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POLYNOMIAL ESTIMATES TOWARDS A SHARP HELLY-TYPE THEOREM FOR THE DIAMETER OF CONVEX SETS

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ABSTRACT. We discuss a problem posed by Bárány, Katchalski and Pach: if $\{P_i : i \in I\}$ is a family of closed convex sets in $\mathbb{R}^n$ such that $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$ then there exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that

$$\mathrm{diam}\,(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant C_n,$$

where $C_n \leqslant c\sqrt{n}$ for an absolute constant $c > 0$. We prove that this statement holds true with $C_n \leqslant cn^{11/2}$. All the previously known estimates for C_n were exponential or superexponential in the dimension n.

1. Introduction

In this note we provide a polynomial estimate for a question of Bárány, Katchalski and Pach on the quantitative version of Helly's theorem for the diameter of convex sets in Euclidean space. Helly's theorem states that, if $\mathcal{P} = \{P_i : i \in I\}$ is a finite family of at least $n + 1$ convex sets in $\mathbb{R}^n$ and if every $n + 1$ or fewer members of $\mathcal{P}$ have non-empty intersection, then $\bigcap_{i\in I} P_i \neq \emptyset$. This classical result and its variants have found important applications in discrete and computational geometry (see e.g. [9], [10] and [1]).

Bárány, Katchalski and Pach obtained in [3] a quantitative version of Helly's theorem for the diameter:

Theorem 1.1. *Let $\{P_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that* $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. *There exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that*

$$\mathrm{diam}\,(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant C_n, \tag{1.1}$$

where $C_n > 0$ is a constant depending only on the dimension.

The example of the cube $[-1, 1]^n$ in $\mathbb{R}^n$, expressed as an intersection of exactly $2n$ closed half-spaces, shows that one cannot replace $2n$ by $2n - 1$ in the statement above. The optimal growth of the constant C_n as a function of n is not completely understood. In [3] the authors established the bound $C_n \leqslant (cn)^{n/2}$ and conjectured that the bound should be polynomial in n; in fact they asked if $(cn)^{n/2}$ can be replaced by $c\sqrt{n}$.

In [8] we proved that there exists an absolute constant $\alpha > 2$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$, then there exist $s \leqslant \alpha n$ and $i_1, \dots, i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{3/2}, \tag{1.2}$$

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where $c > 0$ is an absolute constant. Note that the estimate is polynomial in the dimension but the restriction $s \leqslant 2n$ is relaxed to $s \leqslant \alpha n$ for some absolute constant $\alpha > 2$. In this note we consider the original question of Bárány, Katchalski and Pach, and provide a polynomial estimate.

Theorem 1.2. *Let $\{P_i : i \in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. *We can find $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{11/2}, \tag{1.3}$$

where $c > 0$ is an absolute constant.

All the previously known estimates for the question were exponential or superexponential in the dimension. The main step for the proof of Theorem 1.2 is a Helly-type inclusion theorem.

Theorem 1.3. *Let $\{P_i : i \in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$. *For any $k > n$ there exist $z \in \mathbb{R}^n$, $s \leqslant k + n$ and $i_1, \dots, i_s \in I$ such that*

$$z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n} n(n+2) \left(z + \bigcap_{i\in I} P_i \right), \tag{1.4}$$

where $\gamma_{k,n} = \left(\frac{\sqrt{k}+\sqrt{n}}{\sqrt{k}-\sqrt{n}}\right)^2$.

A main tool for the proof of Theorem 1.3 is the following theorem of Batson, Spielman and Srivastava [6]: If $v_1, \dots, v_m \in S^{n-1}$ and $a_1, \dots, a_m > 0$ satisfy "John's decomposition of the identity" $I_n = \sum_{j=1}^m a_j v_j \otimes v_j$, where $(v_j \otimes v_j)(y) = \langle v_j, y\rangle v_j$, then for every $d > 1$ there exists a subset $\sigma \subseteq \{1, \dots, m\}$ with $|\sigma| \leqslant dn$ and $b_j > 0$, $j \in \sigma$, such that $I_n \preceq \sum_{j\in\sigma} b_j a_j v_j \otimes v_j \preceq \gamma_d^2 I_n$, where $\gamma_d := \frac{\sqrt{d}+1}{\sqrt{d}-1}$.

It is clear that if we apply Theorem 1.3 with $k = n + 1$ then we obtain polynomial estimates (of order $O(n^4)$) for the diameter with $s \leqslant 2n + 1$. In order to reduce the number of the bodies P_{i_j} from $2n+1$ to $2n$, and get the precise statement of Theorem 1.2, we use the idea of a lemma from [3] (see Lemma 2.3 in the next section).

In a different direction, Soberón proved in [15] that for any finite family of convex sets in $\mathbb{R}^n$ with the property that the intersection of every $2n$ of them has diameter at least 1, one can partition the family into a fixed number of subfamilies (depending only on n and $\varepsilon > 0$), each having an intersection with diameter at least $1 - \varepsilon$.

Closing this introductory section we mention that Bárány, Katchalski and Pach in [3] obtained also a quantitative Helly-type result for volume (see also [4]). They proved that if $\{P_i : i \in I\}$ is a family of closed convex sets in $\mathbb{R}^n$ such that $\left|\bigcap_{i\in I} P_i\right| > 0$ then we may find $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant D_n \left| \bigcap_{i\in I} P_i \right|,$$

where $D_n > 0$ is a constant depending only on n. The bound in [3] was $O(n^{2n^2})$ and it was conjectured that one might actually have $D_n \leqslant n^{cn}$ for an absolute constant $c > 0$. Naszódi [13] has recently proved a volume version of Helly's theorem with $D_n \leqslant (cn)^{2n}$, where $c > 0$ is an absolute constant. In fact, a slight modification of Naszódi's argument leads to the exponent $\frac{3n}{2}$ instead of $2n$. In [7], relaxing the requirement that $s \leqslant 2n$ to the weaker one that $s = O(n)$, we showed that there exists an absolute constant $\alpha > 2$ with the following property: for every family $\{P_i : i \in I\}$ of closed convex sets in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \ldots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (cn)^n\, |P|, \tag{1.5}$$

where $c > 0$ is an absolute constant.

Notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean inner product $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$ and the circumradius of K is the radius of the smallest ball which is centered at the origin and contains K:

$$R(K) = \max\{\|x\|_2 : x \in K\}.$$

Finally, given two symmetric positive definite matrices A and B we write $A \preceq B$ if $\langle Ax, x\rangle \leqslant \langle Bx, x\rangle$ for all $x \in \mathbb{R}^n$. We refer to the books [14] and [2] for basic facts from convex geometry.

2. Proof of the theorem

The proof of Theorem 1.3 is based on an extension to the non-symmetric case of the following fact, obtained by Gluskin-Litvak in [11] and Barvinok in [5]: If K is a symmetric convex body in $\mathbb{R}^n$ then for any $k > n$ there exist $N \leqslant k$ points $x_1, \ldots, x_N \in K$ such that

$$\mathrm{absconv}(\{x_1, \ldots, x_N\}) \subseteq K \subseteq \gamma_{k,n}\sqrt{n}\,\mathrm{absconv}(\{x_1, \ldots, x_N\}). \tag{2.1}$$

We shall first prove the next proposition, which is a variant of a result from [8].

Proposition 2.1. *Let K be a convex body in $\mathbb{R}^n$, such that the ellipsoid of minimal volume containing K is the Euclidean unit ball B_2^n. For every $k > n$ there is a subset $X \subseteq K \cap S^{n-1}$ of cardinality* $\mathrm{card}(X) \leqslant k + n$ *such that*

$$K \subseteq B_2^n \subseteq \left(\frac{\sqrt{k}+\sqrt{n}}{\sqrt{k}-\sqrt{n}}\right)^2 n(n+2)\,\mathrm{conv}(X). \tag{2.2}$$

Proof. Since B_2^n is the minimal volume ellipsoid of K, by John's theorem [12] we may find $v_j \in K \cap S^{n-1}$ and $a_j > 0$, $j \in J$, such that

$$I_n = \sum_{j\in J} a_j v_j \otimes v_j \quad \text{and} \quad \sum_{j\in J} a_j v_j = 0. \tag{2.3}$$

It is well-known that (2.3) implies that

$$\mathrm{conv}\{v_1, \ldots, v_m\} \supseteq \frac{1}{n} B_2^n. \tag{2.4}$$

Set $d = k/n > 1$ and $\gamma_{k,n} := \gamma_d = \left(\frac{\sqrt{k}+\sqrt{n}}{\sqrt{k}-\sqrt{n}}\right)^2$ and apply the theorem of Batson, Spielman and Srivastava to find a subset $\sigma \subseteq J$ with $|\sigma| \leqslant k$ and positive scalars b_j, $j \in \sigma$, such that $T := \sum_{j\in\sigma} b_j v_j \otimes v_j$ satisfies

$$I_n \preceq \sum_{j\in\sigma} b_j v_j \otimes v_j \preceq \gamma_{k,n} I_n.$$

Taking traces we see that

$$b := \sum_{j\in\sigma} b_j \leqslant \gamma_{k,n} n.$$

Note that the vector $w=-\frac{1}{bn}\sum_{j\in\sigma}b_jv_j$ has length $\|w\|_2\leqslant\frac{1}{bn}\sum_{j\in\sigma}b_j=\frac{1}{n}$, and hence $w\in\mathrm{conv}\{v_j,j\in J\}$ by (2.4). Therefore, we may find $\kappa\geqslant 1$ such that κw belongs to some facet of $\mathrm{conv}\{v_j,j\in J\}$. Then, we apply Carathéodory's theorem to find $\tau\subseteq J$ with $|\tau|\leqslant n$ and $\rho_i>0$, $i\in\tau$ such that

$$\kappa w=\sum_{i\in\tau}\rho_iv_i\quad\text{and}\quad\sum_{i\in\tau}\rho_i=1. \tag{2.5}$$

We will show that

$$C:=\mathrm{conv}(\{v_j:j\in\sigma\cup\tau\})\supseteq\frac{1}{\gamma_{k,n}n(n+2)}B_2^n. \tag{2.6}$$

Recall that the Minkowski functional of C, defined by $p_C(y)=\min\{t\geqslant 0: y\in tC\}$, is subadditive and positively homogeneous. Given $x\in S^{n-1}$ we set $\delta=\min\{\langle x,v_j\rangle:j\in\sigma\}$ and observe that $|\delta|\leqslant 1$ and $\langle x,v_j\rangle-\delta\leqslant 2$ for all $j\in\sigma$. If $\delta<0$, we write

$$\begin{aligned}p_C(T(x))&\leqslant p_C\left(T(x)-\delta\sum_{j\in\sigma}b_jv_j\right)+p_C\left(\delta\sum_{j\in\sigma}b_jv_j\right)\\&=p_C\left(\sum_{j\in\sigma}b_j(\langle x,v_j\rangle-\delta)v_j\right)+p_C\left(|\delta|bnw\right)\\&\leqslant\sum_{j\in\sigma}b_j(\langle x,v_j\rangle-\delta)p_C(v_j)+|\delta|bnp_C(w).\end{aligned}$$

Since $p_C(v_j)\leqslant 1$ and $\langle x,v_j\rangle-\delta\leqslant 2$ for all $j\in\sigma$, we see that $\sum_{j\in\sigma}b_j(\langle x,v_j\rangle-\delta)p_C(v_j)\leqslant 2\sum_{j\in\sigma}b_j=2b$. Since $w\in C$ we have $p_C(w)\leqslant 1$ and we also have $|\delta|bnp_K(w)\leqslant bn$. Therefore, if $\delta<0$ then we finally get

$$p_C(T(x))\leqslant 2b+bn=b(n+2)\leqslant\gamma_{k,n}n(n+2).$$

If $\delta\geqslant 0$ then $\langle x,v_j\rangle\geqslant 0$ for all $j\in\sigma$, therefore

$$p_C(T(x))=p_C\left(\sum_{j\in\sigma}b_j\langle x,v_j\rangle v_j\right)\leqslant\sum_{j\in\sigma}b_j\langle x,v_j\rangle p_C(v_j)\leqslant\sum_{j\in\sigma}b_j\leqslant\gamma_{k,n}n. \tag{2.7}$$

In any case,

$$p_{T^{-1}(C)}(x)\leqslant\gamma_{k,n}n(n+2)p_{B_2^n}(x) \tag{2.8}$$

for all $x\in S^{n-1}$. Since $I_n\preceq T$, we also have $B_2^n\subseteq T(B_2^n)$, and hence

$$K\subseteq B_2^n\subseteq T(B_2^n)\subseteq\gamma_{k,n}n(n+2)C. \tag{2.9}$$

Since $\mathrm{card}(\sigma\cup\tau)\leqslant k+n$, the proof is complete. □

Theorem 2.2. *Let $\{P_i:i\in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{int}\left(\bigcap_{i\in I}P_i\right)\neq\emptyset$. *For any $k>n$ there exist $z\in\mathbb{R}^n$, $s\leqslant k+n$ and $i_1,\dots i_s\in I$ such that*

$$z+P_{i_1}\cap\cdots\cap P_{i_s}\subseteq\gamma_{k,n}n(n+2)\left(z+\bigcap_{i\in I}P_i\right), \tag{2.10}$$

In particular, assuming that $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$ *we get that for every* $k > n$ *there exist* $s \leqslant k+n$ *and* $i_1, \dots i_s \in I$ *such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant \gamma_{k,n} n(n+2). \tag{2.11}$$

Therefore, if we choose $k = n+1$, *we get that there exist* $s \leqslant 2n+1$ *and* $i_1, \dots i_s \in I$ *such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant 16n(n+2)(n+1)^2. \tag{2.12}$$

Proof. Let $P = \bigcap_{i\in I} P_i$. We may assume that $0 \in \mathrm{int}(P)$ and that the minimal volume ellipsoid of the polar body

$$P^\circ = \mathrm{conv}\left(\bigcup_{i\in I} P_i^\circ\right) \tag{2.13}$$

of P is the Euclidean unit ball. Using Proposition 2.1 for $K = P^\circ$ we may find $X = \{v_1, \dots, v_s\} \subset P^\circ \cap S^{n-1}$ with $\mathrm{card}(X) = s \leqslant k+n$ such that

$$P^\circ \subseteq \gamma_{k,n} n(n+2)\mathrm{conv}(\{v_1, \dots, v_s\}). \tag{2.14}$$

Since $v_1, \dots, v_s$ are contact points of P° with B_2^n, we can easily check that we actually have $v_j \in \bigcup_{i\in I} P_i^\circ$ for all $j = 1, \dots, s$. In other words, we may find $i_1, \dots, i_s \in I$ such that $v_j \in P_{i_j}^\circ$, $j = 1, \dots, s$. Then, (2.14) implies that

$$P^\circ \subseteq \gamma_{k,n} n(n+2)\mathrm{conv}(P_{i_1}^\circ \cup \cdots \cup P_{i_s}^\circ), \tag{2.15}$$

and passing to the polar bodies, we get

$$P_{i_1} \cap \cdots \cap P_{i_s} \subseteq \gamma_{k,n} n(n+2)P \tag{2.16}$$

as claimed. Since $\gamma_{n+1,n} = (\sqrt{n+1} + \sqrt{n})^4 \leqslant 16(n+1)^2$, the proof is complete. □

For the final step of the proof of Theorem 1.2 we use the idea of a lemma from [3] which will allow us to further reduce the number of the bodies P_{i_j} from $2n+1$ to $2n$. We include a sketch of its proof for the reader's convenience.

Lemma 2.3. *Let* $P_1, \dots, P_{2n+1}$ *be convex bodies in* $\mathbb{R}^n$ *such that* $0 \in P_1 \cap \cdots \cap P_{2n+1}$. *If the circumradius of* $P_1 \cap \cdots \cap P_{2n+1}$ *is equal to* R *then we can find* $1 \leqslant j \leqslant 2n+1$ *such that the circumradius of* $\bigcap_{i=1, i\neq j}^{2n+1} P_i$ *is at most* $2n^{3/2}R$.

Proof. If C is a spherical cap such that $\mathrm{dist}(0, \mathrm{conv}(C)) = t$ then we can write it as a geodesic ball $C = B(v, \pi/2 - \delta)$ (for some $v \in S^{n-1}$) where $t = \sin\delta$. Then,

$$\sigma(C) = \sigma(B(v,\delta)) = \frac{1}{2I_{n-1}} \int_0^{\frac{\pi}{2}-\delta} (\sin\theta)^{n-1} d\theta,$$

where σ is the standard rotationally invariant probability measure on the sphere and $I_k = \int_0^{\pi/2} (\cos\theta)^k d\theta$ (see e.g. [2, Chapter 3]). Therefore, we will have $\sigma(C) > \frac{n}{2n+1}$ if

$$\frac{1}{I_{n-1}} \int_0^{\frac{\pi}{2}-\delta} (\sin\theta)^{n-1} d\theta = \frac{1}{I_{n-1}} \int_\delta^{\pi/2} (\cos u)^{n-1} du > \frac{2n}{2n+1},$$

or equivalently

$$\frac{1}{I_{n-1}}\int_0^{\delta}(\cos u)^{n-1}du < \frac{1}{2n+1}.$$

It is known that $\sqrt{k}I_k \geqslant 1$ for all $k \geqslant 1$ and we trivially have $\cos u \leqslant 1$ for all $u \in [0,\delta]$. If we choose $t_n = \frac{1}{2n^{3/2}}$ and $\delta_n = \arcsin t_n$, assuming that $n \geqslant 2$ we get

$$\int_0^{\delta_n}(\cos u)^{n-1}du \leqslant \int_0^{\delta_n}\cos u\,du = \sin\delta_n = \frac{1}{2n^{3/2}} < \frac{1}{(2n+1)\sqrt{n-1}} \leqslant I_{n-1}\cdot\frac{1}{2n+1}.$$

Therefore, if $\mathrm{dist}(0,\mathrm{conv}(C)) = t_n$ we have that

$$\sigma(C) > \frac{n}{2n+1}.$$

We assume that for any $1 \leqslant j \leqslant 2n+1$ the circumradius of $\bigcap_{i=1,i\neq j}^{2n+1} P_i$ is greater than 1 and we will show that the circumradius of $P_1 \cap \cdots \cap P_{2n+1}$ is greater than t_n. We can choose $y_j \in \bigcap_{i=1,i\neq j}^{2n+1} P_i$ with $\|y_j\|_2 = 1$ and then we consider the spherical cap C_j with center y_j and $\mathrm{dist}(0,\mathrm{conv}(C_j)) = t_n$. We claim that there exists $v \in S^{n-1}$ which belongs to at least $n+1$ of the C_j's; otherwise, each point of S^{n-1} would be covered by at most n of the C_j's and this would imply that

$$n \geqslant \sum_{j=1}^{2n+1}\sigma(C_j) > (2n+1)\cdot\frac{n}{2n+1} = n,$$

a contradiction. Now, consider the spherical cap $C(v)$ with center v and $\mathrm{dist}(0,\mathrm{conv}(C(v))) = t_n$. We have at least $n+1$ of the y_j's in $C(v)$, and we may assume that $y_1,\dots,y_{n+1} \in C(v)$. Each line segment $[0,y_j]$, $j \leqslant n+1$, meets the bounding hyperplane H of $C(v)$ at some point $w_j \in \bigcap_{i=1,i\neq j}^{2n+1} P_i$. Applying Radon's theorem for the points $w_1,\dots,w_{n+1}$ in H, we find a point $u \in \bigcap_{j=1}^{n+1}\left(\bigcap_{i=1,i\neq j}^{2n+1} P_i\right) = P_1 \cap \cdots \cap P_{2n+1}$. Since $u \in H$, we have $\|u\|_2 \geqslant t_n$. $\square$

Now, let $\{P_i : i \in I\}$ be a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. We may assume that $0 \in \bigcap_{i\in I} P_i$. First we apply Theorem 2.2 to find $s \leqslant 2n+1$ and $i_1,\dots i_s \in I$ such that

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant c_1 n^4, \tag{2.17}$$

where $c_1 > 0$ is an absolute constant. If $s \leqslant 2n$ then there is nothing to do, otherwise $s = 2n+1$ and then we apply Lemma 2.3 and keep $2n$ of the P_{i_j}'s so that the diameter of their intersection is bounded by

$$c_1 n^4 \cdot 2n^{3/2} \leqslant c_2 n^{11/2}, \tag{2.18}$$

where $c_2 > 0$ is an absolute constant. This completes the proof of Theorem 1.2.

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