',y'')}_{\leq 1}$$

$$\leq q^{-k-l}|E|.$$ □

So now we can bound (3) by

$$q^{2(k+l)}\cdot q^{-l}|S_b^{l-1}|\cdot 2q^{\frac{-k-1}{2}}q^{-k-l}\sqrt{|E||F|} = 2q^{\frac{k-1}{2}}|S_b^{l-1}|\sqrt{|E||F|}.$$

Putting everything together we see that

$$\sum_{\|x'-y'\|=a;\|x''-y''\|=b} E(X)F(Y) = |E||F|\frac{|S_a^{k-1}|}{q^k}\frac{|S_b^{l-1}|}{q^l}+\mathcal{D}, \tag{4}$$

where

$$|\mathcal{D}| \leq 2q^{\frac{k-1}{2}}\sqrt{|E||F|}|S_b^{l-1}| + 2q^{\frac{l-1}{2}}\sqrt{|E||F|}|S_a^{k-1}| + 4q^{\frac{k+l}{2}-1}\sqrt{|E||F|}.$$

By a direct calculation (remembering that $l\geq k$) and using Lemma 2.2, the right hand side of (4) is positive if

$$|E||F| > (1+o(1))16q^{k+2l+1},$$

as desired.

Finally for the sharpness of this result in the case k odd, we need the following theorem from [4].

Theorem 2.4. *There exists $c>0$ and $E\subset\mathbb{F}_q^d$, d odd, such that*

$$|E|\geq cq^{\frac{d+1}{2}} \text{ and } \Delta(E)\neq\mathbb{F}_q.$$

Let $E_1\subset\mathbb{F}_q^k$ be a set as in theorem above and $E_2=\mathbb{F}_q^l$. With $E=E_1\times E_2$ we get $|E|\geq cq^{\frac{2l+k+1}{2}}$ and $B_{k,l}(E,E)=\Delta(E_1)\times\Delta(E_2)=\Delta(E_1)\times\mathbb{F}_q\neq\mathbb{F}_q\times\mathbb{F}_q$ since $\Delta(E_1)\neq\mathbb{F}_q$. Hence our result is sharp if k is odd.

3. Proof of Theorem 1.2

For $a, b \in \mathbb{F}_q$ let

$$s(a,b) := |\{(x', x'', y', y'') \in E \times F : \|x' - y'\| = a, \|x'' - y''\| = b\}|.$$

We observe that

$$\left(\sum_{a,b \in \mathbb{F}_q} s(a,b) \right)^2 = |E|^2 |F|^2,$$

while at the same time Cauchy-Schwarz yields

$$\left(\sum_{a,b \in \mathbb{F}_q} s(a,b) \right)^2 \leq |B_{2,2}(E,F)| \sum_{a,b \in \mathbb{F}_q} s(a,b)^2.$$

Hence,

$$\frac{|E|^2 |F|^2}{\sum_{a,b \in \mathbb{F}_q} s(a,b)^2} \leq |B_{2,2}(E,F)|, \tag{5}$$

so an upper bound on $\sum_{a,b \in \mathbb{F}_q} s(a,b)^2$ will provide a lower bound for $B_{2,2}(E,F)$.

Now

$$s(a,b)^2 = \Big| \{ (x', x'', y', y'', z', z'', w', w'') \in E \times F \times E \times F : \|x' - y'\| = a = \|z' - w'\|, \|x'' - y''\| = b = \|z'' - w''\| \} \Big|,$$

so

$$\sum_{a,b \in \mathbb{F}_q} s(a,b)^2 = \Big| \{ (x', x'', y', y'', z', z'', w', w'') \in E \times F \times E \times F : \|x' - y'\| = \|z' - w'\|, \|x'' - y''\| = \|z'' - w''\| \} \Big|. \tag{6}$$

We now proceed as in [1]. For $\theta, \varphi \in SO_2(\mathbb{F}_q)$ we define $r^E_{\theta,\varphi} : \mathbb{F}_q^2 \times \mathbb{F}_q^2 \to \mathbb{C}$ as

$$r^E_{\theta,\varphi}(u', u'') = \left| \{ (x', x'', z', z'') \in E \times E : x' - \theta z' = u', x'' - \varphi z'' = u'' \} \right|.$$

Therefore

$$\sum_{u', u'' \in \mathbb{F}_q^2} r^E_{\theta,\varphi}(u', u'') r^F_{\theta,\varphi}(u', u'') = \left| \{ (x', x'', z', z'', y', y'', w', w'') \in E^2 \times F^2 : x' - \theta z' = y' - \theta w', x'' - \varphi z'' = y'' - \varphi w'' \} \right|. \tag{7}$$

And we can also calculate the Fourier-transform

$$\begin{aligned}
\widehat{r^E_{\theta,\varphi}}(m',m'') &= q^{-4} \sum_{u',u''\in\mathbb{F}_q^2} r^E_{\theta,\varphi}(u',u'')\chi(-u'\cdot m' - u''\cdot m'') \\
&= q^{-4} \sum_{u',u''\in\mathbb{F}_q^2} \chi(-u'\cdot m' - u''\cdot m'') \sum_{\substack{x',x'',z',z''\in\mathbb{F}_q^2\\ x'-\theta z'=u'\\ x''-\varphi z''=u''}} E(x',x'')E(z',z'') \\
&= q^{-4} \sum_{x',x'',z',z''\in\mathbb{F}_q^2} \chi(-(x'-\theta z')\cdot m' - (x''-\varphi z'')\cdot m'')E(x',x'')E(z',z'') \\
&= q^4 \widehat{E}(m',m'')\overline{\widehat{E}(\theta^{-1}m',\varphi^{-1}m'')}.
\end{aligned}$$

Now our key observation is the following result from [1], contained in the proof of their Theorem 1.5.

Lemma 3.1. *Let q a prime power, $q\equiv 3 \mod 4$. Then for $x,y\in\mathbb{F}_q^2\setminus\{\vec{0}\}$ we have $\|x\|=\|y\|$ if and only if there is a unique $\theta\in SO_2(\mathbb{F}_q)$ such that $x=\theta y$.*

This observation allows us to make the following connection

$$\sum_{a,b\in\mathbb{F}_q} s(a,b)^2 \le \sum_{\substack{u',u''\in\mathbb{F}_q^2\\ \theta,\varphi\in SO_2(\mathbb{F}_q)}} r^E_{\theta,\varphi}(u',u'')r^F_{\theta,\varphi}(u',u'')$$

by comparing (6) and (7) and seeing that

$$\|x'-y'\| = \|z'-w'\| \implies \exists\theta\in SO_2(\mathbb{F}_q) : x'-\theta z' = y'-\theta w'.$$

Next, we observe

$$\begin{aligned}
\sum_{U\in\mathbb{F}_q^2\times\mathbb{F}_q^2} r^E_{\theta,\varphi}(U)r^F_{\theta,\varphi}(U) &= \sum_{U\in\mathbb{F}_q^2\times\mathbb{F}_q^2}\sum_{M\in\mathbb{F}_q^4}\chi(UM)\widehat{r^E_{\theta,\varphi}}(M)\sum_{N\in\mathbb{F}_q^4}\chi(UN)\widehat{r^F_{\theta,\varphi}}(N) \\
&= \sum_{M\in\mathbb{F}_q^4}\widehat{r^E_{\theta,\varphi}}(M)\sum_{N\in\mathbb{F}_q^4}\widehat{r^F_{\theta,\varphi}}(N)\sum_{U\in\mathbb{F}_q^2\times\mathbb{F}_q^2}\chi(U(N+M)) \\
&= q^4\sum_{M\in\mathbb{F}_q^4}\widehat{r^E_{\theta,\varphi}}(M)\overline{\widehat{r^F_{\theta,\varphi}}(M)}
\end{aligned}$$

and it remains to find a bound for the following quantity:

$$\begin{aligned}
\sum_{\theta,\varphi\in SO_2(\mathbb{F}_q)}\sum_{u',u''\in\mathbb{F}_q^2} r^E_{\theta,\varphi}(u',u'')r^F_{\theta,\varphi}(u',u'') &= q^4\sum_{\theta,\varphi\in SO_2(\mathbb{F}_q)}\sum_{m',m''\in\mathbb{F}_q^2}\widehat{r^E_{\theta,\varphi}}(m',m'')\overline{\widehat{r^F_{\theta,\varphi}}(m',m'')} \\
&= q^{12}\sum_{m',m''\in\mathbb{F}_q^2}\sum_{\theta,\varphi\in SO_2(\mathbb{F}_q)}\widehat{E}(m',m'')\overline{\widehat{E}(\theta m',\varphi m'')\widehat{F}(m',m'')}\widehat{F}(\theta m',\varphi m'') \qquad (8)
\end{aligned}$$

where we replaced θ^{-1} and φ^{-1} by θ and φ respectively in the last step.

Again we will need to split the sum into three terms

3.1. The term $m' = \vec{0}, m'' = \vec{0}$. Plugging into (8) we get

$$q^{12} \sum_{\theta,\varphi \in SO_2(\mathbb{F}_q)} |\widehat{E}(\vec{0},\vec{0})|^2 |\widehat{F}(\vec{0},\vec{0})|^2 = q^{-4}|E|^2|F|^2|SO_2(\mathbb{F}_q)|^2.$$

3.2. The term $m' \neq \vec{0}, m'' \neq \vec{0}$.

$$\begin{aligned} q^{12} &\sum_{m',m'' \in \mathbb{F}_q^2 \setminus \{\vec{0}\}} \widehat{E}(m',m'')\overline{\widehat{F}(m',m'')} \sum_{\theta,\varphi \in SO_2(\mathbb{F}_q)} \overline{\widehat{E}(\theta m', \varphi m'')} \widehat{F}(\theta m', \varphi m'') \\ &= q^{12} \sum_{a,b \in \mathbb{F}_q \setminus \{0\}} \sum_{\|m'\|=a, \|m''\|=b} \widehat{E}(m',m'')\overline{\widehat{F}(m',m'')} \sum_{\theta,\varphi \in SO_2(\mathbb{F}_q)} \overline{\widehat{E}(\theta m', \varphi m'')} \widehat{F}(\theta m', \varphi m'') \\ &= q^{12} \sum_{a,b \in \mathbb{F}_q \setminus \{0\}} \left| \sum_{\|m'\|=a, \|m''\|=b} \widehat{E}(m',m'')\overline{\widehat{F}(m',m'')} \right|^2 \end{aligned}$$

where we used Lemma 3.1 in the last step.

We continue with a trivial estimate on one of the inner factors

$$\begin{aligned} q^{12} &\sum_{a,b \in \mathbb{F}_q \setminus \{0\}} \left| \sum_{\|m'\|=a, \|m''\|=b} \widehat{E}(m',m'')\overline{\widehat{F}(m',m'')} \right|^2 \\ &\leq q^{12} \sum_{a,b \in \mathbb{F}_q \setminus \{0\}} \sum_{\|m'\|=a, \|m''\|=b} |\widehat{E}(m',m'')|^2 \sum_{\|n'\|=a, \|n''\|=b} |\widehat{F}(n',n'')|^2 \\ &\leq q^{12} \left(\sum_{a,b \in \mathbb{F}_q \setminus \{0\}} \sum_{\|m'\|=a, \|m''\|=b} |\widehat{E}(m',m'')|^2 \right) \sum_{n',n'' \in \mathbb{F}_q^2 \setminus \{\vec{0}\}} |\widehat{F}(n',n'')|^2 \\ &\leq q^{12} \left(\sum_{m',m'' \in \mathbb{F}_q^2} |\widehat{E}(m',m'')|^2 \right) \left(\sum_{n',n'' \in \mathbb{F}_q^2} |\widehat{F}(n',n'')|^2 \right) \\ &= q^{12} \left(q^{-4} \sum_{u',u'' \in \mathbb{F}_q^2} E(u',u'') \right) \left(q^{-4} \sum_{u',u'' \in \mathbb{F}_q^2} F(u',u'') \right) \\ &= q^4 |E||F|. \end{aligned}$$

3.3. The term $m' \neq \vec{0}, m'' = \vec{0}$. As in the two previous cases we see

$$q^{12} \sum_{m' \in \mathbb{F}_q^2 \setminus \{\vec{0}\}} \sum_{\theta,\varphi \in SO_2(\mathbb{F}_q)} \widehat{E}(m',\vec{0}) \overline{\widehat{E}(\theta m',\vec{0}) \widehat{F}(m',\vec{0})} \widehat{F}(\theta m',\vec{0})$$

$$= q^{12}|SO_2(\mathbb{F}_q)| \sum_{m' \in \mathbb{F}_q^2 \setminus \{\vec{0}\}} \widehat{E}(m',\vec{0})\overline{\widehat{F}(m',\vec{0})} \sum_{\theta \in SO_2(\mathbb{F}_q)} \overline{\widehat{E}(\theta m',\vec{0})} \widehat{F}(\theta m',\vec{0}). \tag{9}$$

We will deal with the inner sum first. Let $0 \neq a = \|m'\|$, then

$$\left|\sum_{\theta \in SO_2(\mathbb{F}_q)} \overline{\widehat{E}(\theta m', \vec{0})} \widehat{F}(\theta m', \vec{0})\right| = \left|\sum_{\|m\|=a} \overline{\widehat{E}(m, \vec{0})} \widehat{F}(m, \vec{0})\right|$$
$$\leq \sqrt{\sum_{\|m\|=a} |\widehat{E}(m, \vec{0})|^2 \sum_{\|n\|=a} |\widehat{F}(n, \vec{0})|^2}. \tag{10}$$

Lemma 3.2. *For $E \subset \mathbb{F}_q^2$, $0 \neq a \in \mathbb{F}_q$ we get*

$$\sum_{\|m\|=a} |\widehat{E}(m, \vec{0})|^2 \leq 3^{\frac{1}{2}} q^{-6} |E|^{\frac{3}{2}}.$$

Proof. With the notation introduced in Lemma 2.1 and $g : \mathbb{F}_q^2 \to \mathbb{C}$ where $g(m) = \overline{\widehat{E}(m, \vec{0}) S_a(m)}$ we can write this as

$$\sum_{m \in \mathbb{F}_q^2} \widehat{E}(m, \vec{0}) S_a(m) g(m) = \sum_{m \in \mathbb{F}_q^2} q^{-4} \sum_{x', x'' \in \mathbb{F}_q^2} \chi(-x' \cdot m) E(x', x'') S_a(m) g(m)$$
$$= q^{-2} \sum_{x' \in \mathbb{F}_q^2} \left(\sum_{x'' \in \mathbb{F}_q^2} E(x', x'') \right) \widehat{S_a g}(x').$$

Using Hölder's Inequality with $q = \frac{4}{3}, r = 4$ we can bound this by

$$\leq q^{-2} \left(\sum_{x' \in \mathbb{F}_q^2} \left(\sum_{x'' \in \mathbb{F}_q^2} E(x', x'') \right)^{\frac{4}{3}} \right)^{\frac{3}{4}} \left(\sum_{x' \in \mathbb{F}_q^2} |\widehat{S_a g}(x')|^4 \right)^{\frac{1}{4}}. \tag{11}$$

We will first find an estimate for the latter factor. By using the definition of the Fourier transform we get:

$$\sum_{x' \in \mathbb{F}_q^2} |\widehat{S_a g}(x')|^4 = q^{-6} \sum_{\substack{u, v, u', v' \in S_a \\ u + v = u' + v'}} g(u) g(v) \overline{g(u') g(v')}. \tag{12}$$

Here we use the Fefferman trick. For fixed $u, v \in S_a$, $u \neq -v$ we want to find $u', v' \in S_a$ such that $u + v = u' + v'$. In other words we want to find $u' \in S_a$ such that $(u + v - u') \in S_a$, so u' is in the intersection of the circles $\{x \in \mathbb{F}_q^2 : \|x\| = a\}$ and $\{x \in \mathbb{F}_q^2 : \|x - (u + v)\| = a\}$ which has at most two solutions as the circles are not identical (since $u + v \neq 0$). But we already know two solutions, namely u and v. So either $u' = u$ and $v' = v$ or $u' = v$ and $v' = u$. If $u = -v$ we get $u' \in S_a$ and $v' = -u'$. Therefore (and by noting that $g(-u) = \overline{g(u)}$) we can bound (12) by

$$\begin{aligned} & q^{-6}\left(\sum_{u,v\in S_a} 2g(u)g(v)\overline{g(u)g(v)} + \sum_{u,u'\in S_a} g(u)g(-u)\overline{g(u')g(-u')}\right) \\ \leq & 3q^{-6}\sum_{u,v\in S_a}|g(u)|^2|g(v)|^2 \\ = & 3q^{-6}\left(\sum_{u\in S_a}|g(u)|^2\right)^2 \\ = & 3q^{-6}\left(\sum_{\|u\|=a}|\widehat{E}(u,\vec{0})|^2\right)^2. \end{aligned}$$

The other factor of (11) can be dealt with as follows

$$\left(\sum_{x'\in\mathbb{F}_q^2}\left(\sum_{x''\in\mathbb{F}_q^2}E(x',x'')\right)^{\frac{4}{3}}\right)^{\frac{3}{4}} = \left(\sum_{x'\in\mathbb{F}_q^2}\left(\sum_{x''\in\mathbb{F}_q^2}E(x',x'')\right)\left(\sum_{x''\in\mathbb{F}_q^2}E(x',x'')\right)^{\frac{1}{3}}\right)^{\frac{3}{4}} \leq q^{\frac{1}{2}}|E|^{\frac{3}{4}}.$$

Therefore we have

$$\sum_{\|m\|=a}|\widehat{E}(m,\vec{0})|^2 \leq 3^{\frac{1}{4}}q^{-2}q^{\frac{1}{2}}|E|^{\frac{3}{4}}q^{-\frac{3}{2}}\left(\sum_{\|m\|=a}|\widehat{E}(m,\vec{0})|^2\right)^{\frac{1}{2}}$$

so

$$\sum_{\|m\|=a}|\widehat{E}(m,\vec{0})|^2 \leq 3^{\frac{1}{2}}q^{-4}q|E|^{\frac{3}{2}}q^{-3} = 3^{\frac{1}{2}}q^{-6}|E|^{\frac{3}{2}}.$$ □

Continuing from (9) and using (10) and Lemma 3.2 we see

$$\begin{aligned} & q^{12}|SO_2(\mathbb{F}_q)|\left|\sum_{m'\in\mathbb{F}_q^2\setminus\{\vec{0}\}}\widehat{E}(m',\vec{0})\overline{\widehat{F}(m',\vec{0})}\right|\sum_{\theta\in SO_2(\mathbb{F}_q)}\overline{\widehat{E}(\theta m',\vec{0})}\widehat{F}(\theta m',\vec{0})\right|\right| \\ \leq & q^{12}|SO_2(\mathbb{F}_q)|\left|\sum_{m'\in\mathbb{F}_q^2\setminus\{\vec{0}\}}\widehat{E}(m',\vec{0})\overline{\widehat{F}(m',\vec{0})}\cdot 3^{\frac{1}{2}}q^{-6}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}}\right|. \end{aligned}$$

Next we need to deal with

$$\begin{aligned} \sum_{m'\in\mathbb{F}_q^2\setminus\{\vec{0}\}}\left|\widehat{E}(m',\vec{0})\overline{\widehat{F}(m',\vec{0})}\right| &\leq \sqrt{\sum_{m'\in\mathbb{F}_q^2\setminus\{\vec{0}\}}\left|\widehat{E}(m',\vec{0})\right|^2\sum_{m'\in\mathbb{F}_q^2\setminus\{\vec{0}\}}\left|\widehat{F}(m',\vec{0})\right|^2} \\ &\leq \sqrt{q^{-8}|E||F|}. \end{aligned}$$

And finally we need the following, which can be obtained from Lemma 6.24. in [7].

Lemma 3.3. *For a prime power q with $q \equiv 3 \mod 4$ we have*

$$|SO_2(\mathbb{F}_q)| = q + 1.$$

Putting those results together we find that (9) is bounded by

$$Cq^{12}qq^{-4}\sqrt{|E||F|}q^{-6}|E|^{\frac{3}{4}}|F|^{\frac{3}{4}} = Cq^3|E|^{\frac{5}{4}}|F|^{\frac{5}{4}}.$$

So we can bound the whole sum (8) by

$$q^4|E||F| + Cq^3\big(|E||F|\big)^{\frac{5}{4}} + q^{-4}|E|^2|F|^2|SO_2(\mathbb{F}_q)|^2.$$

Therefore we get from (5)

$$\min\left\{\frac{|E||F|}{3q^4}, \frac{\big(|E||F|\big)^{\frac{3}{4}}}{3Cq^3}, \frac{q^4}{3|SO_2(\mathbb{F}_q)|^2}\right\} \leq |B_{2,2}(E,F)|.$$

Hence it is enough that (with some unspecified constants c)

$$cq^2 \leq \frac{\big(|E||F|\big)^{\frac{3}{4}}}{3Cq^3} \iff cq^5 \leq \big(|E||F|\big)^{\frac{3}{4}} \iff cq^{\frac{20}{3}} \leq |E||F|$$

since in this case also

$$\frac{|E||F|}{3q^4} \geq \frac{cq^{\frac{20}{3}}}{q^4} \geq cq^2.$$

Remark 3.4 (Sharpness of results). Let p a prime, with $p \equiv 3 \mod 4$. Consider $E = \mathbb{F}_p^2 \times L$, where

$$L = \{(a, 0) : a \in \mathbb{F}_p, 0 \leq a \leq p^{1-\varepsilon}\}.$$

Then $|E| \approx p^{3-\varepsilon}$ and $|\Delta(L)| \approx 2p^{1-\varepsilon}$, so $|B_{2,2}(E,E)| = o(p^2)$. Hence the $6 + \frac{2}{3}$ exponent in Theorem 1.2 is potentially not best possible, but we definitely cannot go below 6.

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