POLYNOMIAL ESTIMATES TOWARDS A SHARP HELLY-TYPE THEOREM FOR THE DIAMETER OF CONVEX SETS

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ABSTRACT. We discuss a problem posed by Bárány, Katchalski and Pach: if \( \{ P_i : i \in I \} \) is a family of closed convex sets in \( \mathbb{R}^n \) such that \( \text{diam} \left( \bigcap_{i \in I} P_i \right) = 1 \) then there exist \( s \leq 2n \) and \( i_1, \ldots, i_s \in I \) such that
\[
\text{diam} \left( P_{i_1} \cap \cdots \cap P_{i_s} \right) \leq C_n,
\]
where \( C_n \leq c\sqrt{n} \) for an absolute constant \( c > 0 \). We prove that this statement holds true with \( C_n \leq cn^{11/2} \). All the previously known estimates for \( C_n \) were exponential or superexponential in the dimension \( n \).

1. Introduction

In this note we provide a polynomial estimate for a question of Bárány, Katchalski and Pach on the quantitative version of Helly’s theorem for the diameter of convex sets in Euclidean space. Helly’s theorem states that, if \( \mathcal{P} = \{ P_i : i \in I \} \) is a finite family of at least \( n + 1 \) convex sets in \( \mathbb{R}^n \) and if every \( n + 1 \) or fewer members of \( \mathcal{P} \) have non-empty intersection, then \( \bigcap_{i \in I} P_i \neq \emptyset \). This classical result and its variants have found important applications in discrete and computational geometry (see e.g. [9], [10] and [1]).

Bárány, Katchalski and Pach obtained in [3] a quantitative version of Helly’s theorem for the diameter:

**Theorem 1.1.** Let \( \{ P_i : i \in I \} \) be a family of closed convex sets in \( \mathbb{R}^n \) such that \( \text{diam} \left( \bigcap_{i \in I} P_i \right) = 1 \). There exist \( s \leq 2n \) and \( i_1, \ldots, i_s \in I \) such that
\[
(1.1) \quad \text{diam} \left( P_{i_1} \cap \cdots \cap P_{i_s} \right) \leq C_n,
\]
where \( C_n > 0 \) is a constant depending only on the dimension.

The example of the cube \([-1,1]^n \) in \( \mathbb{R}^n \), expressed as an intersection of exactly \( 2n \) closed half-spaces, shows that one cannot replace \( 2n \) by \( 2n - 1 \) in the statement above. The optimal growth of the constant \( C_n \) as a function of \( n \) is not completely understood. In [8] the authors established the bound \( C_n \leq (cn)^{n/2} \) and conjectured that the bound should be polynomial in \( n \); in fact they asked if \( (cn)^{n/2} \) can be replaced by \( c\sqrt{n} \).

In [8] we proved that there exists an absolute constant \( \alpha > 2 \) with the following property: if \( \{ P_i : i \in I \} \) is a finite family of convex bodies in \( \mathbb{R}^n \) with \( \text{diam} \left( \bigcap_{i \in I} P_i \right) = 1 \), then there exist \( s \leq \alpha n \) and \( i_1, \ldots, i_s \in I \) such that
\[
(1.2) \quad \text{diam} \left( P_{i_1} \cap \cdots \cap P_{i_s} \right) \leq cn^{3/2},
\]

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where \( c > 0 \) is an absolute constant. Note that the estimate is polynomial in the dimension but the restriction \( s \leq 2n \) is relaxed to \( s \leq \alpha n \) for some absolute constant \( \alpha > 2 \). In this note we consider the original question of Bárány, Katchalski and Pach, and provide a polynomial estimate.

**Theorem 1.2.** Let \( \{P_i : i \in I\} \) be a finite family of convex bodies in \( \mathbb{R}^n \) with \( \text{diam}(\bigcap_{i \in I} P_i) = 1 \). We can find \( s \leq 2n \) and \( i_1, \ldots, i_s \in I \) such that

\[
\text{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leq cn^{11/2},
\]

where \( c > 0 \) is an absolute constant.

All the previously known estimates for the question were exponential or superexponential in the dimension. The main step for the proof of Theorem 1.2 is a Helly-type inclusion theorem.

**Theorem 1.3.** Let \( \{P_i : i \in I\} \) be a finite family of convex bodies in \( \mathbb{R}^n \) with \( \text{int}(\bigcap_{i \in I} P_i) \neq \emptyset \). For any \( k > n \) there exist \( z \in \mathbb{R}^n \), \( s \leq k + n \) and \( i_1, \ldots, i_s \in I \) such that

\[
z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n} n(n + 2) \left( z + \bigcap_{i \in I} P_i \right),
\]

where \( \gamma_{k,n} = \left( \frac{\sqrt{k+n} \sqrt{n}}{\sqrt{k} - \sqrt{n}} \right)^2 \).

A main tool for the proof of Theorem 1.3 is the following theorem of Batson, Spielman and Srivastava [6]: If \( v_1, \ldots, v_m \in S^{n-1} \) and \( a_1, \ldots, a_m > 0 \) satisfy “John’s decomposition of the identity” \( I_n = \sum_{j=1}^m a_j v_j \otimes v_j \), where \( (v_j \otimes v_j)(y) = (v_j, y)v_j \), then for every \( d > 1 \) there exists a subset \( \sigma \subseteq \{1, \ldots, m\} \) with \( |\sigma| \leq dn \) and \( b_j > 0 \), \( j \in \sigma \), such that \( I_n \leq \sum_{j \in \sigma} b_j a_j v_j \otimes v_j \leq \gamma_d^2 I_n \), where \( \gamma_d := \frac{\sqrt{d+1}}{\sqrt{d-1}} \).

It is clear that if we apply Theorem 1.3 with \( k = n + 1 \) then we obtain polynomial estimates (of order \( O(n^4) \)) for the diameter with \( s \leq 2n + 1 \). In order to reduce the number of the bodies \( P_{i_j} \) from \( 2n + 1 \) to \( 2n \), and get the precise statement of Theorem 1.2 we use the idea of a lemma from [3] (see Lemma 2.3 in the next section).

In a different direction, Soberón proved in [15] that for any finite family of convex sets in \( \mathbb{R}^n \) with the property that the intersection of every \( 2n \) of them has diameter at least 1, one can partition the family into a fixed number of subfamilies (depending only on \( n \) and \( \varepsilon > 0 \)), each having an intersection with diameter at least \( 1 - \varepsilon \).

Closing this introductory section we mention that Bárány, Katchalski and Pach in [3] obtained also a quantitative Helly-type result for volume (see also [4]). They proved that if \( \{P_i : i \in I\} \) is a family of closed convex sets in \( \mathbb{R}^n \) such that \( |\bigcap_{i \in I} P_i| > 0 \) then we may find \( s \leq 2n \) and \( i_1, \ldots, i_s \in I \) such that

\[
|P_{i_1} \cap \cdots \cap P_{i_s}| \leq D_n |\bigcap_{i \in I} P_i|,
\]

where \( D_n > 0 \) is a constant depending only on \( n \). The bound in [3] was \( O(n^{2n^2}) \) and it was conjectured that one might actually have \( D_n \leq c n^\alpha \) for an absolute constant \( c > 0 \). Naszódi [13] has recently proved a volume version of Helly’s theorem with \( D_n \leq (cn)^{2n} \), where \( c > 0 \) is an absolute constant. In fact, a slight modification of Naszódi’s argument leads to the exponent \( \frac{3n}{4} \) instead of \( 2n \). In [7], relaxing the requirement that \( s \leq 2n \) to the weaker one that \( s = O(n) \), we showed that there exists an absolute constant \( \alpha > 2 \) with the following property: for every
family $\{P_i : i \in I\}$ of closed convex sets in $\mathbb{R}^n$, such that $P = \bigcap_{i \in I} P_i$ has positive volume, there exist $s \leq \alpha n$ and $i_1, \ldots, i_s \in I$ such that
\begin{equation}
|P_{i_1} \cap \cdots \cap P_{i_s}| \leq (cn)^n |P|,
\end{equation}
where $c > 0$ is an absolute constant.

**Notation.** We work in $\mathbb{R}^n$, which is equipped with a Euclidean inner product $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write $B^n_2$ for the Euclidean unit ball and $S^{n-1}$ for the unit sphere. Volume is denoted by $|\cdot|$ and the circumradius of $K$ is the radius of the smallest ball which is centered at the origin and contains $K$:
\[ R(K) = \max \{ \|x\|_2 : x \in K \}. \]
Finally, given two symmetric positive definite matrices $A$ and $B$ we write $A \preceq B$ if $\langle Ax, x \rangle \leq \langle Bx, x \rangle$ for all $x \in \mathbb{R}^n$. We refer to the books [14] and [2] for basic facts from convex geometry.

### 2. Proof of the theorem

The proof of Theorem 1.3 is based on an extension to the non-symmetric case of the following fact, obtained by Gluskin-Litvak in [11] and Barvinok in [5]: If $K$ is a symmetric convex body in $\mathbb{R}^n$ then for any $k > n$ there exist $N \leq k$ points $x_1, \ldots, x_N \in K$ such that
\begin{equation}
\text{absconv}\{x_1, \ldots, x_N\} \subseteq K \subseteq \gamma_{k,n} \sqrt{n} \text{absconv}\{x_1, \ldots, x_N\}.
\end{equation}
We shall first prove the next proposition, which is a variant of a result from [8].

**Proposition 2.1.** Let $K$ be a convex body in $\mathbb{R}^n$, such that the ellipsoid of minimal volume containing $K$ is the Euclidean unit ball $B^n_2$. For every $k > n$ there is a subset $X \subseteq K \cap S^{n-1}$ of cardinality $\text{card}(X) \leq k + n$ such that
\begin{equation}
K \subseteq B^n_2 \subseteq \left( \frac{\sqrt{k} + \sqrt{n}}{\sqrt{k} - \sqrt{n}} \right)^2 \gamma_{k,n} \sqrt{n} \text{conv}(X).
\end{equation}

**Proof.** Since $B^n_2$ is the minimal volume ellipsoid of $K$, by John’s theorem [12] we may find $v_j \in K \cap S^{n-1}$ and $a_j > 0$, $j \in J$, such that
\begin{equation}
I_n = \sum_{j \in J} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j \in J} a_j v_j = 0.
\end{equation}
It is well-known that $(2.3)$ implies that
\begin{equation}
\text{conv}\{v_1, \ldots, v_m\} \supseteq \frac{1}{n} B^n_2.
\end{equation}
Set $d = k/n > 1$ and $\gamma_{k,n} := \gamma_d = \left( \frac{\sqrt{k} + \sqrt{n}}{\sqrt{k} - \sqrt{n}} \right)^2$ and apply the theorem of Batson, Spielman and Srivastava to find a subset $\sigma \subseteq J$ with $|\sigma| \leq k$ and positive scalars $b_j$, $j \in \sigma$, such that $T := \sum_{j \in \sigma} b_j v_j \otimes v_j$ satisfies
\begin{equation}
I_n \preceq \sum_{j \in \sigma} b_j v_j \otimes v_j \preceq \gamma_{k,n} I_n.
\end{equation}
Taking traces we see that
\[ b := \sum_{j \in \sigma} b_j \leq \gamma_{k,n} n. \]
Note that the vector $w = -\frac{1}{bn} \sum_{j \in \sigma} b_j v_j$ has length $\|w\|_2 \leq \frac{1}{bn} \sum_{j \in \sigma} b_j = \frac{1}{n}$, and hence $w \in \text{conv}\{v_j, j \in J\}$ by (2.4). Therefore, we may find $\kappa \geq 1$ such that $\kappa w$ belongs to some facet of $\text{conv}\{v_j, j \in J\}$. Then, we apply Carathéodory’s theorem to find $\tau \subseteq J$ with $|\tau| \leq n$ and $\rho_i > 0$, $i \in \tau$ such that

$$(2.5) \quad \kappa w = \sum_{i \in \tau} \rho_i v_i \text{ and } \sum_{i \in \tau} \rho_i = 1.$$ 

We will show that

$$(2.6) \quad C := \text{conv}\{v_j : j \in \sigma \cup \tau\} \supseteq \frac{1}{\gamma_{k,n}n(n+2)} B^n_2.$$ 

Recall that the Minkowski functional of $C$, defined by $p_C(y) = \min\{t \geq 0 : y \in tC\}$, is subadditive and positively homogeneous. Given $x \in S^{n-1}$ we set $\delta = \min\{\langle x, v_j \rangle : j \in \sigma\}$ and observe that $|\delta| \leq 1$ and $\langle x, v_j \rangle - \delta \leq 2$ for all $j \in \sigma$. If $\delta < 0$, we write

$$p_C(T(x)) \leq p_C\left(T(x) - \delta \sum_{j \in \sigma} b_j v_j\right) + p_C\left(\delta \sum_{j \in \sigma} b_j v_j\right)$$

$$= p_C\left(\sum_{j \in \sigma} b_j (\langle x, v_j \rangle - \delta) v_j\right) + p_C(|\delta| b_n w)$$

$$\leq \sum_{j \in \sigma} b_j (\langle x, v_j \rangle - \delta) p_C(v_j) + |\delta| b_n p_C(w).$$

Since $p_C(v_j) \leq 1$ and $\langle x, v_j \rangle - \delta \leq 2$ for all $j \in \sigma$, we see that $\sum_{j \in \sigma} b_j (\langle x, v_j \rangle - \delta) p_C(v_j) \leq 2 \sum_{j \in \sigma} b_j = 2b$. Since $w \in C$ we have $p_C(w) \leq 1$ and we also have $|\delta| b_n p_K(w) \leq b_n$. Therefore, if $\delta < 0$ then we finally get

$$p_C(T(x)) \leq 2b + b_n = b(n+2) \leq \gamma_{k,n}n(n+2).$$

If $\delta \geq 0$ then $\langle x, v_j \rangle \geq 0$ for all $j \in \sigma$, therefore

$$(2.7) \quad p_C(T(x)) = p_C\left(\sum_{j \in \sigma} b_j (\langle x, v_j \rangle) v_j\right) \leq \sum_{j \in \sigma} b_j (\langle x, v_j \rangle) p_C(v_j) \leq \sum_{j \in \sigma} b_j \leq \gamma_{k,n}n.$$ 

In any case,

$$(2.8) \quad p_T^{-1}(C)(x) \leq \gamma_{k,n}n(n+2)p_{B^n_2}(x)$$

for all $x \in S^{n-1}$. Since $I_n \subseteq T$, we also have $B^n_2 \subseteq T(B^n_2)$, and hence

$$(2.9) \quad K \subseteq B^n_2 \subseteq T(B^n_2) \subseteq \gamma_{k,n}n(n+2)C.$$ 

Since $\text{card}(\sigma \cup \tau) \leq k + n$, the proof is complete.

$\square$

**Theorem 2.2.** Let $\{P_i : i \in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with $\text{int}(\bigcap_{i \in I} P_i) \neq \emptyset$. For any $k > n$ there exist $z \in \mathbb{R}^n$, $s \leq k + n$ and $i_1, \ldots, i_s \in I$ such that

$$(2.10) \quad z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n}n(n+2) \left(z + \bigcap_{i \in I} P_i\right).$$
In particular, assuming that \( \text{diam}(\bigcap_{i \in I} P_i) = 1 \) we get that for every \( k > n \) there exist \( s \leq k + n \) and \( i_1, \ldots, i_s \in I \) such that
\[
\text{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leq \gamma_{k,n} n(n + 2).
\]
Therefore, if we choose \( k = n + 1 \), we get that there exist \( s \leq 2n + 1 \) and \( i_1, \ldots, i_s \in I \) such that
\[
\text{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leq 16n(n + 2)(n + 1)^2.
\]

Proof. Let \( P = \bigcap_{i \in I} P_i \). We may assume that \( 0 \in \text{int}(P) \) and that the minimal volume ellipsoid of the polar body
\[
P^o = \text{conv} \left( \bigcup_{i \in I} P^o_i \right)
\]
of \( P \) is the Euclidean unit ball. Using Proposition 2.1 for \( K = P^o \) we may find \( X = \{v_1, \ldots, v_s\} \subset P^o \cap S^{n-1} \) with \( \text{card}(X) = s \leq k + n \) such that
\[
P^o \subseteq \gamma_{k,n} n(n + 2)\text{conv}(\{v_1, \ldots, v_s\}).
\]
Since \( v_1, \ldots, v_s \) are contact points of \( P^o \) with \( B^n_2 \), we can easily check that we actually have \( v_j \in \bigcup_{i \in I} P^o_i \) for all \( j = 1, \ldots, s \). In other words, we may find \( i_1, \ldots, i_s \in I \) such that \( v_j \in P^o_{i_j}, \)
\( j = 1, \ldots, s \). Then, (2.14) implies that
\[
P^o \subseteq \gamma_{k,n} n(n + 2)\text{conv}(P^o_{i_1} \cup \cdots \cup P^o_{i_s}),
\]
and passing to the polar bodies, we get
\[
P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n} n(n + 2)P
\]
as claimed. Since \( \gamma_{n+1,n} = (\sqrt{n+1} + \sqrt{n})^4 \leq 16(n+1)^2 \), the proof is complete.

For the final step of the proof of Theorem 1.2 we use the idea of a lemma from [3] which will allow us to further reduce the number of the bodies \( P_{i_j} \) from \( 2n + 1 \) to \( 2n \). We include a sketch of its proof for the reader’s convenience.

Lemma 2.3. Let \( P_1, \ldots, P_{2n+1} \) be convex bodies in \( \mathbb{R}^n \) such that \( 0 \in P_1 \cap \cdots \cap P_{2n+1} \). If the circumradius of \( P_1 \cap \cdots \cap P_{2n+1} \) is equal to \( R \) then we can find \( 1 \leq j \leq 2n + 1 \) such that the circumradius of \( \bigcap_{i=1, i \neq j} P_i \) is at most \( 2n^{3/2}R \).

Proof. If \( C \) is a spherical cap such that \( \text{dist}(0, \text{conv}(C)) = t \) then we can write it as a geodesic ball \( C = B(v, \pi/2 - \delta) \) (for some \( v \in S^{n-1} \)) where \( t = \sin \delta \). Then,
\[
\sigma(C) = \sigma(B(v, \delta)) = \frac{1}{2I_{n-1}} \int_0^{\pi/2} (\sin \theta)^{n-1} d\theta,
\]
where \( \sigma \) is the standard rotationally invariant probability measure on the sphere and \( I_k = \int_0^{\pi/2} (\cos \theta)^k d\theta \) (see e.g. [2] Chapter 3)). Therefore, we will have \( \sigma(C) > \frac{n}{2n+1} \) if
\[
\frac{1}{2I_{n-1}} \int_0^{\pi/2} (\sin \theta)^{n-1} d\theta = \frac{1}{2I_{n-1}} \int_{\delta}^{\pi/2} (\cos u)^{n-1} du > \frac{2n}{2n+1},
\]

or equivalently
\[
\frac{1}{I_{n-1}} \int_0^\delta (\cos u)^{n-1} du < \frac{1}{2n+1}.
\]
It is known that \(\sqrt{k} I_k \geq 1\) for all \(k \geq 1\) and we trivially have \(\cos u \leq 1\) for all \(u \in [0, \delta]\). If we choose \(t_n = \frac{1}{2n^{3/2}}\) and \(\delta_n = \arcsin t_n\), assuming that \(n \geq 2\) we get
\[
\int_0^{\delta_n} (\cos u)^{n-1} du \leq \int_0^{\delta_n} \cos u du = \sin \delta_n = \frac{1}{2n^{3/2}} < \frac{1}{(2n+1)\sqrt{n-1}} \leq I_{n-1} \cdot \frac{1}{2n+1}.
\]
Therefore, if \(\text{dist}(0, \text{conv}(C)) = t_n\) we have that
\[
\sigma(C) > \frac{n}{2n+1}.
\]
We assume that for any \(1 \leq j \leq 2n+1\) the circumradius of \(\bigcap_{i=1, i \neq j}^{2n+1} P_i\) is greater than 1 and we will show that the circumradius of \(P_1 \cap \cdots \cap P_{2n+1}\) is greater than \(t_n\). We can choose \(y_j \in \bigcap_{i=1, i \neq j}^{2n+1} P_i\) with \(\|y_j\|_2 = 1\) and then we consider the spherical cap \(C_j\) with center \(y_j\) and \(\text{dist}(0, \text{conv}(C_j)) = t_n\). We claim that there exists \(v \in S^{n-1}\) which belongs to at least \(n + 1\) of the \(C_j\)’s; otherwise, each point of \(S^{n-1}\) would be covered by at most \(n\) of the \(C_j\)’s and this would imply that
\[
n \geq \sum_{j=1}^{2n+1} \sigma(C_j) > (2n+1) \cdot \frac{n}{2n+1} = n,
\]
a contradiction. Now, consider the spherical cap \(C(v)\) with center \(v\) and \(\text{dist}(0, \text{conv}(C(v))) = t_n\). We have at least \(n + 1\) of the \(y_j\)’s in \(C(v)\), and we may assume that \(y_1, \ldots, y_{n+1} \in C(v)\). Each line segment \([0, y_j]\), \(j \leq n + 1\), meets the bounding hyperplane \(H\) of \(C(v)\) at some point \(w_j \in \bigcap_{i=1, i \neq j}^{2n+1} P_i\). Applying Radon’s theorem for the points \(w_1, \ldots, w_{n+1}\) in \(H\), we find a point \(u \in \bigcap_{j=1}^{n+1} \left( \bigcap_{i=1, i \neq j}^{2n+1} P_i \right) = P_1 \cap \cdots \cap P_{2n+1}\). Since \(u \in H\), we have \(\|u\|_2 \geq t_n\). \(\square\)

Now, let \(\{P_i : i \in I\}\) be a finite family of convex bodies in \(\mathbb{R}^n\) with \(\text{diam} \left( \bigcap_{i \in I} P_i \right) = 1\). We may assume that \(0 \in \bigcap_{i \in I} P_i\). First we apply Theorem 2.2 to find \(s \leq 2n + 1\) and \(i_1, \ldots, i_s \in I\) such that
\[
\text{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leq c_1 n^4,
\]
where \(c_1 > 0\) is an absolute constant. If \(s \leq 2n\) then there is nothing to do, otherwise \(s = 2n + 1\) and then we apply Lemma 2.3 and keep \(2n\) of the \(P_{i_j}\)’s so that the diameter of their intersection is bounded by
\[
c_1 n^4 \cdot 2n^{3/2} \leq c_2 n^{11/2},
\]
where \(c_2 > 0\) is an absolute constant. This completes the proof of Theorem 1.2.
References


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