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ON A PARTIALLY OVERDETERMINED PROBLEM IN A CONE

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Abstract. We prove a rigidity result for Serrin's overdetermined problem in a cone that is contained in a half-space in arbitrary dimensions. In the special case where the cone is an epigraph, this result was shown previously in low dimensions with a different approach.

Introduction and proof of the main result

In Corollary 9 of their paper [7], Farina and Valdinoci considered the following partially overdetermined problem in the cone

$$\Omega = \left\{x = (x', x_n) \in \mathbb{R}^n,\ n \geq 2\ :\ x_n > \alpha|x'|\right\} \text{ with } \alpha \geq 0 :$$

$$\text{(1)} \qquad \begin{cases} \Delta u + f(u) = 0,\ u > 0 \text{ in } \Omega; \\ u = 0,\ \partial_\nu u = c \text{ on } \partial\Omega \setminus \{0\}, \end{cases}$$

with

$$\text{(2)} \qquad u \in C^2(\Omega) \cap C^1\left(\bar{\Omega} \setminus \{0\}\right) \cap W^{1,\infty}(\Omega),$$

where ν denotes the exterior unitary normal vector on $\partial\Omega \setminus \{0\}$, $c \in \mathbb{R}$ is any fixed constant, and $f \in C^1(\mathbb{R})$. It is shown therein that $\alpha = 0$, *provided that* $n \leq 3$. It is worth mentioning that the above problem was initially studied in [6], motivated by a question of Vazquez. Their result represents a further extension of the famous *Serrin's problem* [12] in unbounded epigraphs, in the spirit of [3]. We point out that the characterization "partially overdetermined" comes from the fact that the overdetermined boundary conditions are not prescribed in the entire $\partial\Omega$.

In this short note, using a completely different approach, we prove a generalization of this result in any dimension. In fact, as will be apparent, our arguments go through with a bit weaker regularity assumptions on f. Moreover, $W^{1,\infty}_{loc}(\Omega)$ in (2) suffices for our purposes (see also (3) below). Our approach is greatly motivated from the study of the regularity properties of free boundaries in one-phase and obstacle-type problems, and hinges on the fact that $\lambda\Omega \equiv \Omega$ for any cone Ω with vertex at the origin and $\lambda > 0$.

Theorem 1. *Let Ω be an open cone in $\mathbb{R}^n$, $n \geq 2$, that is contained in the half-space $\{x_n > 0\}$ and such that $\partial\Omega \setminus \{0\}$ has $C^{1,\beta}$ regularity for some $\beta > 0$. Moreover, let c, f be as above and let u satisfy (1) and (2). Then, the cone Ω coincides with the half-space $\{x_n > 0\}$.*

Proof. We first consider the case

$$c \neq 0 \text{ i.e. } c < 0.$$

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Motivated from the study of one-phase free boundary problems [1], for $r > 0$ small, we consider the following *blow-up* of u:

$$u_r(y) = \frac{u(ry)}{r}, \quad y \in \Omega.$$

We readily find that

$$\begin{cases} \Delta u_r + rf(ru_r) = 0, \ u_r > 0 \text{ in } \Omega; \\ \\ u_r = 0, \ \partial_\nu u_r = c \text{ on } \partial\Omega \setminus \{0\}. \end{cases}$$

By virtue of (2), which implies that

$$u(x) \leq C|x| \text{ for some } C > 0 \text{ near the origin (see also [8, Thm. 4.1])}, \tag{3}$$

and standard elliptic estimates [9, Ch. 9-10] (interior $W^{2,p}$ estimates and boundary $C^{1,\beta}$ Schauder estimates, keeping in mind that the cone becomes flatter and flatter at infinity), along a sequence $r_j \to 0$, u_{r_j} converges in $C^1_{loc}\left(\bar{\Omega} \setminus \{0\}\right)$ to some blow-up limit $u_0 \in C^2(\Omega) \cap C\left(\bar{\Omega}\right) \cap C^1\left(\bar{\Omega} \setminus \{0\}\right)$ which solves

$$\begin{cases} \Delta u_0 = 0, \ u_0 > 0 \text{ in } \Omega; \\ \\ u_0 = 0, \ \partial_\nu u_0 = c \text{ on } \partial\Omega \setminus \{0\}. \end{cases} \tag{4}$$

We point out that we got a nontrivial limit u_0 because of the assumption that $c < 0$.

By a result of [2], all positive harmonic functions in Ω that vanish on $\partial\Omega$ must be homogeneous, i.e.,

$$u_0(y) = |y|^\gamma \Phi_0\left(\frac{y}{|y|}\right),$$

for some $\gamma > 0$ and $\Phi_0 \in C\left(\overline{\mathbb{S}^{n-1} \cap \Omega}\right) \cap C^2\left(\mathbb{S}^{n-1} \cap \Omega\right)$ which vanishes on $\mathbb{S}^{n-1} \cap \partial\Omega$. Observe that, since ∇u_0 is a homogeneous function of degree $\gamma - 1$, in order for the overdetermined boundary conditions in (4) to be satisfied, we must have

$$\gamma = 1.$$

So, Φ_0 is a positive eigenfunction with Dirichlet boundary conditions to the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1} \cap \Omega$, corresponding to the eigenvalue $n-1$ (see for instance the proof of [3, Lem. 2.1]). Hence, recalling that $n-1$ is the principal Dirichlet eigenvalue of $-\Delta_{\mathbb{S}^{n-1}}$ on the upper half-sphere $\mathbb{S}^{n-1}_+$, we deduce that

$$\left|\mathbb{S}^{n-1} \cap \Omega\right| = \left|\mathbb{S}^{n-1}_+\right| = \frac{1}{2}\left|\mathbb{S}^{n-1}\right|.$$

The above relation, however, is only possible if Ω coincides with the half-space $\{x_n > 0\}$ as desired.

It remains to consider the case

$$c = 0.$$

Firstly, by Hopf's boundary point lemma (at some point on $\partial\Omega \setminus \{0\}$ which belongs to the boundary of a ball contained in Ω), we deduce that

$$f(0) < 0. \tag{5}$$

It follows readily from (1) and (2) that u (extended trivially outside of Ω) satisfies, in the weak sense, the following problem:

$$\Delta u = -H(u)f(u) \quad \text{in } \mathbb{R}^n, \tag{6}$$

where H stands for the usual Heaviside function. We just point out that near the origin one uses that

$$\lim_{\rho\to 0} \int_{B_\rho(0)} \nabla u \nabla \varphi dx = 0,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n)$, which holds since $u \in W^{1,\infty}$ (see also [5, Thm. 1.3] for a related argument).

Using (5) and the assumed regularity on f, u, it follows from [10, Ch. 2] that

$$u \in C^{1,1}\left(B_\delta(0)\right) \quad \text{for some small } \delta > 0, \tag{7}$$

(see also [11] for a more general approach).

This time, as in the study of free boundary problems of obstacle type [10], for $r > 0$ small, we consider the following blow-up of u:

$$u_r(y) = \frac{u(ry)}{r^2}, \quad y \in \Omega.$$

We readily find that

$$\begin{cases} \Delta u_r + f(r^2 u_r) = 0, \ u_r > 0 \text{ in } \Omega; \\ u_r = 0, \ \partial_\nu u_r = 0 \text{ on } \partial\Omega \setminus \{0\}. \end{cases}$$

By virtue of (7) and standard elliptic estimates, along a sequence $r_j \to 0$, u_{r_j} converges in $C^1_{loc}(\mathbb{R}^n)$ to some blow-up limit $u_0 \in C^2(\Omega) \cap C^{1,1}(\mathbb{R}^n)$ (globally) which satisfies

$$\begin{cases} \Delta u_0 = -f(0), \ u_0 > 0 \text{ in } \Omega; \\ u_0 = 0, \ \partial_\nu u_0 = 0 \text{ on } \partial\Omega \setminus \{0\}. \end{cases}$$

We point out that we got a nontrivial limit u_0 because of (5).

Let e be an arbitrary direction in $\mathbb{R}^n$ and let

$$v = \partial_e u_0.$$

Then, the function v is harmonic in the cone Ω, vanishes on its boundary, and is globally Lipschitz continuous. To conclude, let us suppose, to the contrary, that the cone Ω does not coincide with the half-space $\{x_n > 0\}$. Then, since v has at most linear growth, we get from Lemma 2.1 in [3] and the comments after it that

$$v \equiv 0.$$

On the other hand, recalling that the direction e is arbitrary, this contradicts the fact that u_0 is nontrivial.

The proof of the theorem is complete. □

Remark 1. The initial value problem

$$\ddot{u} = -f(u); \ u(0) = 0, \ \dot{u}(0) = c,$$

has a unique local solution U (not necessarily positive) with maximal interval of existence $[0, T)$, $T \leq +\infty$. By Hopf's boundary lemma, applied to the difference $u(x) - U(x_n)$, we infer that u coincides with U for $0 \leq x_n < T$. In particular, the partial derivatives $\partial_{x_i} u$, $i < n$

are identically zero in this strip, and thus in the entire half-space $\{x_n > 0\}$ (by the unique continuation principle applied in the linear equation that each one satisfies). Consequently, u depends only on x_n.

Remark 2. It is well known that, in any dimension $n \geq 3$, there exists an $\alpha_n > 0$ such that the cone described by $|x_n| < \alpha_n |x'|$ supports a one-homogeneous solution to (1) with $f \equiv 0$, see [1, 4].

Remark 3. Our assumption that the boundary of the cone is smooth except from its vertex is not restrictive by means of a dimension reduction argument as in [13].

References

[1] H.W. Alt and L.A. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, J. Reine Angew. Math. **325** (1981), 105–144.

[2] A. Ancona, *On positive harmonic functions in cones and cylinders*, Rev. Mat. Iberoam. **28** (2012), 201–230.

[3] H. Berestycki, L.A. Caffarelli and L. Nirenberg, *Monotonicity for elliptic equations in unbounded Lipschitz domains*, Comm. Pure Appl. Math. **50** (1997), 1089–1111.

[4] L.A. Caffarelli, D. Jerison and C. Kenig, *Global energy minimizers for free boundary problems and full regularity in three dimensions*, Contemp. Math. **350**, Amer. Math. Soc., Providence, RI (2004), 83–97.

[5] X. Cabré and J. Terra, *Saddle-shaped solutions of bistable diffusion equations in all of* $\mathbb{R}^{2m}$, J. Eur. Math. Soc. (JEMS) **11** (2009), 819–843.

[6] A. Farina and E. Valdinoci, *Overdetermined problems in unbounded domains with Lipschitz singularities*, Rev. Mat. Iberoam. **26** (2010), 965–974.

[7] A. Farina and E. Valdinoci, *On partially and globally overdetermined problems of elliptic type*, Amer. J. Math. **135** (2013), 1699–1726.

[8] J. Heinonen, Lectures on Lipschitz Analysis, University of Jyvaskyla, Department of Mathematics and Statistics, Report **100**, 2005.

[9] J. Jost, Partial differential equations, Springer-Verlag, 2002.

[10] A. Petrosyan, H. Shahgholian and N. Uraltseva, Regularity of free boundaries in obstacle-type problems, Graduate Studies in Mathematics **136**, American Mathematical Society, Providence, RI, 2012.

[11] H. Shahgholian, $C^{1,1}$ *regularity in semilinear elliptic problems*, Comm. Pure Appl. Math. **56** (2003), 278–281.

[12] J. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal. **43** (1971), 304–318.

[13] G. Weiss, *Partial regularity for a minimum problem with free boundary*, Journal Geom. Anal. **9** (1999), 317–326.

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