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JACOBIANS OF QUATERNION POLYNOMIALS

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Abstract. A map f from the quaternion skew field $\mathbb{H}$ to itself, can also be thought as a transformation $f : \mathbb{R}^4 \to \mathbb{R}^4$. In this manuscript, the Jacobian $J(f)$ of f is computed, in the case where f is a quaternion polynomial. As a consequence, the Cauchy-Riemann equations for f are derived. It is also shown that the Jacobian determinant of f is non negative over $\mathbb{H}$. The above commensurates well with the theory of analytic functions of one complex variable.

1. Introduction

Let $\mathbb{H}$ denote the skew field of quaternions. Its elements are of the form $c = c_0+\mathbf{i}c_1+\mathbf{j}c_2+\mathbf{k}c_3$, where $c_m \in \mathbb{R}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are such that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$. The real part of c is $Re(c) = c_0$ while the imaginary part $\mathrm{Im}(c) = \mathbf{i}c_1 + \mathbf{j}c_2 + \mathbf{k}c_3$. The norm of c, $|c| = \sqrt{c_0^2 + c_1^2 + c_2^2 + c_3^2}$, its conjugate $c^* = c_0 - \mathbf{i}c_1 - \mathbf{j}c_2 - \mathbf{k}c_3$ while its inverse is $c^{-1} = c^* \cdot |c|^{-2}$, provided that $|c| \neq 0$. c is called an *imaginary* unit if $\mathrm{Re}(c) = 0$ and $|c| = 1$, and it has the property $c^2 = -1$. In that regard, $\mathbb{H}$ is a real normed division (non commutative) algebra. An element $c = c_0 + \mathbf{i}c_1 + \mathbf{j}c_2 + \mathbf{k}c_3$ of $\mathbb{H}$ can also be represented via a real 4×4 matrix $\mathcal{C}$ as follows:

$$\mathcal{C} = \begin{bmatrix} c_0 & -c_1 & -c_2 & -c_3 \\ c_1 & c_0 & -c_3 & c_2 \\ c_2 & c_3 & c_0 & -c_1 \\ c_3 & -c_2 & c_1 & c_0 \end{bmatrix}.$$

Notice that $|\mathcal{C}| = |c|^4$. The following notation will be frequently used in the sequel:

Definition 1.1. *For any 4×4 real matrix B and a quaternion c, we define: $cB \equiv \mathcal{C}B$ and $Bc \equiv B\mathcal{C}$.*

We may identify $\mathbb{H}$ with $\mathbb{R}^4$ via the map $(x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w) \to (x, y, z, w)$. Let $f : \mathbb{H} \to \mathbb{H}$. In view of this identification, we can also think of f as a map from $\mathbb{R}^4 \to \mathbb{R}^4$. Indeed, if $f(x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w) = f_1(x, y, z, w) + \mathbf{i}f_2(x, y, z, w) + \mathbf{j}f_3(x, y, z, w) + \mathbf{k}f_4(x, y, z, w)$ we define $f : \mathbb{R}^4 \to \mathbb{R}^4$ by $f(x, y, z, w) = (f_1, f_2, f_3, f_4)$. In that case, we have

Definition 1.2. *The Jacobian $J(f)(c)$, $c \in \mathbb{H}$ is the matrix $\left[\frac{\partial f_i}{\partial x_j}\right]$, $i = 1, 2, 3, 4$, $x_1 = x, x_2 = y, x_3 = z, x_4 = w$ evaluated at c. The determinant of $J(f)$ will be denoted by $|J(f)|$.*

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In this manuscript, we are concerned with the computation of $J(f)$ and its determinant at a point $c \in \mathbb{H}$, in the case where f is a polynomial with coefficients in $\mathbb{H}$. More specifically, we will consider polynomials of the form

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0, \text{ where } a_k \in \mathbb{H}. \tag{1}$$

We will first show that $|J(t^n)| \geq 0$ at all points $c \in \mathbb{H}$. By extending the above, we will find the form of $J(f)$ at a complex number $r + \mathbf{i}s$ (the complex number i is to be identified with the quaternion $\mathbf{i}$ throughout this note). Using this we deduce the form of $J(f)(c),\ c \in \mathbb{H}$. From the above, the Cauchy-Riemann equations for f are derived. Finally, we prove that $|J(f)| \geq 0$ over $\mathbb{H}$, a fact similar to the one in the theory of analytic functions of a complex variable.

2. A brief overview of quaternion polynomials

In this paragraph we will recall some facts, needed for the rest of the paper, concerning quaternion polynomials. For more details, the reader is referred to [2, 3, 4].

Due to the non commutative nature of $\mathbb{H}$, polynomials over $\mathbb{H}$ are usually distinguished into the following three types: *left*, *right* and *general*, [3]. A left polynomial is an expression of the form (1). If $a_n \neq 0$, n is called the degree of f. Here we shall consider, unless otherwise stated, left polynomials only and call them simply polynomials. If $g(t) = b_m t^m + b_{m-1} t^{m-1} + \cdots + b_0$ is another polynomial, their product $fg(t)$ is defined in the usual way:

$$fg(t) = \sum_{k=0}^{m+n} c_k t^k, \quad \text{where } c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Note that in the above setting the multiplication is performed as if the coefficients were chosen in a commutative field.

An equivalent representation of f is $f(t) = a(t) + \mathbf{i}b(t) + \mathbf{j}c(t) + \mathbf{k}d(t) = a + \mathbf{i}b + \mathbf{j}(c - \mathbf{i}d)$, where $a, b, c, d \in \mathbb{R}[t]$. If $c \in \mathbb{H}$, we define $f(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0$; if $f(c) = 0$, c is called a *zero* or a *root* of f. According to Theorem 1 of [3] an element $c \in \mathbb{H}$ is a zero of f if and only if there exists a (left) polynomial $g(t)$ such that $f(t) = g(t)(t - c)$. In their nominal paper [1], Eilenberg and Niven, using a degree argument, proved the fundamental theorem of algebra for quaternions, namely that any quaternion polynomial of positive degree n of the form $f(t) = a_0 t a_1 t \cdots t a_n + \phi(t), a_i \in \mathbb{H}, a_i \neq 0$ and $\phi(t)$ is a sum of finite number of similar monomials $b_0 t b_1 t \cdots t b_k, k < n$, has a root in $\mathbb{H}$.

Roots of f are distinguished into two types: (i) *isolated* and (ii) *spherical.* A root c of f is called spherical if and only if its characteristic polynomial $q_c(t) = t^2 - 2t\,\mathrm{Re}(c) + |c|^2$ divides f; for any such polynomial, call $\alpha_c \pm \mathbf{i}\beta_c$ its complex roots. In that case any quaternion γ similar to c, is also a root of f; ($c_1, c_2 \in \mathbb{H}$ are called *similar*, and denoted by $c_1 \sim c_2$, if $c_1\,\eta = \eta\,c_2$ for a non zero $\eta \in \mathbb{H}$). For example, if $f(t) = t^2 + 1$, any imaginary unit quaternion $c = \alpha_1\,\mathbf{i} + \alpha_2\,\mathbf{j} + \alpha_3\,\mathbf{k}$ is a root of f.

Remark 2.1. *The polynomial $f(t)$ has a spherical root if and only if it has roots $\alpha + \mathbf{i}\beta, \alpha - \mathbf{i}\beta,\ \alpha, \beta \in \mathbb{R}, \beta \neq 0$.*

The above facts, as well as the representation of f as in (1), allows us to factor f into a product of linear factors $(t - c_i),\ c_i \in \mathbb{H}$. Indeed, since $f(t) = g(t)(t - c)$ and $g(t)$ has a root, simple induction shows that

$$f(t) = a_n(t - c_n)(t - c_{n-1}) \cdots (t - c_1),\ c_j \in \mathbb{H}. \tag{2}$$

A word of caution: In the above factorization, while c_1 is necessarily a root of f, $c_j, j = 2, \cdots n$, might not be roots of f. For example, the polynomial $f(t) = (t+\mathbf{k})(t+\mathbf{j})(t+\mathbf{i}) = t^3 + (\mathbf{i}+\mathbf{j}+\mathbf{k})t^2 + (-\mathbf{i}+\mathbf{j}-\mathbf{k})t + 1$ has only one root, namely $t = -\mathbf{i}$. Theorem 2.1 of [2] provides a more detailed version of the above factorization.

If we write f in the form $f(t) = a(t) + \mathbf{i}b(t) + \mathbf{j}c(t) + \mathbf{k}d(t)$ we see that f has no spherical roots if and only if $\gcd(a, b, c, d) = 1$. Such an f will be called *primitive.* Then, it is known (Corollary 3.3, [4]) that a primitive $f(t)$ of degree n, has at most n distinct roots in $\mathbb{H}$.

The conjugate f^* of f is defined as $f^* = a(t) - \mathbf{i}b(t) - \mathbf{j}c(t) - \mathbf{k}d(t)$. Note that $f^*f = a^2 + b^2 + c^2 + d^2$, which is a real positive polynomial. Observe that if $\alpha + \mathbf{i}\beta$ is a root of f^*f, then there exists a $c \in H$, similar to $\alpha + \mathbf{i}\beta$ so that $f(c) = 0$.

Definition 2.1. *Let $\phi(t) \in \mathbb{C}[t]$ and $\zeta \in \mathbb{C}$ be a root of ϕ. We denote by $\mu(\phi)(\zeta)$ the multiplicity of ζ. Now let $c \in \mathbb{H}$ be a root of f and let $m = \mu(f^*f)(\alpha_c + \mathbf{i}\beta_c)$. Then, (1) if c is isolated, we define its multiplicity $\mu(f)(c)$, as a root of f, to be m; (2) if c is spherical, its multiplicity is set to be $2m$.*

Note that the above notion of multiplicity agrees with the one given in Definition 2.6 of [2], page 23. From the above we have:

Criterion 1. *Let c be a root of the* primitive *polynomial f. Then c has multplicity k if and only if* $\gcd(q_c^k, f^*f) = q_c^k$.

3. The Jacobian determinant of t^n

Eilenberg and Niven in [1], proved the fundamental theorem of algebra for quaternions, using a degree argument. The key ingredient was Lemma 2 whose proof was depended on the positiveness of $|J(t^n)|$ at the roots of the equation $t^n = \mathbf{i}$. In this section, we will show that $|J(t^n)|$ is non negative at any point $(x, y, z, w) \in \mathbb{R}^4$.

First, we need a lemma.

Lemma 3.1. *Let $u, v, p, q \in \mathbb{R}[x, y, z, w]$ be homogeneous polynomials of the same degree $m \geq 1$ so that $yp = zv$ and $yq = wv$. Let*

$$F : \mathbb{R}^4 \to \mathbb{R}^4, \qquad F(x, y, z, w) = (u, v, p, q).$$

Then, $|J(F)|$ *is equal to* $y^{-3}mv^2(vu_x - uv_x)$.

Proof: Since $yp = zv$ and $yq = wv$, we see that

$$p_x = \frac{z}{y}v_x,\ p_y = \frac{-z}{y^2}v + \frac{z}{y}v_y,\ p_z = \frac{v}{y} + \frac{z}{y}v_z,\ p_w = \frac{z}{y}v_w,$$

$$q_x = \frac{w}{y}v_x,\ q_y = \frac{-w}{y^2}v + \frac{w}{y}v_y,\ q_z = \frac{w}{y}v_z,\ q_w = \frac{v}{y} + \frac{w}{y}v_w.$$

Then, the Jacobian of F is

$$J(F) = \begin{bmatrix} u_x & u_y & u_z & u_w \\ v_x & v_y & v_z & v_w \\ \frac{z}{y}v_x & \frac{-z}{y^2}v + \frac{z}{y}v_y & \frac{v}{y} + \frac{z}{y}v_z & \frac{z}{y}v_w \\ \frac{w}{y}v_x & \frac{-w}{y^2}v + \frac{w}{y}v_y & \frac{w}{y}v_z & \frac{v}{y} + \frac{w}{y}v_w \end{bmatrix}. \tag{3}$$

Now using the fact that u, v are homogeneous of degree m, an easy calculation shows that $|J(F)| = y^{-3}mv^2(vu_x - uv_x)$. Note that $|J(F)| \in \mathbb{R}[x, y, z, w]$ since y is a factor of u and v. ■

For u, v, p, q as in Lemma 3.1, define new polynomials U, V, P, Q by the formula

$$U + \mathbf{i}V + \mathbf{j}P + \mathbf{k}Q = (x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w)(u + \mathbf{i}v + \mathbf{j}p + \mathbf{k}q) = (u + \mathbf{i}v + \mathbf{j}p + \mathbf{k}q)(x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w).$$

Notice that U, V, P, Q are homogeneous of degree $m + 1$, $yP = zV, yQ = wV$ and $U = xu - yv - zp - wq, V = yu + xv$. Using the fact that $yP = zV, yQ = wV$ we observe that U takes the form

$$U = xu - v\left(\frac{y^2 + z^2 + w^2}{y}\right).$$

Corollary 3.1. *Let* U, V, P, Q *be as above and define* $\Phi = (U, V, P, Q)$. *Then,*

$$|J(\Phi)| = \frac{(m+1)V^2[y(x^2 + y^2 + z^2 + w^2)(vu_x - uv_x) + y^2u^2 + v^2(y^2 + z^2 + w^2)]}{y^4}.$$

Proof: We have

$$U_x = u + xu_x - v_x\left(\frac{y^2 + z^2 + w^2}{y}\right), \quad V_x = yu_x + v + xv_x.$$

Now, a calculation shows that

$$VU_x - UV_x = \frac{y(x^2 + y^2 + z^2 + w^2)(vu_x - uv_x) + y^2u^2 + v^2(y^2 + z^2 + w^2)}{y}.$$

Thus, the result follows from Lemma 3.1. ■

Corollary 3.2. *Let* $f(t) = t^n = (x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w)^n = f_1 + \mathbf{i}f_2 + \mathbf{j}f_3 + \mathbf{k}f_4 = (f_1, f_2, f_3, f_4)$. *Then,* $|J(f)| \geq 0$.

Proof: For induction purposes, we will rename $f = t^n = f^{[n]} = (f^{[n]_r}),\ r = 1, 2, 3, 4$. Observe first that $f^{[n]_r}$ are homogeneous of degree n and satisfy the conditions $yf^{[n]_3} = zf^{[n]_2}, yf^{[n]_4} = wf^{[n]_2}$. If $n = 1$ we get $|J(t)| = 1$, while if $n = 2$ we see that $|J(t^2)| = 16x^2(x^2+y^2+z^2+w^2) \geq 0$. Let now $n \geq 2$. Then, Corollary 3.1 shows that $|J(t^{n+1})|$ is equal to

$$\frac{(n+1)(f^{[n]_2})^2[y|f^{[1]}|^2(f^{[n]_2}f_x^{[n]_1} - f^{[n]_1}f_x^{[n]_2}) + y^2u^2 + v^2(y^2+z^2+w^2)]}{y^4}.$$

From Lemma 3.1 we see that $y(x^2 + y^2 + z^2 + w^2)(f^{[n]_2}f_x^{[n]_1} - f^{[n]_1}f_x^{[n]_2}) \geq 0$ and thus by induction the result follows. ■

4. The Jacobian of $f(t)$

In this section we will compute the Jacobian of a quaternion polynomial $f(t)$. First, we compute the Jacobian of a complex polynomial $a(t) + \mathbf{i}b(t),\ a, b \in \mathbb{R}[t]$ at the complex number $t_0 = r + \mathbf{i}s$ (in the sequel, for a complex polynomial ϕ, we will denote its Jacobian over $\mathbb{C}$ by $J_{\mathbb{C}}(\phi)$ to differentiate it from the Jacobian of ϕ over $\mathbb{H}$). Using this, we will calculate $J(f)(t_0)$. Finally, utilizing the above we compute the Jacobian of f at a point $c_0 \in \mathbb{H}$.

We begin with the following:

Lemma 4.1. *For $n \in \mathbb{N}$, define $a_n = 2n - 1,\ b_n = 2n$ and $c_n = (-1)^{n+1}$. Then,*

$$J(t^{2n-1})(\mathbf{i}) = \begin{bmatrix} c_n a_n & 0 & 0 & 0 \\ 0 & c_n a_n & 0 & 0 \\ 0 & 0 & c_n & 0 \\ 0 & 0 & 0 & c_n \end{bmatrix} \quad \text{and} \quad J(t^{2n})(\mathbf{i}) = \begin{bmatrix} 0 & -c_n b_n & 0 & 0 \\ c_n b_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4}$$

Proof: We will use the notation of Corollary 3.2. From (3) we see that $f_{x_j}^{[n]_3}(\mathbf{i}) = 0$ for $x_j = x, y, w$ and $f_{x_j}^{[n]_4}(\mathbf{i}) = 0$ for $x_j = x, y, z$. Recall that

$$f^{[k+1]_1} = xf^{[k]} - f^{[k]_2}\left(\frac{y^2+z^2+w^2}{y}\right) \quad \text{and} \quad f^{[k+1]_2} = yf^{[k]_1} + xf^{[k]_2}. \tag{5}$$

By using induction and taking into account (5) we get

$$f^{[2n-1]_1}(\mathbf{i}) = 0,\ f^{[2n]_1}(\mathbf{i}) = (-1)^n,\ f^{[2n-1]_2}(\mathbf{i}) = (-1)^{n+1},\ f^{[2n]_2}(\mathbf{i}) = 0.$$

Now $f_x^{[2n+1]_1} = f^{[2n]_1} + xf_x^{[2n]_1} - f_x^{[2n]_2}\left(\frac{y^2+z^2+w^2}{y}\right)$. By induction $f_x^{[2n]_2}(\mathbf{i}) = (-1)^{n+1}(2n)$ and thus $f_x^{[2n+1]_1}(\mathbf{i}) = (-1)^n - (-1)^{n+1}(2n) = (-1)^{n+1+1}(2n+1)$ as required. Similarly, we get $f_y^{[2n-1]_2}(\mathbf{i}) = (-1)^{n+1}(2n-1),\ f_y^{[2n]_1}(\mathbf{i}) = -c_n b_n$ and $f_x^{[2n]_2}(\mathbf{i}) = c_n b_n$. Finally, it is easy to see that $f_{x_j}^{[k]_r}(\mathbf{i}) = 0$ when $r = 3, 4$ and $x_j = z, w$. This finishes the proof. ■

Note that if $s \in \mathbb{R}, s \neq 0,\ J(t^k)(\mathbf{i}s) = s^{k-1}J(t^k)(\mathbf{i})$.

Corollary 4.1. *Let $g(t) = a(t) + \mathbf{i}b(t),\ a(t), b(t) \in \mathbb{R}[t]$ be a complex polynomial and $r + \mathbf{i}s \in \mathbb{C} - \mathbb{R}$. Then*

$$J(g)(r + \mathbf{i}s) = \begin{bmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \gamma & -\delta \\ 0 & 0 & \delta & \gamma \end{bmatrix}, \quad \text{for } \alpha, \beta, \gamma, \delta \in \mathbb{R}. \tag{6}$$

Proof: Since $J(g)(r + \mathbf{i}s) = J(g(t + r)(\mathbf{i}s)$, we may assume that $r = 0$. We write

$$g(t) = a_n t^{2n-1} + a_{n-1} t^{2n-3} + \cdots + a_1 t + b_m t^{2m} + b_{m-1} t^{2m-2} + \cdots + b_1 t^2 + b_0, \tag{7}$$

$a_i, b_j \in \mathbb{C}$. Then,

$$\begin{aligned} J(g)(\mathbf{i}s) &= \textstyle\sum_{k=1}^n a_k\, J(t^{2k-1})(\mathbf{i}s) + \sum_{l=1}^m b_l\, J(t^{2l})(\mathbf{i}s) \\ &= \textstyle\sum_{k=1}^n a_k\, s^{2k-2} J(t^{2k-1})(\mathbf{i}) + \sum_{l=1}^m = b_l\, s^{2m-1} J(t^{2l})(\mathbf{i}). \end{aligned}$$

Now use Lemma 4.1 to get

$$\alpha = \sum_{k=1}^{n} (-1)^{k+1} s^{2k-2} (2k-1) a_k^1 - \sum_{l=1}^{m} (-1)^l s^{2l-1} (2l) b_l^2,$$

$$\beta = \sum_{k=1}^{n} (-1)^{k+1} s^{2k-2} (2k-1) a_k^2 + \sum_{l=1}^{m} (-1)^l s^{2l-1} (2l) b_l^1,$$

$$\gamma = \sum_{k=1}^{n} (-1)^{k+1} s^{2k-2} a_k^1 \ \text{ and } \ \delta = \sum_{k=1}^{n} (-1)^{k+1} s^{2k-2} a_k^2,$$

where $a_k = a_k^1 + \mathbf{i}a_k^2,\ b_l = b_l^1 + \mathbf{i}b_l^2$. ∎

Note that $|J(g)(t)| \geq 0$ for all $t \in \mathbb{C}$. On the other hand, $J_{\mathbb{C}}(g)(r + \mathbf{i}s) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$. Thus, $|J(g)(r + \mathbf{i}s)| = |J_{\mathbb{C}}(g)(r + \mathbf{i}s)| \cdot (\gamma^2 + \delta^2)$. In view of this it may happen that $|J_{\mathbb{C}}(g)(r + \mathbf{i}s)| > 0$ while $|J(g)(r + \mathbf{i}s)| = 0$. For example, if $g(t) = 2t^5 + t^3 - t + t^2 + \mathbf{i}(3t^5 + t^3 - 2t + 5)$ and $r = 0, s = 1$, and $\alpha = -6, \beta = -12$ but $\gamma = \delta = 0$. Things, however, become more interesting when $r + \mathbf{i}s$ is a root of g. Indeed, we have:

Remark 4.1. *For $g, r + \mathbf{i}s$ as in Corollary 4.1, if $|J_{\mathbb{C}}(g)(r + \mathbf{i}s)| > 0$ and $|J(g)(r + \mathbf{i}s)| = 0$, g is non primitive.*

Proof: Without loss of generality, suppose that $r + \mathbf{i}s = \mathbf{i}$. Write g as in (7). Then, $a(\mathbf{i}) = \mathbf{i}\gamma + \sum_{l=1}^m (-1)^l b_l^1 + b_0^1$ and $b(\mathbf{i}) = \mathbf{i}\delta + \sum_{l=1}^m (-1)^l b_l^2 + b_0^2$. Since $0 = g(\mathbf{i}) = a(\mathbf{i}) + \mathbf{i}b(\mathbf{i})$ we get $a(\mathbf{i}) = b(\mathbf{i}) = 0$. Thus, $a(t), b(t)$ are both divisible by $t^2 + 1$ which makes g non primitive. ∎

Now let $f(t)$ be as in (1). We write $f(t) = a(t) + \mathbf{i}b(t) + \mathbf{j}(c(t) - \mathbf{i}d(t)) = g(t) + \mathbf{j}h(t)$ $a, b, c, d \in \mathbb{R}[t]$.

Proposition 4.1. *For $r + \mathbf{i}s \in \mathbb{C} - \mathbb{R}, f, g, h$ as above,*

$$J(f)(r+\mathbf{i}s) = \begin{bmatrix} \alpha_1 & -\beta_1 & -\alpha_2 & \beta_2 \\ \beta_1 & \alpha_1 & \beta_2 & \alpha_2 \\ \alpha_3 & -\beta_3 & \alpha_4 & -\beta_4 \\ -\beta_3 & -\alpha_3 & \beta_4 & \alpha_4 \end{bmatrix}, \quad \textit{for } \alpha_k, \beta_k \in \mathbb{R}. \tag{8}$$

Proof: We have $J(f)(t) = J(g)(t) + \mathbf{j}J(h)(t)$. Note that $\mathbf{j}J(h)(r+\mathbf{i}s)$ is equal to

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_3 & -\beta_3 & 0 & 0 \\ \beta_3 & \alpha_3 & 0 & 0 \\ 0 & 0 & \alpha_2 & -\beta_2 \\ 0 & 0 & \beta_2 & \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\alpha_2 & \beta_2 \\ 0 & 0 & \beta_2 & \alpha_2 \\ \alpha_3 & -\beta_3 & 0 & 0 \\ -\beta_3 & -\alpha_3 & 0 & 0 \end{bmatrix}.$$

Now the result follows from Corollary 4.1. ■

We finally proceed with the computation of $J(f)$ at $t_0 = x_0 + \mathbf{i}y_0 + \mathbf{j}z_0 + \mathbf{k}w_0 \in \mathbb{H} - \mathbb{C}$. Assuming, as usual, that $x_0 = 0$, we will "push" t_0 to $\mathbf{i}s$, with $\mathbf{i}s \sim t_0$ and use Proposition 4.1 to find the said Jacobian.

To achieve the above, for a $c \in \mathbb{H}, |c| = 1$, consider the transformation $h_c(t) = ctc^*$. This is a rotation in $\mathbb{R}^4$, realized by an orthogonal matrix A_c in the sense that $h_c(x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w) = c(x+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)c^* = (h_1+\mathbf{i}h_2+\mathbf{j}h_3+\mathbf{k}h_4)$, then $h(x, y, z, w) = (h_1, h_2, h_3, h_4) = A_c\,(x, y, z, w)^T$. On the other hand, observe that ctc^* commutes with itself and preserves the root structure of $f(t)$. Indeed, if $f(t)$ is as in (1), consider a new polynomial $\phi(u)$,

$$\phi(u) = cf(u)c^* = ca_nc^*u^n + ca_{n-1}c^*u^{n-1} + \cdots + ca_1c^*u + ca_0c^*. \tag{9}$$

Note now that if c_0 is a root of f, cc_0c^* is a root of ϕ of the same structure. Divide f by $(t-t_0)$ to get $f(t) = g(t)(t-t_0) + f(t_0)$ and let $\gamma \in \mathbb{H}$ be so that $\gamma t_0 \gamma^* = \mathbf{i}s$. Consider the map $F(x, y, z, w) = (f \circ h_\gamma)(x, y, z, w)$. Then, $JF = J(f)(h_\gamma) \cdot J(h_\gamma) = J(f)(h_\gamma) \cdot A_\gamma$. Therefore,

$$J(f)(t_0) = J(f)(\mathbf{i}s)A_\gamma. \tag{10}$$

This along with formula (8) finishes the task of computing $J(f)(t_0)$.

4.1. **Cauchy-Riemann equations.** With the help from the previous section, we are now able to write Cauchy-Riemann (CR) equations for quaternion polynomials. However, the results concerning the CR equations over $\mathbb{R}$ can also be proven independently of the previous ones. The motivation for this is an elementary proof of CR equations for complex polynomials:

Let $f(t) \in \mathbb{C}[t]$. We write $f(x + \mathbf{i}y) = a(x, y) + \mathbf{i}b(x, y)$. Then, $a_x = b_y$ and $a_y = -b_x$.

Proof: Let $t_1 = (x_1, y_1) \in \mathbb{C}$. Then, $f(t) = f^1(t)(t - t_1) + f(t_1)$. In that case, if $f^1(x + \mathbf{i}y) = a^1(x, y) + \mathbf{i}b^1(x, y)$ we get $a(x, y) + \mathbf{i}b(x, y) = (a^1(x, y) + \mathbf{i}b^1(x, y))(x - x_1 + \mathbf{i}(y - y_1)) + f(t_1) = (x-x_1)a^1-(y-y_1)b^1+\mathbf{i}((y-y_1)a^1+(x-x_1)b^1)+f(t_1)$. Then, $a_x(t_1) = a^1(t_1)$ and $b_y(t_1) = a^1(t_1)$. Similarly, $a_y(t_1) = -b_x(t_1)$. ■

The same technique can be applied to any monic quaternion polynomial $f(t)$ for any $t_0 \in \mathbb{R}$. Indeed, divide f by $t - t_0$ to get $f(t) = g(t)(t - t_0) + f(t_0) = (t^{n-1} + a_{n-2}t^{n-2} + \cdots a_1t +$

$a_0)(t-t_0)+f(t_0)$. Since t_0 is real, it commutes with any quaternion, and thus we also have $f(t)=(t-t_0)g(t)+f(t_0)$. Thus, if $t=x+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w$ and $g(x+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)=g_1+\mathbf{i}g_2+\mathbf{j}g_3+\mathbf{k}g_4$, we see that $f(x+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)=f_1+\mathbf{i}f_2+\mathbf{j}f_3+\mathbf{k}f_4=(g_1+\mathbf{i}g_2+\mathbf{j}g_3+\mathbf{k}g_4)(x-t_0+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)+f(t_0)=(x-t_0+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)(g_1+\mathbf{i}g_2+\mathbf{j}g_3+\mathbf{k}g_4)+f(t_0)$. Now an easy calculation shows that

$$J(f)(t_0)=\begin{bmatrix} g_1(t_0) & -g_2(t_0) & 0 & 0 \\ g_2(t_0) & g_1(t_0) & 0 & 0 \\ 0 & 0 & g_1(t_0) & 0 \\ 0 & 0 & 0 & g_1(t_0) \end{bmatrix}. \tag{11}$$

with all the other partials equal to zero. This provides the CR equations at a real point t_0:

$$\frac{\partial f_1}{\partial x}=\frac{\partial f_2}{\partial y},\quad \frac{\partial f_1}{\partial y}=-\frac{\partial f_2}{\partial x},\quad \frac{\partial f_3}{\partial z}=\frac{\partial f_4}{\partial w}=\frac{\partial f_1}{\partial x}. \tag{12}$$

The above method, however, fails when $t_0\in\mathbb{H}-\mathbb{R}$. The reason is that t–being a quaternion variable now–does not commute with t_0 and thus $f_1+\mathbf{i}f_2+\mathbf{j}f_3+\mathbf{k}f_4\neq(g_1+\mathbf{i}g_2+\mathbf{j}g_3+\mathbf{k}g_4)(x-t_0+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)+f(t_0)$. For example, if $f(t)=t^2+(\mathbf{i}+\mathbf{j})t-\mathbf{k}=(t+\mathbf{j})(t+\mathbf{i})$, there is no linear polynomial $g_1+\mathbf{i}g_2+\mathbf{j}g_3+\mathbf{k}g_4$ so that $f(x+\mathbf{i}y+\mathbf{j}z+\mathbf{k}w)=(g_1+\mathbf{i}g_2+\mathbf{j}g_3+\mathbf{k}g_4)(x+\mathbf{i}(y+1)+\mathbf{j}z+\mathbf{k}w)$.

Next, formula (6) provides the CR equations for an $f(t)\in\mathbb{C}[t]$ and $t_0=r+\mathbf{i}s$:

$$\frac{\partial f_1}{\partial x}=\frac{\partial f_2}{\partial y},\quad \frac{\partial f_1}{\partial y}=-\frac{\partial f_2}{\partial x},\quad \frac{\partial f_3}{\partial z}=\frac{\partial f_4}{\partial w},\quad \frac{\partial f_3}{\partial w}=-\frac{\partial f_4}{\partial z}. \tag{13}$$

Observe that $|J(f)(t_0)|\geq 0$.

Finally, for the general case, pick $\gamma\in\mathbb{H}$ so that $\gamma t_0\gamma^*=r+\mathbf{i}s$. With the aid of (8) and (10) we get the CR equations for f at t_0:

$$J(f)(t_0)=\begin{bmatrix} \alpha_1 & -\beta_1 & -\alpha_2 & \beta_2 \\ \beta_1 & \alpha_1 & \beta_2 & \alpha_2 \\ \alpha_3 & -\beta_3 & \alpha_4 & -\beta_4 \\ -\beta_3 & -\alpha_3 & \beta_4 & \alpha_4 \end{bmatrix}\cdot A_\gamma,\quad \text{where } \alpha_k,\beta_k\in\mathbb{R}. \tag{14}$$

4.2. **$|J(f)|$ at a root of f.** In this section we will show that $|J(f)|$ is non negative over $\mathbb{H}$. In particular, we will prove that if t_0 is a root of f, t_0 is simple if and only if $|J(f)(t_0)|>0$. Thus, at a multiple root $|J(f)|$ vanishes.

Let $t_0=x_0+\mathbf{i}y_0+\mathbf{j}z_0+\mathbf{k}w_0\in\mathbb{H}$. Divide $f(t)$ by $t-t_0$ to get $f(t)=g(t)(t-t_0)+f(t_0)$. After making a suitable transformation, we will assume that $t_0=\mathbf{i}$. Let $g(t)=b_mt^m+\cdots+b_1t+b_0$, $b_k\in\mathbb{H}$. We write $f(t)=b_0(t-\mathbf{i})+b_1t(t-\mathbf{i})+\cdots+b_mt^m(t-\mathbf{i})+f(t_0)$. Let A be the matrix

$$A=\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Notice that $A^2 = -I$. Furthermore, from (4) we get $J(t^k(t-\mathbf{i}))(\mathbf{i}) = A^k$ for $k \geq 0$. Therefore,

$$J(f)(\mathbf{i}) = b_0 I + b_1 A + b_2 A^2 + \cdots + b_m A^m = \sum_{k=0}^{m} (-1)^k b_{2k} I + \sum_{l=0}^{m} (-1)^l b_{2l+1} A.$$

Set $\sum_{k=0}^{m} (-1)^k b_{2k} = B_e$, $\sum_{l=0}^{m} (-1)^l b_{2l+1} = B_o$. We claim that $|B_e I + B_o A| \geq 0$. Indeed, if either of B_e, B_o is zero there is nothing to prove. Suppose then $B_e B_o \neq 0$. Then it is enough to show $|I + CA| \geq 0$ for $C = B_o/B_e$. If $C = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$, $I + CA$ takes the form

$$\begin{bmatrix} -b+1 & -a & d & -c \\ a & -b+1 & -c & -d \\ d & -c & b+1 & a \\ -c & -d & -a & b+1 \end{bmatrix},$$

whose determinant $|I + CA| = q(b^2) = b^4 + (-2 + 2a^2 + 2c^2 + 2d^2)b^2 + (1 - 2c^2 - 2d^2 + 2a^2 + 2c^2a^2 + 2d^2a^2 + a^4 + d^4 + c^4 + 2d^2c^2)$. The discriminant of $q(b^2)$ is equal to $-16a^2$ and that proves the claim. Moreover, $q(b^2)$ vanishes precisely when $a = 0$, $b^2 + c^2 + d^2 = 1$. In short, $|B_e I + B_o A| = 0$ if and only $B_e + B_o\,\delta = 0$, for a suitable imaginary unit quaternion δ.

Let γ be an imaginary unit quaternion. Recall that $\gamma \sim \mathbf{i}$. Then, $g(\gamma) = B_e + B_o\gamma$, since $\gamma^2 = -1$. Thus, if $g(\gamma) \neq 0$, which in turn says that $\mu(f)(\mathbf{i}) = 1$, $|B_e I + B_o A| > 0$. On the other hand, if $g(\gamma) = 0$, which means that $\mu(f)(\mathbf{i}) \geq 2$, then $|J(f)(\mathbf{i})|$ vanishes, as required.

5. Closure

In this paper Jacobians of (left) quaternion polynomials were computed along with their determinants. As a result, Cauchy-Riemman equations were obtained for this type of functions. Apparently, similar results can be gotten for *right* quaternion polynomials.

It is hoped that the fact, of the Jacobian determinant of f at any point in $\mathbb{H}$ being non negative, would help in finding the zeros of f. Finally, it would be interesting to investigate whether results of the same nature hold true for general quaternion polynomials of the form $f(t) = a_0 t a_1 t \cdots t a_n + \phi(t), a_i \in \mathbb{H}, a_i \neq 0$ and $\phi(t)$ being a sum of a finite number of similar monomials $b_0 t b_1 t \cdots t b_k, k < n$.

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