

ON SPACES OF HOLOMORPHIC FUNCTIONS WITH BOUNDED OR CONTINUOUS UP TO THE BOUNDARY DERIVATIVES

D. MOSCHONAS AND V. NESTORIDIS

ABSTRACT. We consider the spaces $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$ containing all holomorphic functions f on an open set $\Omega \subseteq \mathbb{C}$, such that all derivatives $f^{(l)}$, $l \in F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$, are bounded on Ω , or continuously extendable on $\bar{\Omega}$, respectively. We endow these spaces with their natural topologies and they become Fréchet spaces. We prove that the set S of non-extendable functions in each of these spaces is either void, or dense and G_δ . We give examples where $S = \emptyset$ or not. Furthermore, we examine cases where F can be replaced by $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$, or $\tilde{F}_0 = \{l \in \mathbb{N}_0 : 0 \leq l \leq \sup F\}$ and the corresponding spaces stay unchanged.

1. INTRODUCTION

Suppose that f is a function defined on an interval I and that its derivatives $f^{(a)}$ and $f^{(b)}$ are bounded, where a and b are natural numbers, $0 \leq a < b$. Let l be a natural number such that $a < l < b$ and consider the derivative $f^{(l)}$. A natural question is if $f^{(l)}$ is also bounded, or more generally what can be said about the growth of $f^{(l)}$ on I . This question has been investigated by several mathematicians, such as Landau, Kolmogorov, Hardy, Littlewood and others; see [14], [9], [12], [11]. In particular, if $I = \mathbb{R}$ or $I = (0, +\infty)$, then the boundedness of $f^{(a)}$ and $f^{(b)}$ implies the boundedness of $f^{(l)}$, $a < l < b$ ([6], [14]). It follows that if Ω is an open subset of the complex plane \mathbb{C} which is the union of a family of open half-lines and if f is a holomorphic function on Ω such that $f^{(a)}$ and $f^{(b)}$ are bounded on Ω , then all the intermediate derivatives $f^{(l)}$, $a < l < b$, are also bounded on Ω . For instance, Ω could be an open angle, or a strip, or the union of two meeting angles, or of two meeting strips, etc.

The above consideration led us to consider the space:

$$H_F^\infty(\Omega) = \{f \in H(\Omega) : f^{(l)} \text{ is bounded on } \Omega, \text{ for all } l \in F\},$$

where F is an arbitrary, non-empty subset of $\mathbb{N}_0 = \{0, 1, \dots\}$ and examine whether $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$ or not, where $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$. Indeed, if Ω is a union of open half-lines we will prove that $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$. We believe that this does not hold for the general open set, but we do not have a counter-example. Furthermore, we believe that a complete metric can be defined on the set of all open sets Ω (contained in a disc), so that for the generic open set Ω one has $H_F^\infty(\Omega) \neq H_{\tilde{F}}^\infty(\Omega)$. The space $H_F^\infty(\Omega)$ endowed with its natural topology is a Fréchet space and thus Baire's Category theorem is at our disposal in order to prove some generic results.

In analogy to the space $H_F^\infty(\Omega)$, we consider the space:

$$\mathcal{A}_F(\Omega) = \{f \in H(\Omega) : f^{(l)} \text{ has a continuous extension on } \bar{\Omega}, \text{ for all } l \in F\},$$

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where the closure is taken in \mathbb{C} . This space endowed with its natural topology is also a Fréchet space and Baire's theorem can be applied again in order to prove some generic results. Moreover, we use the completeness of these spaces and a result from [18] to prove that either every function is extendable, or generically every function is non-extendable. We give examples where each horn of the above dichotomy occurs.

We note that if $p \in \mathbb{N}_0 = \{0, 1, \dots\}$ and $F = \{0, 1, \dots, p\}$, the spaces $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$ are denoted by $H_p^\infty(\Omega)$ and $\mathcal{A}_p(\Omega)$, respectively; these spaces have been studied extensively in [5] and elsewhere.

Afterwards, we present another dichotomy result regarding the space $H_F^\infty(\Omega)$, proven using a result from [21]. It states that if F is a non-empty subset of $\mathbb{N}_0 = \{0, 1, \dots\}$, Ω is an open subset of \mathbb{C} and $l \notin F$, then either for every function f in $H_F^\infty(\Omega)$ the derivative $f^{(l)}$ is bounded, or generically for every function f in $H_F^\infty(\Omega)$ the derivative $f^{(l)}$ is unbounded. In other words, either $H_{F \cup \{l\}}^\infty(\Omega) = H_F^\infty(\Omega)$, or $H_{F \cup \{l\}}^\infty(\Omega)$ is meager in $H_F^\infty(\Omega)$. It remains open to find an example of an open set $\Omega \subseteq \mathbb{C}$ for which the equality $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$ fails to hold, though we believe that such examples exist and that this phenomenon is valid for the generic open set Ω . We also prove that for any unbounded open set Ω and any non-empty $F \subseteq \mathbb{N}_0$, generically for every function f in $\mathcal{A}_F(\Omega)$ all derivatives $f^{(l)}$, $l \in \mathbb{N}_0$, are unbounded.

Finally, in the last section we provide a stronger result about the space $H_F^\infty(\Omega)$; we broaden the class of all open sets $\Omega \subseteq \mathbb{C}$ for which the equality $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$ holds, where $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ is non-empty and $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$. More precisely, the above holds for all open sets Ω which are equal to the union of a family of segments whose length is uniformly far away from zero. Additionally, we present a candidate for an open set Ω which may render the previous equality invalid.

A preliminary version of the present article can be found in [15].

2. THE SPACES $H_F^\infty(\Omega)$ AND $\mathcal{A}_F(\Omega)$

If $\Omega \subseteq \mathbb{C}$ is a domain then $H^\infty(\Omega)$ is the space of bounded holomorphic functions on Ω . This space endowed with the supremum norm becomes a Banach space. Let $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ be a non-empty set. We wish to consider the space $H_F^\infty(\Omega)$ containing all holomorphic functions f on Ω whose derivatives $f^{(l)}$ belong to $H^\infty(\Omega)$ for all $l \in F$. Namely:

$$H_F^\infty(\Omega) = \{f \in H(\Omega) : f^{(l)} \in H^\infty(\Omega), \text{ for all } l \in F\}.$$

We define a natural topology on this space via the seminorms:

$$\sup_{z \in \Omega} |f^{(l)}(z)|, \text{ for } l \in F \quad \text{and} \quad |f^{(l)}(z_0)|, \text{ for } 0 \leq l < \min F,$$

where z_0 is an arbitrary, yet fixed point in Ω . We will show that $H_F^\infty(\Omega)$ is a complete metric space, hence a Fréchet space. In fact, if F is finite it is a Banach space. In any case, Baire's theorem is at our disposal.

For the proof we need the following propositions:

Proposition 2.1. *Let $\Omega \subseteq \mathbb{C}$ be a domain and z_0 be a fixed point in Ω . Also, let f_n, f , $n \in \mathbb{N}$, be holomorphic functions on Ω such that $f'_n \rightarrow f'$ uniformly on compact subsets of Ω and $f_n(z_0) \rightarrow f(z_0)$. Then, $f_n \rightarrow f$ uniformly on compact subsets of Ω .*

Proof. Let $D(z, r)$ be a disc such that $\overline{D(z, r)} \subseteq \Omega$. If $f_n(z) \rightarrow f(z)$, then by writing:

$$f(w) = f(z) + \int_{[z, w]} f'(\zeta) d\zeta$$

for all $w \in \overline{D(z, r)}$, one can easily show that $f_n \rightarrow f$ uniformly on $\overline{D(z, r)}$. Therefore, the set $G = \{z \in \Omega : f_n(z) \rightarrow f(z)\}$ can easily be seen to be open and closed in Ω . In addition, it is non-empty because $z_0 \in G$ by the hypothesis. Since Ω is connected, it follows that $G = \Omega$ and thus the convergence $f_n \rightarrow f$ is uniform on every closed disc contained in Ω . Since every compact subset of Ω can be covered by a finite union of such discs, we conclude that $f_n \rightarrow f$ uniformly on compact subsets of Ω . ■

One can also easily see that the following corollary holds using the fact that:

$$g(z) = g(z_0) + \int_{[z_0, z]} g'(\zeta) d\zeta$$

for all holomorphic functions g on Ω on any bounded convex domain Ω and any $z_0, z \in \Omega$.

Corollary 2.2. *Let $\Omega \subseteq \mathbb{C}$ be a bounded convex domain and z_0 be a fixed point in Ω . Also, let f_n, f , $n \in \mathbb{N}$, be holomorphic functions on Ω such that $f'_n \rightarrow f'$ uniformly on Ω and $f_n(z_0) \rightarrow f(z_0)$. Then, $f_n \rightarrow f$ uniformly on Ω .*

Proposition 2.3. *Let $\Omega \subseteq \mathbb{C}$ be a domain and z_0 be a fixed point in Ω . Also, let $(f_n)_n$ be a sequence of holomorphic functions on Ω such that the sequence $(f'_n)_n$ is uniformly Cauchy on compact subsets of Ω and the sequence $(f_n(z_0))_n$ is Cauchy. Then, there exists a holomorphic function f on Ω such that $f_n \rightarrow f$ uniformly on compact subsets of Ω .*

Proof. Let $g(z) = \lim f'_n(z)$, where the convergence is uniform on every compact subset of Ω . If we show that g has a primitive f on Ω , then by adding a constant we can obtain $f(z_0) = \lim f_n(z_0)$. Then, Proposition 2.1 yields the result. Therefore, it remains to prove that g has a primitive on Ω , even though Ω is not assumed to be simply connected. It suffices to show that:

$$\int_{\gamma} g(\zeta) d\zeta = 0,$$

for all closed polygonal curves γ in Ω (see [1]). Let γ be such a curve. Since $f'_n \rightarrow g$ uniformly on the compact set γ and γ is a closed curve, it follows that:

$$\int_{\gamma} g(\zeta) d\zeta = \lim \int_{\gamma} f'_n(\zeta) d\zeta = 0. \quad \blacksquare$$

A combination of Propositions 2.1 and 2.3 easily implies the following:

Theorem 2.4. *Let $\Omega \subseteq \mathbb{C}$ be a domain and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, the space $H_F^\infty(\Omega)$ endowed with its natural topology is a Fréchet space.*

Proof. Let $(f_n)_n$ be a Cauchy sequence in $H_F^\infty(\Omega)$. Then, $(f_n^{(l)})_n$ is uniformly Cauchy on Ω for all $l \in F$ and $(f_n^{(l)}(z_0))_n$ is a Cauchy sequence for all l satisfying $0 \leq l < \min F$. Therefore, every $(f_n^{(l)})_n$ converges uniformly on Ω towards some function g_l holomorphic on Ω , where $l \in F$. Firstly, we cover the case of all indices l satisfying $0 \leq l < \min F$, assuming that

$\min F > 0$. Pick $a = \min F$, then $f_n^{(a)} \rightarrow g_a$ uniformly on Ω and $(f_n^{(a-1)}(z_0))_n$ is a Cauchy sequence. By Proposition 2.3 we have that $(f_n^{(a-1)})_n$ converges uniformly on compact subsets of Ω towards some function h which is holomorphic on Ω . By Weierstrass's theorem, we have that $f_n^{(a)} \rightarrow h'$ uniformly on compact subsets of Ω , therefore $h' = g_a$. Also, $f_n^{(a-1)}(z_0) \rightarrow h(z_0)$. By a finite induction argument we obtain the existence of a function f holomorphic on Ω such that $f^{(a)} = g_a$. Thus, $f_n^{(a)} \rightarrow f^{(a)}$ uniformly on Ω and $f_n^{(l)}(z_0) \rightarrow f^{(l)}(z_0)$ for $0 \leq l < \min F$. By applying Weierstrass's theorem again, we have that $f_n^{(l)} \rightarrow f^{(l)}$ uniformly on compact subsets of Ω for all $l > \min F$. But if $l \in F$, this convergence is uniform on all of Ω , because by assumption the sequence $(f_n^{(l)})_n$ is uniformly Cauchy on Ω . Hence, $f_n^{(l)} \rightarrow f^{(l)}$ uniformly on Ω for all $l \in F$. In the simpler case where $\min F = 0$, the result follows by using the same arguments, without the use of Proposition 2.3. The fact that $f \in H_F^\infty(\Omega)$ is easily obtained since each derivative $f^{(l)}$, $l \in F$, is the uniform limit of bounded functions. This shows that $f_n \rightarrow f$ in the topology of $H_F^\infty(\Omega)$. ■

We now turn our attention to the second space at hand. If $\Omega \subseteq \mathbb{C}$ is a domain then $\mathcal{A}(\Omega)$ is the space of holomorphic functions on Ω possessing a continuous extension on $\bar{\Omega}$, where the closure is taken in \mathbb{C} . If Ω is bounded, then this space endowed with the supremum norm becomes a Banach space. If Ω is unbounded, then the topology of $\mathcal{A}(\Omega)$ is defined by the seminorms:

$$\sup_{\substack{z \in \bar{\Omega} \\ |z| \leq n}} |f(z)|, \text{ for } n \in \mathbb{N},$$

and then it is a Fréchet space. Let $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ be a non-empty set. We wish to consider the space $\mathcal{A}_F(\Omega)$ containing all holomorphic functions f on Ω whose derivatives $f^{(l)}$ belong to $\mathcal{A}(\Omega)$ for all $l \in F$. Namely:

$$\mathcal{A}_F(\Omega) = \{f \in H(\Omega) : f^{(l)} \in \mathcal{A}(\Omega), \text{ for all } l \in F\}.$$

The natural topology of $\mathcal{A}_F(\Omega)$ is the one defined by the seminorms:

$$\sup_{\substack{z \in \bar{\Omega} \\ |z| \leq n}} |f^{(l)}(z)|, \text{ for } l \in F, n \in \mathbb{N} \quad \text{and} \quad |f^{(l)}(z_0)|, \text{ for } 0 \leq l < \min F,$$

where z_0 is an arbitrary, yet fixed point in Ω . We will show that $\mathcal{A}_F(\Omega)$ is a complete metric space, hence a Fréchet space. In fact, if Ω is bounded and F is finite it is a Banach space. Thus, Baire's Theorem can be applied once again.

Theorem 2.5. *Let $\Omega \subseteq \mathbb{C}$ be a domain and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, the space $\mathcal{A}_F(\Omega)$ endowed with its natural topology is a Fréchet space.*

The proof of the above theorem is omitted, it is essentially identical to that of Theorem 2.4. The reasoning exhibited in that proof works here as well.

We proceed to the setting of general open sets. Let $\Omega \subseteq \mathbb{C}$ be a (possibly non-connected) open set. Then, Ω has countably many connected components Ω_i , $i \in I$, that is I is either finite, or $I = \mathbb{N}$. For every $i \in I$ we fix a point z_i in Ω_i . Let $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ be a non-empty set. We consider the spaces:

$$H_F^\infty(\Omega) = \{f \in H(\Omega) : f^{(l)} \in H^\infty(\Omega), \text{ for all } l \in F\}$$

and

$$\mathcal{A}_F(\Omega) = \{f \in H(\Omega) : f^{(l)} \in \mathcal{A}(\Omega), \text{ for all } l \in F\}.$$

The topology of $H_F^\infty(\Omega)$ is induced by the seminorms:

$$\sup_{z \in \Omega} \left| f^{(l)}(z) \right|, \text{ for } l \in F \quad \text{and} \quad \left| f^{(l)}(z_i) \right|, \text{ for } 0 \leq l < \min F, i \in I.$$

The topology of $\mathcal{A}_F(\Omega)$ is induced by the seminorms:

$$\sup_{\substack{z \in \Omega \\ |z| \leq n}} \left| f^{(l)}(z) \right|, \text{ for } l \in F, n \in \mathbb{N} \quad \text{and} \quad \left| f^{(l)}(z_i) \right|, \text{ for } 0 \leq l < \min F, i \in I.$$

By applying the previous results of this section regarding domains to each connected component of the open set Ω , we deduce that $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$ are Fréchet spaces; the proofs of these assertions are similar to the ones in the case where Ω was a domain, only with some minor modifications. Therefore, Theorems 2.4 and 2.5 extend to the case of non-connected open sets Ω . Also, Baire's theorem can be applied.

Theorem 2.6. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, the space $H_F^\infty(\Omega)$ endowed with its natural topology is a Fréchet space.*

Theorem 2.7. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, the space $\mathcal{A}_F(\Omega)$ endowed with its natural topology is a Fréchet space.*

3. ON $H_F^\infty(\Omega)$ FOR Ω OPEN AND CONVEX

In this section, we are interested in open, convex sets $\Omega \subseteq \mathbb{C}$ and we prove some results regarding the space $H_F^\infty(\Omega)$, where $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ is a non-empty set. In particular, we will show that if Ω is unbounded then $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$, where $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$ and if Ω is bounded then $H_F^\infty(\Omega) = H_{\tilde{F}_0}^\infty(\Omega) = H_{F_0}^\infty(\Omega)$, where $\tilde{F}_0 = \{l \in \mathbb{N}_0 : 0 \leq l \leq \sup F\}$. In this section we will also give similar results about two other types of open sets, unions of open half-lines and simply connected domains with finite interior diameter.

We start by presenting the following lemma of geometric nature. Results of similar type, as well as their inverses, are investigated in [16], [2], [3], [10] and [4]. The proof we provide is quite elementary:

Lemma 3.1. *Let $\Omega \subseteq \mathbb{C}$ be an open set. If Ω is unbounded and convex, then it is the union of open half-lines.*

Proof. Fix a point p in Ω . Since Ω is unbounded, there exists a sequence $(z_n)_n$ of points in Ω such that $0 < |z_n - p| \rightarrow +\infty$. Let $h_n = \frac{z_n - p}{|z_n - p|}$, then $|h_n| = 1$ for all $n \in \mathbb{N}$. That is, every h_n belongs to the compact set $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, therefore we can extract a convergent subsequence $(h_{k_n})_n$ of $(h_n)_n$. Let $h = \lim h_{k_n} \in S^1$. We will show that $p + th \in \Omega$ for all $t > 0$. Indeed, let $t > 0$. Since $|z_{k_n} - p| \rightarrow +\infty$, for large enough n we have that $|z_{k_n} - p| > 2t$ and therefore $0 < \frac{2t}{|z_{k_n} - p|} < 1$. Since $p \in \Omega$, $z_n \in \Omega$ for all $n \in \mathbb{N}$ and Ω is convex, we deduce that:

$$w_n = \left(1 - \frac{2t}{|z_{k_n} - p|}\right) p + \frac{2t}{|z_{k_n} - p|} z_{k_n} \in \Omega$$

for large enough n . Observe that:

$$w_n = p + 2t \frac{z_{k_n} - p}{|z_{k_n} - p|} = p + 2th_{k_n} \rightarrow p + 2th, \text{ as } n \rightarrow +\infty.$$

Therefore, $w_n - 2th \rightarrow p$ and thus $2p + 2th - w_n \rightarrow p$, as $n \rightarrow +\infty$. Since $p \in \Omega$ and Ω is an open set, we deduce that $2p + 2th - w_n \in \Omega$ for large enough n . So far we have showed that

$w_n \in \Omega$ and that $2p + 2th - w_n \in \Omega$ for large enough n . For these points, the convexity of Ω implies the following:

$$\frac{1}{2}w_n + \frac{1}{2}(2p + 2th - w_n) = p + th \in \Omega.$$

Thus, the closed half-line $\{p + th : t \geq 0\}$ is contained in Ω . In order to find an open half-line contained in Ω , we pick $r > 0$ sufficiently small so that the disc $D(p, r)$ is contained in Ω . Then, by extending the closed half-line $\{p + th : t \geq 0\}$ towards the point p by an open line segment of length r , we conclude that the open half-line $\{p + th : t > -r\}$, which is parallel to h , is contained in Ω . This completes the proof. ■

Next, we present an interpolation inequality involving derivatives of functions of one real variable, which gave us the motivation for Lemma 3.1; we refer to [14]:

Theorem 3.2 (Landau-Kolmogorov inequality). *Let f be a function defined on I , where $I = \mathbb{R}$ or $I = (0, +\infty)$, taking real or complex values. Assume that f is n -times differentiable on I and let:*

$$M_k = \sup_{x \in I} |f^{(k)}(x)|,$$

for $k = 0, 1, \dots, n$. If both M_0 and M_n are finite, then the following bounds are valid:

$$M_k \leq C(n, k, I) \cdot M_0^{1-k/n} \cdot M_n^{k/n},$$

for $k = 1, \dots, n-1$, where $0 < C(n, k, I) < +\infty$ are constants dependent only on n, k and I .

It follows that if f and $f^{(n)}$ are bounded, then all the intermediate derivatives $f^{(k)}$, $k = 1, \dots, n-1$, are bounded as well. Clearly, such an inequality holds for functions defined on any open unbounded interval, when comparing derivatives of any order. The constants mentioned above are sharp and depend on I only in the following sense: $C(n, k, \mathbb{R})$ and $C(n, k, (0, +\infty))$ are different, but in each case they depend only on n and k . So from now on we will denote them simply by $C(n, k)$, since their dependency on I is of no true significance. We also note that we shall make use of this inequality in the simpler form

$$M_k \leq C(n, k) \cdot \max\{M_0, M_n\},$$

for $k = 1, \dots, n-1$, which we easily derive from the previous one.

We make some additional remarks for the interested reader. Each $C(n, k)$ lies between 1 and $\frac{\pi}{2}$ and can be expressed in terms of some series of numbers; these results, among others, are due to Kolmogorov, see [14] for details as well as bounds, estimates and results on the asymptotic behaviour of $C(n, k)$. Most of the literature surrounding this inequality is concerned with the study of $C(n, k)$, finding an explicit formula for them and describing the extremizing functions of the inequality; see [20] for an example of such a work. Also, similar inequalities are valid for other norms. None of these facts are relevant as far as our purposes are concerned.

Inspired by Theorem 3.2, we are now ready to prove the following:

Theorem 3.3. *Let $\Omega \subseteq \mathbb{C}$ be an open set, which is the union of open half-lines and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$ as sets and as topological spaces, where $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$.*

Proof. Clearly, $H_{\tilde{F}}^\infty(\Omega) \subseteq H_F^\infty(\Omega)$. For the inverse inclusion, let $f \in H_F^\infty(\Omega)$. Fix any point w in Ω , then there exists an open half-line L contained entirely in Ω on which the point w lies. We can identify L with the interval $I := (-\varepsilon, +\infty)$ for any fixed $\varepsilon > 0$; that is, I will serve

as a parametrization of L . Choose $h \in \mathbb{C}$ parallel to L with $|h| = 1$. Consider the function $g : I \rightarrow \mathbb{C}$ defined by $g(t) = f(w + th)$. Since f is holomorphic on Ω , g is of class C^∞ on I and $g^{(k)}(t) = f^{(k)}(w + th) \cdot h^k$ for all $t \in I$ and $k \in \mathbb{N}_0$. Therefore, $|g^{(k)}(0)| = |f^{(k)}(w)|$ for all $k \in \mathbb{N}_0$, since $|h| = 1$. Observe that:

$$M_k := \sup_{x \in I} |g^{(k)}(x)| = \sup_{z \in L} |f^{(k)}(z)|,$$

for all $k \in \mathbb{N}_0$. We will show that $f \in H_F^\infty(\Omega)$. To this end, we pick $l \in \tilde{F}$ and we will show that $f^{(l)} \in H^\infty(\Omega)$. If $l \in F$ we have nothing to prove, so assume that $l \notin F$. Next, pick any $a_1, a_2 \in F$ such that $a_1 < l < a_2$; for instance, choose $a_1 = \max\{n \in F : n < l\}$ and $a_2 = \min\{n \in F : n > l\}$. We have that $f^{(a_1)}$ and $f^{(a_2)}$ are bounded on Ω , thus they are also bounded on $L \subseteq \Omega$. Consequently, $g^{(a_1)}$ and $g^{(a_2)}$ are bounded on I , that is $M_{a_1} < +\infty$ and $M_{a_2} < +\infty$. By invoking Theorem 3.2 we obtain the existence of some constants $C(a_2 - a_1, k)$ satisfying:

$$M_k \leq C(a_2 - a_1, k) \cdot \max\{M_{a_1}, M_{a_2}\},$$

for all $k \in \mathbb{N}$ such that $a_1 < k < a_2$. Since $a_1 < l < a_2$, it follows that $M_l < +\infty$ which means that $g^{(l)}$ is also bounded on I . Hence:

$$|f^{(l)}(w)| = |g^{(l)}(0)| \leq M_l \leq C(a_2 - a_1, l) \cdot \max\{M_{a_1}, M_{a_2}\}$$

and for $j = a_1, a_2$ we have that:

$$M_{a_j} = \sup_{z \in L} |f^{(a_j)}(z)| \leq \sup_{z \in \Omega} |f^{(a_j)}(z)| < +\infty.$$

The last two inequalities combined imply that:

$$|f^{(l)}(w)| \leq C(a_2 - a_1, l) \cdot \max\left\{\sup_{z \in \Omega} |f^{(a_1)}(z)|, \sup_{z \in \Omega} |f^{(a_2)}(z)|\right\},$$

from which we obtain the following:

$$\sup_{z \in \Omega} |f^{(l)}(z)| \leq C(a_2 - a_1, l) \cdot \max\left\{\sup_{z \in \Omega} |f^{(a_1)}(z)|, \sup_{z \in \Omega} |f^{(a_2)}(z)|\right\} < +\infty.$$

Hence, $f^{(l)}$ is bounded on Ω and $l \in \tilde{F}$ was arbitrary. This implies that $f \in H_F^\infty(\Omega)$, thus $H_F^\infty(\Omega) \subseteq H_{\tilde{F}}^\infty(\Omega)$ and the set equality $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$ is thereby established.

It remains to show that the corresponding topologies coincide as well. Since Ω is an open set, it has a countable number of connected components Ω_i , $i \in I$, that is I is countable. After fixing points z_i in each Ω_i the space $H_{\tilde{F}}^\infty(\Omega)$ is topologized via the seminorms:

$$\sup_{z \in \Omega} |f^{(l)}(z)|, \text{ for } l \in F \quad \text{and} \quad |f^{(l)}(z_i)|, \text{ for } 0 \leq l < \min F, \quad i \in I.$$

Since $F \subseteq \tilde{F}$ and $\min F = \min \tilde{F}$, the topology of $H_F^\infty(\Omega)$ is induced by the same seminorms, in addition to the following ones:

$$\sup_{z \in \Omega} |f^{(l)}(z)|, \text{ for } l \in \tilde{F} \setminus F,$$

for which we gave the following bounds:

$$\sup_{z \in \Omega} |f^{(l)}(z)| \leq C(a_2 - a_1, l) \cdot \max \left\{ \sup_{z \in \Omega} |f^{(a_1)}(z)|, \sup_{z \in \Omega} |f^{(a_2)}(z)| \right\},$$

where $a_1, a_2 \in F$ such that $a_1 < l < a_2$. Thus, every seminorm from one topology is bounded by some seminorm from the other topology, meaning that each topology is finer than the other. It follows that these topologies are indeed the same, completing the proof. ■

Remark 3.4. Another proof of the equivalence of these topologies can be given by using the Open Mapping theorem for Fréchet spaces, since the identity map $I : H_{\tilde{F}}^\infty(\Omega) \rightarrow H_F^\infty(\Omega)$ is a continuous bijection, hence an isomorphism.

Remark 3.5. The result obtained in Theorem 3.3 is best possible in the sense that we cannot obtain the same result for larger \tilde{F} . For example, consider the function $f : \Omega = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 1\} \rightarrow \mathbb{C}$ defined by $f(z) = z$. Then, $f \in H_{\{1\}}^\infty(\Omega)$ but $f \notin H_{\{0,1\}}^\infty(\Omega)$.

If $\Omega \subseteq \mathbb{C}$ is an unbounded convex domain, then by Lemma 3.1 it is the union of open half-lines. As a consequence, Theorem 3.3 is valid for such a domain Ω . Therefore, we combine Lemma 3.1 and Theorem 3.3 and obtain the following:

Theorem 3.6. *Let $\Omega \subseteq \mathbb{C}$ be an unbounded convex domain and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, $H_{\tilde{F}}^\infty(\Omega) = H_F^\infty(\Omega)$ as sets and as topological spaces, where $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$.*

We proceed to the setting of bounded convex domains and begin with an elementary observation. Let $\Omega \subseteq \mathbb{C}$ be a bounded convex domain and f be a bounded holomorphic function on Ω . Using the convexity and boundedness of Ω , it is easy to see that the primitive:

$$F(z) = \int_{[a,z]} f(\zeta) d\zeta$$

of f is Lipschitz continuous on Ω , where a is an arbitrary, yet fixed point in Ω determining the path of integration. Thus, F is uniformly continuous on Ω which implies that F is continuously extendable on the compact set $\bar{\Omega}$. This in turn implies that F is bounded on Ω . Since every other primitive of f differs from F only by a constant, we deduce that every primitive of f is bounded and uniformly continuous on Ω .

By making use of these statements, we prove the following:

Theorem 3.7. *Let $\Omega \subseteq \mathbb{C}$ be a bounded convex domain and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, $H_{\tilde{F}}^\infty(\Omega) = H_F^\infty(\Omega)$ as sets and as topological spaces, where $\tilde{F}_0 = \{l \in \mathbb{N}_0 : 0 \leq l \leq \sup F\}$.*

Proof. Clearly, $H_{\tilde{F}_0}^\infty(\Omega) \subseteq H_F^\infty(\Omega)$. For the inverse inclusion, let $f \in H_F^\infty(\Omega)$. We will show that $f \in H_{\tilde{F}_0}^\infty(\Omega)$. To this end, we pick an $l \in \tilde{F}_0$ and we will show that $f^{(l)} \in H^\infty(\Omega)$. If $l \in F$ we have nothing to prove, so assume that $l \notin F$. Next, pick any $a \in F$ such that $a > l$; for instance, choose $a = \min\{n \in F : n > l\}$. We have that $f^{(a)}$ is bounded on Ω and by integrating the bounded function $f^{(a)}$ repeatedly we deduce that $f^{(l)}$ is also bounded on Ω ; this follows from the earlier discussion. Hence $f^{(l)}$ is bounded on Ω and $l \in \tilde{F}_0$ was arbitrary. This implies that $f \in H_{\tilde{F}_0}^\infty(\Omega)$, thus $H_F^\infty(\Omega) \subseteq H_{\tilde{F}_0}^\infty(\Omega)$ and the set equality $H_F^\infty(\Omega) = H_{\tilde{F}_0}^\infty(\Omega)$ is thereby established.

It remains to show that the corresponding topologies coincide as well. Since Ω is a domain, for a fixed point z_0 in Ω the space $H_F^\infty(\Omega)$ is topologized via the seminorms:

$$\sup_{z \in \Omega} |f^{(l)}(z)|, \text{ for } l \in F \quad \text{and} \quad |f^{(l)}(z_0)|, \text{ for } 0 \leq l < \min F,$$

while the topology of $H_{\tilde{F}_0}^\infty(\Omega)$ is induced by the seminorms:

$$\sup_{z \in \Omega} |f^{(l)}(z)|, \text{ for } l \in \tilde{F}_0.$$

Clearly, every seminorm from the first topology is bounded by some seminorm from the second topology. Hence, the topology of $H_{\tilde{F}_0}^\infty(\Omega)$ is finer than the topology of $H_F^\infty(\Omega)$. For the inverse inclusion, pick a sequence $(f_n)_n$ in $H_{\tilde{F}_0}^\infty(\Omega)$ and an $f \in H_F^\infty(\Omega)$ such that $f_n \rightarrow f$ in the topology of $H_{\tilde{F}_0}^\infty(\Omega)$. In order to show that $f_n \rightarrow f$ in the topology of $H_F^\infty(\Omega)$, it suffices to prove that $f_n^{(l)} \rightarrow f^{(l)}$ uniformly on Ω for all $l \in \tilde{F}_0$. This is already true for $l \in F$, which leaves the case where $l \notin F$. If $l < \min F$, then use Corollary 2.2 for $(f_n^{(b)})_n$ and $f^{(b)}$, where $b = \min F$, to obtain the result. If $l > \min F$, then choose $b = \min\{n \in F : n > l\}$ and use the same reasoning, after using Weierstrass's theorem to guarantee that $f_n^{(b-1)}(z_0) \rightarrow f^{(b-1)}(z_0)$. It follows that these topologies are indeed the same, completing the proof. ■

Remark 3.8. Another proof of the equivalence of these topologies can be given by using the Open Mapping theorem for Fréchet spaces.

Remark 3.9. In [19], see also [13], a Jordan domain $\Omega \subseteq \mathbb{C}$ has been constructed, supporting a bounded holomorphic function g , so that its primitive G is unbounded. Thus, for this domain Ω , the spaces $H_F^\infty(\Omega)$ and $H_{\tilde{F}_0}^\infty(\Omega)$ are different for some non-empty set $F \subseteq \mathbb{N}_0$ such that $0 \notin F$ and $1 \in F$. This is certainly true for $F = \{1\}$ and $\tilde{F}_0 = \{0, 1\}$.

Suppose that $\Omega \subseteq \mathbb{C}$ is a simply connected domain for which a constant $0 < M < +\infty$ exists with the property that any two points $p, q \in \Omega$ can be joined by a rectifiable curve $\gamma_{p,q}$ in Ω with length bounded by M (then, Ω is bounded with $\text{diam}(\Omega) \leq M$). Equivalently, Ω has finite interior diameter:

$$\text{diam}_I(\Omega) = \sup_{p,q \in \Omega} \inf_{\gamma \in \Gamma(p,q)} \text{length}(\gamma) < +\infty,$$

where $\Gamma(p, q)$ is the set of all rectifiable curves in Ω joining the points p and q . Examples of such sets Ω are given by convex or star-like bounded domains and many others. Then, Theorem 3.7 is still valid for such a domain Ω , because if f is a bounded holomorphic function on Ω , its primitive F is bounded by $M \cdot \sup_{z \in \Omega} |f(z)| + |F(z_0)|$, where z_0 is a fixed point in Ω . This condition has been used in [19]. More recently, it has been proven in [22] that this condition is necessary and sufficient for a simply connected domain Ω , in order for the primitive of any bounded holomorphic function on Ω to be also bounded; this condition is equivalent to the boundedness of the integration operator.

This discussion furnishes the proof for the following:

Theorem 3.10. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain with finite interior diameter and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, $H_F^\infty(\Omega) = H_{\tilde{F}_0}^\infty(\Omega)$ as sets and as topological spaces, where $\tilde{F}_0 = \{l \in \mathbb{N}_0 : 0 \leq l \leq \sup F\}$.*

We close this section with some remarks about Lemma 3.1:

Remark 3.11. The result obtained in Lemma 3.1 can be strengthened in the following sense: if $\Omega \subseteq \mathbb{C}$ is an open, unbounded, convex set, then it is the union of open half-lines which are parallel to each other. Indeed, pick $p, (z_n)_n, (h_n)_n$ as in the proof of Lemma 3.1, let $h = \lim h_{k_n}$ for some subsequence $(h_{k_n})_n$ of $(h_n)_n$ and construct the open half-line $L_{p,h} = \{p + th : t > -r_1\}$ for some $r_1 > 0$ such that $L_{p,h} \subseteq \Omega$. This half-line is contained in Ω , passes through the point p and is parallel to h . Now, pick any point q in Ω . Since $z_n \rightarrow \infty$, we have that $z_{k_n} \neq q$ for all but finitely many $n \in \mathbb{N}$. Considering the sequence $d_n = \frac{z_{k_n} - q}{|z_{k_n} - q|}$, for large enough n , we have that $d_n \rightarrow h$, since a short calculation shows that:

$$h_n - d_n = \frac{z_{k_n} - p}{|z_{k_n} - p|} - \frac{z_{k_n} - q}{|z_{k_n} - q|} \rightarrow 0.$$

Hence, the open half-line $L_{q,h} = \{q + th : t > -r_2\}$ for a sufficiently small $r_2 > 0$ is contained in Ω , passes through point q and is parallel to h .

This means that h can be thought of as a direction of the set Ω . Also, if we consider the set of all open half-lines contained in Ω emanating from a fixed point p , then this set defines, in general, an open cone contained in Ω with a certain angle θ . The only exception is the case of an open strip, where the above set contains only two opposite half-lines. Clearly, θ is independent of the choice of the starting point p . If θ is greater than 180° , then $\Omega = \mathbb{C}$ and if $\theta = 180^\circ$, then Ω is a open half-plane. If θ is less than 180° , not much can be said about the geometry of Ω ; it could be an open strip, the interior of an angle, the interior of a parabola or the interior of the branch of a hyperbola etc.

Remark 3.12. The inverse of Lemma 3.1 does not hold, an open set $\Omega \subseteq \mathbb{C}$ which is the union of open half-lines is certainly unbounded, but not necessarily convex (not even star-like), even if Ω is assumed to be connected; this fact is illustrated by picking $\Omega = \{z \in \mathbb{C} : |z| > 1\}$.

Furthermore, Lemma 3.1 and Remark 3.11 are also valid for any open, unbounded, convex set $\Omega \subseteq \mathbb{R}^d$ or \mathbb{C}^d , $d \in \mathbb{N}$, since the unit sphere is compact in each case. However, the compactness of the unit sphere, which is equivalent to the finiteness of the dimension of the ambient space, is essential for Lemma 3.1 and Remark 3.11 to hold, since neither of these statements is true in general if such an Ω is a subset of an infinite-dimensional normed vector space. In fact, a convex subset of a normed vector space X contains a half-line (ray) if and only if the dimension of X is finite; see [16], [2], [3], [10] and [4]. We will neither concern ourselves with these type of results in this paper, nor will we provide counter-examples.

Remark 3.13. If $\Omega \subseteq \mathbb{C}$ is a closed, unbounded, convex set, then it is the union of closed half-lines, since by repeating the proof of Lemma 3.1 and replacing $2t$ with t , we have that $w_n \in \Omega$ for large enough n and that $w_n \rightarrow p + th$, thus $p + th \in \Omega$ by the closedness of Ω . Clearly, the last step of that proof cannot be repeated when dealing with boundary points of Ω , though it can be repeated for its interior points. Thus, for any point $p \in \Omega$ we can find a closed half-line entirely contained in Ω , containing the point p , and if p is an interior point of Ω , then the half-line can be chosen to be open. However, if Ω is assumed neither open, nor closed, then Lemma 3.1 does not hold in general; for instance, let Ω be the strip $\{z \in \mathbb{C} : 0 < \text{Im}z < 1\} \cup \{0\}$ which has 0 as a boundary point, but no half-line entirely contained in Ω exists containing 0 . In light of this remark, one can easily formulate results analogous to Lemma 3.1 and Remarks 3.11 and 3.12 for closed, unbounded, convex sets.

4. ON $\mathcal{A}_F(\Omega)$ FOR Ω OPEN AND CONVEX

In this section, we prove that analogues of Theorems 3.6 and 3.7 are valid for the space $\mathcal{A}_F(\Omega)$, where $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ is a non-empty set and Ω is a convex domain. One would expect that $\mathcal{A}_F(\Omega) = \mathcal{A}_{\tilde{F}}(\Omega)$ when Ω is unbounded and $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$ and that $\mathcal{A}_F(\Omega) = \mathcal{A}_{\tilde{F}_0}(\Omega)$ when Ω is bounded and $\tilde{F}_0 = \{l \in \mathbb{N}_0 : 0 \leq l \leq \sup F\}$. We will actually show that $\mathcal{A}_F(\Omega) = \mathcal{A}_{\tilde{F}}(\Omega) = \mathcal{A}_{\tilde{F}_0}(\Omega)$ for any convex domain $\Omega \subseteq \mathbb{C}$, regardless of whether Ω is bounded or unbounded.

Theorem 4.1. *Let $\Omega \subseteq \mathbb{C}$ be a convex domain (bounded or unbounded) and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, $\mathcal{A}_F(\Omega) = \mathcal{A}_{\tilde{F}}(\Omega)$ as sets and as topological spaces, where $\tilde{F}_0 = \{l \in \mathbb{N}_0 : 0 \leq l \leq \sup F\}$.*

Proof. Clearly $\mathcal{A}_{\tilde{F}_0}(\Omega) \subseteq \mathcal{A}_F(\Omega)$. For the inverse inclusion, let $f \in \mathcal{A}_F(\Omega)$. We will show that $f \in \mathcal{A}_{\tilde{F}}(\Omega)$. To this end, we pick $l \in \tilde{F}_0$ and we will show that $f^{(l)} \in \mathcal{A}(\Omega)$. If $l \in F$ we have nothing to prove, so assume that $l \notin F$. Next, pick any $a \in F$ such that $a > l$; for instance, choose $a = \min\{n \in F : n > l\}$. We have that $f^{(a)}$ is continuously extendable on $\overline{\Omega}$. If Ω is bounded, then $\overline{\Omega}$ is compact and thus $f^{(a)}$ is bounded on Ω . The result follows from the discussion preceding Theorem 3.7; $f^{(l)}$ is continuously extendable on $\overline{\Omega}$. If Ω is unbounded, we work in a similar way but only locally. Fix any point $\zeta \in \partial\Omega$ and consider the set $\Omega_r = \Omega \cap D(\zeta, r)$, for any fixed $r > 0$. Since $f^{(a)}$ is continuously extendable on $\overline{\Omega}$, it is also continuously extendable on the compact set $\overline{\Omega}_r$ and thus $f^{(a)}$ is bounded on the bounded convex domain Ω_r . The result follows from the previous case, that is $f^{(l)}$ is continuously extendable on $\overline{\Omega}_r$ and thus extends continuously at point ζ which was arbitrary and belongs to $\overline{\Omega}^\circ$, where the interior is relative to $\overline{\Omega}$. Hence, $f^{(l)}$ is continuously extendable on $\overline{\Omega}$ and $l \in \tilde{F}_0$ was arbitrary. This implies that $f \in \mathcal{A}_{\tilde{F}_0}(\Omega)$, thus $\mathcal{A}_F(\Omega) \subseteq \mathcal{A}_{\tilde{F}_0}(\Omega)$ and the set equality $\mathcal{A}_F(\Omega) = \mathcal{A}_{\tilde{F}_0}(\Omega)$ is thereby established.

It remains to show that the corresponding topologies coincide as well. This can be done by hand by showing that they have the same convergent sequences, after comparing the seminorms in both topologies. However, it is shorter to consider the identity map $I : \mathcal{A}_{\tilde{F}}(\Omega) \rightarrow \mathcal{A}_F(\Omega)$, which is a continuous bijection among two Fréchet spaces, hence an isomorphism. The proof is complete. ■

Remark 4.2. For the Jordan domain $\Omega \subseteq \mathbb{C}$ mentioned in Remark 3.9, we have that the function g constructed in [19] is continuously extendable on $\overline{\Omega}$, but its primitive G is not, since G is unbounded on Ω and $\overline{\Omega}$ is compact. Thus, for this domain Ω , the spaces $\mathcal{A}_F(\Omega)$ and $\mathcal{A}_{\tilde{F}_0}(\Omega)$ are different for some non-empty set $F \subseteq \mathbb{N}_0$ such that $0 \notin F$ and $1 \in F$. This is certainly true for $F = \{1\}$ and $\tilde{F}_0 = \{0, 1\}$.

5. NON-EXTENDABILITY IN $H_F^\infty(\Omega)$ AND $\mathcal{A}_F(\Omega)$

In this section, we deal with the notion of non-extendability of holomorphic functions in the spaces $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$, where $\Omega \subseteq \mathbb{C}$ is an open set and $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ is a non-empty set.

We give the following preliminary definition:

Definition 5.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function. We say that f is extendable (in the sense of Riemann surfaces) if there exist two open discs D_1 and D_2 , such that $D_2 \cap \Omega \neq \emptyset$, $D_2 \cap \Omega^c \neq \emptyset$ and $D_1 \subseteq \overline{D_1} \subseteq D_2 \cap \Omega$ and a (bounded) holomorphic

function $F : D_2 \rightarrow \mathbb{C}$, such that $F|_{D_1} = f|_{D_1}$. Otherwise, we say that f is non-extendable, or that it is holomorphic exactly on Ω .

Remark 5.2. The reason for which F can be chosen to be bounded is that, if needed, we can replace the disc D_2 by another disc D'_2 compactly contained in D_2 , that is $\overline{D'_2} \subseteq D_2$, such that $\overline{D_1} \subseteq D'_2 \cap \Omega$. Also, in Definition 5.1, one can replace D_2 with any non-empty domain $U \subseteq \mathbb{C}$ satisfying $U \cap \Omega \neq \emptyset$ and $U \cap \Omega^c \neq \emptyset$ and replace D_1 with a connected component V of $U \cap \Omega$, resulting in an equivalent definition of extendability; see [18] for a proof of this fact.

The next theorem can be found in [18], proven using Baire's and Montel's theorems:

Theorem 5.3. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $H(\Omega)$ be the set of holomorphic functions on Ω . Also, let $X(\Omega) \subseteq H(\Omega)$ be a topological vector space endowed with the usual operations $+, \cdot$, whose topology is induced by a complete metric. Suppose that the convergence $f_n \rightarrow f$ in $X(\Omega)$ implies the pointwise convergence $f_n(z) \rightarrow f(z)$, for all $z \in \Omega$. Then, there exists an $F \in X(\Omega)$ which is non-extendable, if and only if, for any two discs D_1 and D_2 as in Definition 5.1, there exists a function $F_{D_1, D_2} \in X(\Omega)$ so that the restriction $F_{D_1, D_2}|_{D_1}$ on D_1 does not possess a (bounded) holomorphic extension on D_2 . If the previous assumptions hold, then the set $S = S(X(\Omega)) = \{f \in X(\Omega) : f \text{ is non-extendable}\}$ is a dense and G_δ subset of $X(\Omega)$.*

Corollary 5.4. *The set $S(X(\Omega))$ from Theorem 5.3 is always a G_δ subset of $X(\Omega)$, because either $S(X(\Omega)) = \emptyset$, or $S(X(\Omega)) \neq \emptyset$ and then it is dense and G_δ from the previous theorem.*

An immediate corollary of Proposition 2.1 is the following:

Proposition 5.5. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $F \subseteq \mathbb{N}_0$ be a non-empty set. Also, let f_n, f , $n \in \mathbb{N}$, be holomorphic functions on Ω . If either:*

- (i) $f_n, f \in H_F^\infty(\Omega)$ for $n \in \mathbb{N}$ and $f_n \rightarrow f$ in the topology of $H_F^\infty(\Omega)$, or
 - (ii) $f_n, f \in \mathcal{A}_F(\Omega)$ for $n \in \mathbb{N}$ and $f_n \rightarrow f$ in the topology of $\mathcal{A}_F(\Omega)$,
- then $f_n \rightarrow f$ uniformly on compact subsets of Ω , hence $f_n \rightarrow f$ pointwise on Ω .

This enables us to prove these two generic results:

Theorem 5.6. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, the set $S(H_F^\infty(\Omega))$ of functions in $H_F^\infty(\Omega)$ which are non-extendable is either void, or a dense and G_δ subset of $H_F^\infty(\Omega)$.*

Proof. Assume that $S(H_F^\infty(\Omega)) \neq \emptyset$. By combining the completeness of the metric space $H_F^\infty(\Omega)$ with condition (i) of Proposition 5.5, we easily deduce that the assumptions of Theorem 5.3 for $X(\Omega) = H_F^\infty(\Omega) \subseteq H(\Omega)$ are verified. Thus, $S(H_F^\infty(\Omega))$ is dense and G_δ in $H_F^\infty(\Omega)$. ■

Theorem 5.7. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, the set $S(\mathcal{A}_F(\Omega))$ of functions in $\mathcal{A}_F(\Omega)$ which are non-extendable is either void, or a dense and G_δ subset of $\mathcal{A}_F(\Omega)$.*

Proof. Similar to the proof of Theorem 5.6. Use the completeness of the metric space $\mathcal{A}_F(\Omega)$ along with condition (ii) of Proposition 5.5. The result follows from Theorem 5.3 for $X(\Omega) = \mathcal{A}_F(\Omega) \subseteq H(\Omega)$. ■

Next, we give examples and examine in each one of them whether $S(H_F^\infty(\Omega))$ and $S(\mathcal{A}_F(\Omega))$ are empty or not:

Example 5.8. Let $U \subseteq \mathbb{C}$ be a domain and $K \subseteq U$ be a compact set which is removable for bounded holomorphic functions; that is, its analytic capacity $\gamma(K)$ is zero. For instance, K

could be a singleton or a planar Cantor-type set, obtained by removing corner quarters; see [8]. Let $\Omega = U \setminus K$. Then, it is easy to see that every $f \in H_F^\infty(\Omega)$ is extendable for any choice of F , provided that $0 \in F$. Thus, $S(H_F^\infty(\Omega)) = \emptyset$.

Example 5.9. Let $U \subseteq \mathbb{C}$ be a domain and $K \subseteq U$ be a compact set with continuous analytic capacity $a(K)$ equal to zero. Let $\Omega = U \setminus K$. Then, it is easy to see that every $f \in \mathcal{A}_F(\Omega)$ is extendable for any choice of F , provided that $0 \in F$. Thus, $S(\mathcal{A}_F(\Omega)) = \emptyset$.

If we use a result from [7], [5], we see that the same holds if K is any closed subset of \mathbb{C} with $a(K) = 0$; for instance, K could be a straight line, a line segment, a circular arc, a circle, an analytic curve or the boundary of a convex set.

Example 5.10. Let $U \subseteq \mathbb{C}$ be a domain and $K \subseteq U$ be a singleton, or more generally, a compact set containing an isolated point. Let $\Omega = U \setminus K$. Then, every holomorphic function f which belongs to $H_F^\infty(\Omega)$ or $\mathcal{A}_F(\Omega)$ is extendable for any choice of F . Thus, $S(H_F^\infty(\Omega)) = S(\mathcal{A}_F(\Omega)) = \emptyset$.

Indeed, let $a = \min F$ and ζ be an isolated point of K . If $f \in H_F^\infty(\Omega)$, then by Riemann's theorem on removable singularities, $f^{(a)}$ is holomorphic on a sufficiently small disc $D(\zeta, r)$ contained in Ω , for some $r > 0$. If $f \in \mathcal{A}_F(\Omega)$, then $f^{(a)}$ is again holomorphic on $D(\zeta, r)$. In any case, since this disc is a bounded convex domain, by integrating $f^{(a)}$ repeatedly we conclude that f is extendable on $D(\zeta, r)$. Hence $S(H_F^\infty(\Omega)) = S(\mathcal{A}_F(\Omega)) = \emptyset$.

Example 5.11. Let $\Omega \subseteq \mathbb{C}$ be a domain, such that every point $\zeta \in \partial\Omega$ is the limit of a sequence $(z_n)_n$ of points contained in $(\overline{\Omega})^c$. Then, for any choice of F , the sets $S(H_F^\infty(\Omega))$ and $S(\mathcal{A}_F(\Omega))$ are dense and G_δ in $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$, respectively.

We will use a result from [18] regarding non-extendability which was stated in Theorem 5.3. Pick any two discs D_1 and D_2 as in Definition 5.1 and a point $\zeta \in \partial\Omega \cap D_2$. By our assumption, there exists a point $w \in D_2 \setminus \overline{\Omega}$. Consider the function $f(z) = \frac{1}{z-w}$ which belongs to $H_F^\infty(\Omega) \cap \mathcal{A}_F(\Omega)$. Since this function restricted to D_1 is equal to $f|_{D_1}(z) = \frac{1}{z-w}$, by analytic continuation we have that its only holomorphic extension on $D_2 \setminus \{w\}$ is the function $g(z) = \frac{1}{z-w}$ which has a pole at $w \in D_2$. Thus, f does not possess a holomorphic extension on D_2 . According to Theorem 5.3 we have that $S(H_F^\infty(\Omega)) \neq \emptyset$ and $S(\mathcal{A}_F(\Omega)) \neq \emptyset$, therefore the sets $S(H_F^\infty(\Omega))$ and $S(\mathcal{A}_F(\Omega))$ are dense and G_δ in $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$, respectively, as Theorems 5.6 and 5.7 indicate.

Example 5.12. Let $\Omega \subseteq \mathbb{C}$ be a domain bounded by a finite set of disjoint Jordan curves. Then, for any choice of F , the sets $S(H_F^\infty(\Omega))$ and $S(\mathcal{A}_F(\Omega))$ are dense and G_δ in $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$, respectively. Clearly, this is a particular case of Example 5.11.

6. TWO MORE DICHOTOMY RESULTS

In this section, we prove two more dichotomy results regarding boundedness or unboundedness of derivatives of functions in the spaces $H_F^\infty(\Omega)$ and $\mathcal{A}_F(\Omega)$, where $\Omega \subseteq \mathbb{C}$ is an open set and $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ is a non-empty set.

We will use the following result from [21]:

Proposition 6.1. *Let V be a topological vector space over the field \mathbb{R} or \mathbb{C} and let X be a non-empty set. We denote by \mathbb{C}^X the set of all complex valued functions on X and consider a linear operator $T : V \rightarrow \mathbb{C}^X$ with the property that the mapping $V \ni a \mapsto T_x(a) = T(a)(x) \in \mathbb{C}$ is continuous, for all $x \in X$; observe that this assumption is weaker than T being continuous.*

Let $S = S(T, V, X) = \{a \in V : T(a) \text{ is unbounded on } X\}$. Then, either $S = \emptyset$, or S is a dense and G_δ subset of V .

Note that in Proposition 6.1, V is not assumed to be a complete metric space.

Let $\Omega \subseteq \mathbb{C}$ be an open set and let V be one of the topological vector spaces $H_F^\infty(\Omega)$ or $\mathcal{A}_F(\Omega)$ endowed with its natural topology, where F is a non-empty subset of \mathbb{N}_0 . Let X be any subset of Ω and fix an element l of \mathbb{N}_0 . Then, the function $V \ni f \mapsto T_l(f)(z) = f^{(l)}(z) \in \mathbb{C}$ is continuous for all $z \in X$; this follows from Weierstrass's theorem if $l \geq \min F$ and from Proposition 2.1 if $0 \leq l < \min F$. By Proposition 6.1, the corresponding set $S = S_l$ is either empty, or dense and G_δ in the space V . In particular, the above holds true for $V = H_F^\infty(\Omega)$ and $X = \Omega$. Therefore, the set $S_l = \{f \in H_F^\infty(\Omega) : f^{(l)} \text{ is unbounded on } \Omega\}$ is either void, or dense and G_δ in $H_F^\infty(\Omega)$.

This constitutes the proof of the first dichotomy result of this section:

Theorem 6.2. *Let $\Omega \subseteq \mathbb{C}$ be an open set, $F \subseteq \mathbb{N}_0$ be a non-empty set and $l \in \mathbb{N}_0$. Then, either for every $f \in H_F^\infty(\Omega)$ the derivative $f^{(l)}$ is bounded on Ω , or generically for every $f \in H_F^\infty(\Omega)$ the derivative $f^{(l)}$ is unbounded on Ω . That is, either $H_{F \cup \{l\}}^\infty(\Omega) = H_F^\infty(\Omega)$, or $H_{F \cup \{l\}}^\infty(\Omega)$ is meager in $H_F^\infty(\Omega)$.*

If $l \in F$, then obviously for every $f \in H_F^\infty(\Omega)$ the derivative $f^{(l)}$ is bounded on Ω . If Ω is a bounded convex domain and $l \leq \sup F$, then for every $f \in H_F^\infty(\Omega)$ the derivative $f^{(l)}$ is bounded on Ω , according to Theorem 3.7.

Next, we give an example where the other horn of the previous dichotomy holds. In [19], see also [13], a Jordan domain Ω was constructed such that a function $g : \overline{\Omega} \rightarrow \mathbb{C}$ continuous on $\overline{\Omega}$ and holomorphic on Ω has an unbounded primitive on Ω . Let us call this primitive G ; then $G \in H_{\{1\}}^\infty(\Omega)$, but for $l = 0$ the function $G^{(0)} = G$ is unbounded on Ω . Thus, in this domain Ω , generically every function $f \in H_{\{1\}}^\infty(\Omega)$ has the property that $f^{(0)} = f$ is unbounded on Ω . It follows that $H_{\{0,1\}}^\infty(\Omega)$ is meager in $H_{\{1\}}^\infty(\Omega)$ for this particular domain Ω .

Let $\Omega = \mathbb{D}$ be the open unit disc and let $w(z) = (z - 1) \cdot \exp \frac{z+1}{z-1}$. Then $w \in \mathcal{A}(\mathbb{D}) \subseteq H^\infty(\mathbb{D}) = H_{\{0\}}^\infty(\mathbb{D})$ and w' is unbounded on \mathbb{D} , thus generically every function $f \in H_{\{0\}}^\infty(\mathbb{D})$ has the property that $f^{(1)} = f'$ is unbounded on \mathbb{D} . It follows that $H_{\{0,1\}}^\infty(\mathbb{D})$ is meager in $H_{\{0\}}^\infty(\mathbb{D})$. More generally, if F is finite and $l > \max F$, then generically for every $f \in H_F^\infty(\mathbb{D})$ the derivative $f^{(l)}$ is unbounded on \mathbb{D} and $H_{F \cup \{l\}}^\infty(\mathbb{D})$ is meager in $H_F^\infty(\mathbb{D})$.

It remains open to give an example of a domain $\Omega \subseteq \mathbb{C}$ supporting a holomorphic function f such that $f^{(0)} = f$ and $f^{(2)} = f''$ are bounded on Ω , but $f^{(1)} = f'$ is unbounded. We believe that such a domain Ω exists. Moreover, we think that a complete metric topology can be defined on the set of all domains (contained in the open unit disc), so that for the generic domain Ω there exists a holomorphic function f on Ω such that f and f'' are bounded, but f' is not. More generally, we think that for every non-empty set $F \subseteq \mathbb{N}_0$ and $l \notin F$, $\min F < l < \sup F$, for the generic domain $\Omega \subseteq \mathbb{C}$ there exists an $f \in H_F^\infty(\Omega)$ such that $f^{(l)}$ is unbounded and $H_{F \cup \{l\}}^\infty(\Omega)$ is meager in $H_F^\infty(\Omega)$. But we do not have a proof and these assertions remain open.

Next, consider the space $\mathcal{A}_F(\Omega)$, where $\Omega \subseteq \mathbb{C}$ is an open set and F is a non-empty subset of \mathbb{N}_0 . If Ω is bounded, then for every $f \in \mathcal{A}_F(\Omega)$ all derivatives $f^{(l)}$, $l \in F$, are also bounded on Ω . Assume that Ω is unbounded, then we apply the result obtained during the discussion coming after Proposition 6.1, this time for $V = \mathcal{A}_F(\Omega)$ and $X = \Omega$. Therefore, the set $S_l = \{f \in \mathcal{A}_F(\Omega) : f^{(l)} \text{ is unbounded on } \Omega\}$, where l is a fixed element of \mathbb{N}_0 , is either void, or dense and G_δ in $\mathcal{A}_F(\Omega)$. But the function $f(z) = z^{l+1}$ belongs to $\mathcal{A}_F(\Omega)$. Therefore, the set S_l is

dense and G_δ in $\mathcal{A}_F(\Omega)$. Baire's theorem implies that the set $S = \bigcap_{l \in \mathbb{N}_0} S_l$ is also dense and G_δ in $\mathcal{A}_F(\Omega)$.

This discussion serves as the proof of the second dichotomy result of this section:

Proposition 6.3. *Let $\Omega \subseteq \mathbb{C}$ be an unbounded open set and $F \subseteq \mathbb{N}_0$ be a non-empty set. Then, the set S of functions $f \in \mathcal{A}_F(\Omega)$ such that all derivatives $f^{(l)}$, $l \in \mathbb{N}_0$, are unbounded on Ω , is dense and G_δ in $\mathcal{A}_F(\Omega)$.*

More generally, if Ω is an unbounded open set and $(z_n)_n$ is a sequence of points in Ω converging to ∞ , then for $X = \{z_n : n \in \mathbb{N}\}$ generically every function $f \in \mathcal{A}_F(\Omega)$ has the property that $f^{(l)}$ is unbounded on X for all $l \in \mathbb{N}_0$. To give an explicit example of such a function $f \in \mathcal{A}_F(\mathbb{C}) = H(\mathbb{C})$, it suffices to set $f(z) = \exp(e^{-i\theta}z)$ for some well-chosen $\theta \in \mathbb{R}$. Indeed, let $c_n = \frac{z_n}{|z_n|}$ and let $c_{k_n} \rightarrow c$ for a subsequence. Then, $|c| = 1$ and pick a $\theta \in \mathbb{R}$ such that $c = e^{-i\theta}$. Notice that $|f^{(l)}(z)| = |f(z)|$ for all $z \in \mathbb{C}$ and $l \in \mathbb{N}_0$ and that $|f(z_{k_n})| \rightarrow +\infty$.

7. A STRONGER RESULT

One of the referees of this paper suggested that instead of Landau-Kolmogorov's inequality we can use a weaker inequality and obtain stronger results than those of Section 3. For the sake of completeness, we include a short existential proof of this weak inequality. Furthermore, the main theorem of this section guides us towards a possible counter-example of an open set $\Omega \subseteq \mathbb{C}$ for which $H_F^\infty(\Omega) \neq H_F^\infty(\Omega)$, where $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ is a non-empty set.

Let $n \geq 1$ be a fixed natural number. We consider the set $C^n([0, 1])$ of all functions $f : [0, 1] \rightarrow \mathbb{C}$ which are n -times continuously differentiable on $[0, 1]$. The usual topology on $C^n([0, 1])$ is defined by the norm:

$$N_1(f) = \|f^{(0)}\|_\infty + \|f^{(1)}\|_\infty + \dots + \|f^{(n)}\|_\infty,$$

where $\|g\|_\infty = \sup_{x \in [0, 1]} |g(x)|$. We denote by X the space $(C^n([0, 1]), N_1)$ which is well-known to be a Banach space. Next, we consider a different norm:

$$N_2(f) = \|f^{(0)}\|_\infty + \|f^{(n)}\|_\infty.$$

The space $(C^n([0, 1]), N_2)$ is denoted by Y .

Proposition 7.1. *The space Y is complete, therefore a Banach space.*

Proof. Consider a Cauchy sequence $(f_m)_m$ in Y . It follows easily that there are two functions $u, v : [0, 1] \rightarrow \mathbb{C}$ such that $f_m \rightarrow u$ and $f_m^{(n)} \rightarrow v$ uniformly on $[0, 1]$. Obviously, v is continuous. It suffices to prove that u is n -times (continuously) differentiable on $[0, 1]$ and that $u^{(n)} = v$. We consider the integration operator $T : C([0, 1]) \rightarrow C([0, 1])$ given by:

$$T(h)(x) = \int_0^x h(t) dt.$$

Clearly, T is bounded. Since $f_m^{(n)} \rightarrow v$ uniformly on $[0, 1]$, it follows that $T^n(f_m^{(n)}) \rightarrow T^n(v)$ uniformly on $[0, 1]$, where T^n is the composition of T with itself n -times. Also, $T^n(v)$ is n -times differentiable on $[0, 1]$ and $(T^n(v))^{(n)} = v$. We have that $T^n(f_m^{(n)}) = f_m + P_m$, where P_m

is a polynomial of degree at most $n - 1$. Since $f_m \rightarrow u$ and $T^n(f_m^{(n)}) \rightarrow T^n(v)$, it follows that $P_m \rightarrow T^n(v) - u$ uniformly on $[0, 1]$. But the space of polynomials of degree at most $n - 1$ is finite-dimensional and hence complete. It follows that $T^n(v) - u = P$, where P is a polynomial of degree at most $n - 1$. Therefore, $u = T^n(v) - P$ which is n -times differentiable and $v^{(n)} = (T^n(v))^{(n)} - P^{(n)} = v$ because $P^{(n)} \equiv 0$. This completes the proof. ■

Corollary 7.2. *The norms N_1 and N_2 are equivalent and there is a constant $C < +\infty$ such that:*

$$\|f^{(l)}\|_\infty \leq C \left(\|f^{(0)}\|_\infty + \|f^{(n)}\|_\infty \right)$$

for all $f \in C^n([0, 1])$ and l satisfying $0 < l < n$.

Proof. We consider the identity map $I : X \rightarrow Y$. Since $N_2(f) \leq N_1(f)$ for all $f \in C^n([0, 1])$, the Open Mapping theorem for the Banach spaces X, Y yields that I is an isomorphism. The result easily follows. ■

By the change of variable $t \rightarrow \delta t$, $\delta > 0$, we easily obtain the following:

Corollary 7.3. *Let $\delta > 0$. Then, there exists a constant $C_\delta < +\infty$ such that if f is n -times continuously differentiable on a closed segment $I \subseteq \mathbb{C}$ of length $|I| = \delta$, then*

$$\|f^{(l)}\|_\infty \leq C_\delta \cdot \left(\|f^{(0)}\|_\infty + \|f^{(n)}\|_\infty \right)$$

for all l satisfying $0 < l < n$, where $\|g\|_\infty = \sup_{z \in I} |g(z)|$. Moreover, if $0 < \delta < \delta_0$, then $C_{\delta_0} \leq C_\delta$.

Proof. Only the last statement needs an explanation. If $|I| = \delta_0 > \delta$, then for every $z \in I$ there exists $w \in I$ such that $|z - w| = \delta$. We consider the closed line segment $J = [z, w] \subset I$ with length $|J| = \delta$. Then, for any l such that $0 < l < n$ and any $f \in C^n(I) \subseteq C^n(J)$ we have that:

$$|f^{(l)}(z)| \leq C_\delta \cdot \left(\sup_{\zeta \in J} |f^{(0)}(\zeta)| + \sup_{\zeta \in J} |f^{(n)}(\zeta)| \right).$$

Taking supremum over all $z \in I$, we obtain that $C_{\delta_0} \leq C_\delta$. ■

The following is proven using the methods of Section 3 and Corollary 7.3:

Theorem 7.4. *Let $\Omega \subseteq \mathbb{C}$ be a domain and $F \subseteq \mathbb{N}_0 = \{0, 1, \dots\}$ be a non-empty set. Suppose that there exists $\delta > 0$ with the property that every $z \in \Omega$ is contained in a closed segment $I_z \subseteq \Omega$ with length $|I_z| \geq \delta$, that is with length uniformly bounded away from zero. Then, $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$ as sets and as topological spaces, where $\tilde{F} = \{l \in \mathbb{N}_0 : \min F \leq l \leq \sup F\}$.*

Remark 7.5. Any star-like open subset Ω of \mathbb{C} and many other sets satisfy the assumptions of Theorem 7.4 and therefore $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega)$; in particular, this holds for all convex domains $\Omega \subseteq \mathbb{C}$. Of course if Ω is a bounded convex domain, then $H_F^\infty(\Omega) = H_{\tilde{F}}^\infty(\Omega) = H_{\tilde{F}_0}^\infty(\Omega)$. Finally, a candidate for an example of a domain Ω such that $H_F^\infty(\Omega) \neq H_{\tilde{F}}^\infty(\Omega)$ for some non-empty $F \subseteq \mathbb{N}_0$ is a Jordan domain spiraling to a point, as the Jordan domain Ω constructed in [19] which supports a function $f \in \mathcal{A}(\Omega)$ whose primitive is unbounded.

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DEPARTMENT OF MATHEMATICS, NATIONAL AND KAPODISTRIAN UNIVERSITY OF ATHENS, PANEPISTEMIOPO-
LIS, 15784 ATHENS, GREECE.

E-mail address: dmoschon@math.uoa.gr

DEPARTMENT OF MATHEMATICS, NATIONAL AND KAPODISTRIAN UNIVERSITY OF ATHENS, PANEPISTEMIOPO-
LIS, 15784 ATHENS, GREECE.

E-mail address: vnestor@math.uoa.gr