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DIRECTION SETS, LIPSCHITZ GRAPHS AND DENSITY

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Abstract. We consider the direction set $\tilde{D}(E)$ determined by various subsets E of Euclidean space and show that there is a trichotomy: Either (i) E is not the graph of a function with respect to any axis-system in which case it determines all directions, (ii) E is the graph of a Lipschitz function with respect to some axis-system, in which case $\tilde{D}(E)$ is not dense in S^{d-1} or (iii) there are axis systems such that E is the graph of a function and in all such axis systems, the functions obtained are non-Lipschitz, in which case $\tilde{D}(E)$ is a dense, proper subset of S^{d-1}. We then explore a variety of results based on this trichotomy under additional assumptions on the set E.

1. Introduction

The purpose of this paper is to study direction sets determined by subsets of Euclidean space. Informally, direction sets consist of direction vectors determined by pairs of vectors from a given set. More precisely, we have the following definitions:

Definition 1.1. Fix a subset $E \subseteq \mathbb{R}^d, d \geq 2$. The (oriented) direction set $\tilde{D}(E)$ determined by E is the set

$$\tilde{D}(E) = \left\{ \frac{y-x}{|y-x|} : x, y \in E, x \neq y \right\} \subseteq S^{d-1}.$$

We say that E determines all directions, or a dense subset of directions, respectively, if $\tilde{D}(E) = S^{d-1}$ or $\tilde{D}(E)$ is a dense subset of S^{d-1}, respectively.

It was shown by the first listed author, Mourgoglou and Senger ([1]; see also [2], Theorem 10.11) that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $d-1$, then the $(d-1)$-dimensional Lebesgue measure of $\tilde{D}(E)$, viewed as a subset of S^{d-1}, is positive. They also obtained a rather precise description of the distribution of these directions. In this paper we turn in a slightly different direction and obtain rather comprehensive qualitative information about the structure of subsets of $\mathbb{R}^d$ for which the direction set is not dense in the sphere.

It is often convenient to consider unoriented directions, which are elements of S^{d-1} taken modulo the antipodal action $a(x) = -x$. It is well known that the quotient of the sphere S^{d-1} under the above action is the projective space $\mathbb{R}P^{d-1}$ whose elements can be thought of as either pairs of antipodal points on the sphere, $\pm\hat{u}$ or lines through the origin in $\mathbb{R}^d$. The natural map $\pi : S^{d-1} \to \mathbb{R}P^{d-1}$ is a double cover map such that for each line $L \in \mathbb{R}P^{d-1}$, $\pi^{-1}(L)$ is the pair of antipodal points where the line L intersects the sphere.

Definition 1.2. Let $E \subseteq \mathbb{R}^d, d \geq 2$. The unoriented direction set $D(E)$ determined by E is the image of the oriented direction set $\tilde{D}(E)$ under the double cover map

$$\pi : S^{d-1} \to \mathbb{R}P^{d-1}.$$

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Note for every set E, the oriented direction set $\tilde{D}(E) \subseteq S^{d-1}$ can be seen to be invariant under the antipodal map a. Due to this it is easily seen that $\tilde{D}(E) = S^{d-1}$ if and only if $D(E) = \mathbb{R}P^{d-1}$ and $\tilde{D}(E)$ is dense in S^{d-1} if and only if $D(E)$ is dense in $\mathbb{R}P^{d-1}$.

In this paper when we refer to a subset of $\mathbb{R}^d$ as the graph set of a function, we will always mean that the set is a rotation of a set of the form

$$\{(x_1, \dots, x_{d-1}, f(x_1, \dots, x_{d-1}))|(x_1, \dots, x_{d-1}) \in U\}$$

for some function $f : U \to \mathbb{R}$ and $U \subseteq \mathbb{R}^{d-1}$.

The reader is cautioned that it is possible that the same set E can be the graph of a Lipschitz function with respect to some rotation but also the graph of a non-Lipschitz function with respect to another rotation. (Recall a function $f : U \to \mathbb{R}$ is Lipschitz, if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in U$.)

Example 1.3. Let $E = \{(x, \sqrt{x})|0 \leq x \leq 1\}$, then E is by definition the graph of $y = f(x) = \sqrt{x}$ over $[0, 1]$. This function is non-Lipschitz as the secant slopes near $x = 0$ can be made to limit to $lim_{t=0} f'(t) = +\infty$. On the other hand reversing the role of the axis, E is also the graph of $x = g(y) = y^2$ over $[0, 1]$ which is a Lipschitz function.

Our main result is the following.

Theorem 1.4. *Let $E \subseteq \mathbb{R}^d$. Exactly one of the following statements holds:*

i) *Up to some rotation, E is the graph of a Lipschitz function $f : A \to \mathbb{R}$ for some $A \subseteq \mathbb{R}^{d-1}$ and $D(E)$ is not dense in $\mathbb{R}P^{d-1}$.*
ii) *Up to some rotation, E is the graph of a function $f : A \to \mathbb{R}$, but that function is never Lipschitz, and $D(E)$ is dense in $\mathbb{R}P^{d-1}$ but is not equal to all of $\mathbb{R}P^{d-1}$.*
iii) *E is not a graph of a scalar valued function under any rotation and $D(E) = \mathbb{R}P^{d-1}$.*

We can say a bit more with a few extra assumptions.

Corollary 1.5. *Let E be a compact subspace of $\mathbb{R}^d$. Then exactly one of the following holds:*

i) *Up to some rotation, E is the graph of a Lipschitz function $f : A \to \mathbb{R}$ for some compact $A \subseteq \mathbb{R}^{d-1}$, and $D(E)$ is not dense in $\mathbb{R}P^{d-1}$.*
ii) *Up to some rotation, E is the graph of a continuous function $f : A \to \mathbb{R}$, but that function is never Lipschitz. A is compact, and $D(E)$ is dense in $\mathbb{R}P^{d-1}$ but is not equal to all of $\mathbb{R}P^{d-1}$.*
iii) *E is not a graph of a scalar valued function and $D(E) = \mathbb{R}P^{d-1}$.*

Using the intermediate value theorem, one can further upgrade the result when E is a compact, connected subset of $\mathbb{R}^2$.

Corollary 1.6. *Let E be a compact, connected subset of $\mathbb{R}^2$. Then exactly one of the following holds:*

i) *Up to some rotation, E is the graph of a Lipschitz function $f : [a, b] \to \mathbb{R}$, and $D(E)$ is not dense in $\mathbb{R}P^1$.*
ii) *Up to some rotation, E is the graph of a continuous function $f : [a, b] \to \mathbb{R}$, but that function is never Lipschitz, and $D(E)$ is $\mathbb{R}P^1$ minus a single point.*
iii) *E is not a graph of a scalar valued function and $D(E) = \mathbb{R}P^1$.*

Note the last corollary shows that once a compact, connected subset of the plane misses two directions, it misses a nonempty open set of directions and up to rotation, it is the graph of a Lipschitz map over a closed interval $[a, b]$ (where $a = b$ is a possibility). Also note, that it is known that when the vector space of continuous real-valued functions on the interval $[a, b]$, $a < b$ is given the sup-norm (uniform convergence norm), the nowhere differentiable continuous functions form a co-meager set (topological analog of full measure subset). Thus "most" continuous functions are nowhere differentiable (and hence not Lipschitz as Lipschitz functions are almost everywhere differentiable), and hence "most" continuous functions $f : [a, b] \to \mathbb{R}$ determine every possible secant slope in $\mathbb{R}$.

We also establish the following:

Proposition 1.7. *Let $E \subseteq \mathbb{R}^d$. Then $D(E)$ is countable if and only if E is countable or E is contained in a line in $\mathbb{R}^d$.*

Putting these results together it follows that if $E \subseteq \mathbb{R}^d, d \geq 2$ and E has Hausdorff dimension $> d - 1$, then $D(E)$ is an uncountable dense subset of $\mathbb{R}P^{d-1}$. This is because the Hausdorff dimension of a graph set of a Lipschitz function $f : A \to \mathbb{R}, A \subseteq \mathbb{R}^{d-1}$ is at most $d - 1$. It also follows immediately that if E is a compact connected subset of the plane of Hausdorff dimension > 1 then $D(E)$ misses at most one direction.

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2. Proofs of the main results

Throughout the paper, x and y are d-dimensional vectors when we are in $\mathbb{R}^d$. In two dimensions, (x, y) denotes a 2-dimensional vector.

2.1. **Proof of Proposition 1.7.** When $E \subseteq \mathbb{R}^d$ is countable or contained in a line, it is clear that $D(E)$ is countable. So let us just prove the converse. Suppose $D(E)$ is countable and fix a point $x \in E$. Then as $D(E)$ is countable, E must be contained in a countable union of lines through x. If E is countable we are done so assume it is uncountable. Then there must be a line L through x such that uncountably many elements of E lie in this line. If there was an element $e \in E$ outside that line, we would obtain uncountably many directions generated by E by noting that the lines through e and the various uncountable elements of $E \cap L$ all have different directions. Thus it follows that if E is uncountable, it must be contained in a single line.

2.2. **Proof of Theorem 1.4.** First consider $E \subset \mathbb{R}^d, d \geq 2$ with $D(E) \neq \mathbb{R}P^{d-1}$. As E fails to determine some direction, after a rotation, we may assume that E fails to determine the x_d-axis direction. The projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ then determines a bijection $E \to \pi(E) = A \subseteq \mathbb{R}^{d-1}$. Let f be the d-th coordinate function of the inverse of this bijection, it follows that $E = Graph(f) = \{(x, f(x)) | x \in A \subseteq \mathbb{R}^{d-1}\}$ is the graph set of a scalar valued-function. Conversely when E is the graph set of a scalar-valued function, by the vertical line test, it misses a direction.

Thus it follows that $D(E) = \mathbb{R}P^{d-1}$ if and only if E is not a graph set of a scalar-valued function.

Thus it suffices for the remainder of the proof to only consider the case $D(E) \neq \mathbb{R}P^{d-1}$.

Given a direction $\alpha \in \mathbb{R}P^{d-1} - D(E)$, it follows as above, that after a rotation, E is the graph of a function $f_\alpha : U_\alpha \to \mathbb{R}$. (Note that the function f_α does in general depend on the direction α chosen in $\mathbb{R}P^{d-1} - D(E)$.) We will show next that the function f_α is Lipschitz if and only if there is an open neighborhood of α in $\mathbb{R}P^{d-1}$ that does not intersect $D(E)$.

If there is an open neighborhood of α in $\mathbb{R}P^{d-1}$ which does not intersect $D(E)$, then after the rotation described above, there is an open set of lines about the x_d-axis, which do not intersect $D(Graph(f_\alpha))$.

In particular, there is an $\epsilon > 0$ such that $D(Graph(f_\alpha))$ does not contain any line within angle ϵ of the x_d-axis. It follows easily that the secants of f_α must all have slopes bounded by $tan(\frac{\pi}{2} - \epsilon)$ and that f_α is Lipschitz. Conversely the Lipschitzness of f_α would provide a finite bound on the slopes of the secants of f_α which would provide an open neighborhood of lines about the x_d-axis which miss $D(Graph(f_\alpha))$. After rotation this would imply that there is open neighborhood about α in $\mathbb{R}P^{d-1}$ that does not intersect $D(E)$.

Thus there are only two possibilities, if $D(E)$ is not dense in $\mathbb{R}P^{d-1}$, then there must exist some $\alpha \in \mathbb{R}P^{d-1}$ such that an open neighborhood of α does not intersect $D(E)$. In this case, E is under a corresponding rotation, the graph of the Lipschitz function f_α. On the other hand if $D(E)$ is a proper dense subset of $\mathbb{R}P^{d-1}$, then E is up to some rotation, the graph of a function, but this function is never Lipschitz.

2.3. **Proof of Corollary 1.5 and Corollary 1.6.** By the proof of Theorem 1.4, when $D(E) \neq \mathbb{R}P^{d-1}$, for every $\alpha \in \mathbb{R}P^{d-1} - D(E)$, E is (up to rotation) the graph set of a function $f_\alpha : A_\alpha \to \mathbb{R}$ where $A_\alpha \subseteq \mathbb{R}^{d-1}$ is a projection of E to $\mathbb{R}^{d-1}$. When E is compact (connected), A_α is also compact (connected). In particular when $d = 2$, A_α is a closed subinterval of $\mathbb{R}$ as every compact, connected subset of $\mathbb{R}$ is a closed interval (to see this just apply the intermediate and extreme value theorems to the inclusion map $i : A \to \mathbb{R}$.)

When this graph set E is compact, it follows from the topological closed graph theorem ([3]) that each f_α is continuous. Finally when $d = 2$, and E is compact and connected, then (up to any such rotation) E is the graph set of $f_\alpha : [a, b] \to \mathbb{R}$. The secant formula

$$\frac{f_\alpha(x) - f_\alpha(y)}{x - y}$$

defines a continuous function on the open triangle

$$\{(x, y) \in [a, b] \times [a, b] : x < y\}.$$

Since this triangle is connected it follows that $D(E)$ is connected in $\mathbb{R}P^1$. Thus in the case that $D(E)$ is dense, it can miss at most one point, because $D(E)$ is connected and ($\mathbb{R}P^1 -$two points) has two connected components.

This concludes the proofs of these corollaries.

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