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ON 2-SKELETA OF PLATONIC POLYTOPES

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Abstract. Euler showed that every graph whose vertices have even degree can be decomposed into edge-disjoint cycles. We prove that an analogous result holds for the 2-skeleta of Platonic polytopes, where cycles are replaced by surfaces.

1. Introduction

Coxeter defines "polytope" as general term of the sequence: line segment, polygon, polyhedron. A d-polytope is topologically a ball and its boundary is a $d-1$-sphere.

Among all d-polytopes, for $d \geq 3$, we consider the **Platonic d-polytopes** which are those polytopes in $\mathbb{R}^d$ which satisfy a maximum set of symmetries. See [2].

Platonic polytopes come in three infinite families: the simplex, the (hyper)cube, and the cross-polytope, which exist for all values of d. There are also five sporadic cases, all considered below. A polytope with its faces determines a **complex**.

For the corresponding Platonic complexes, we study the 2-skeleton and show that it behaves in some ways like the 1-skeleton. For $t = 1$ and 2, the t-skeleton decomposes into regular arrangements of (subcomplexes homeomorphic to) closed t-manifolds intersecting pairwise in $t-1$-manifolds. For t=1, these are cycle-subgraphs and vertices, resp.; the decomposition holds for nontrivial, connected even graphs.

For $t = 2$, a much more nuanced decomposition is possible because there are distinct types of manifold, but simultaneously our arguments apply only to the very special Platonic case. Necessarily, *every edge is a face of a positive even number of* 2*-cells*, holds for both odd-dimensional hypercubes and simplexes and for cross-polytope of all dimensions.

Under the above evenness condition, we decompose the 2-skeleton of the simplex into (minimum) spheres of two distinct geometric types: *boundaries of tetrahedra* and *boundaries of octahedra*. For the 2-skeleton of the cross-polytope, we have a **factorization** as all the surfaces are *boundaries of octahedra* (3-dimensional cross-polytope). Note that the latter spheres are not face-boundaries, as all proper faces of a cross-polytope are simplexes.

In [7], the 2-skeleta of odd-dimensional hypercubes were decomposed into pairwise face-disjoint tori and spheres. Here we give a different derivation of the decomposition of the hypercube's 2-skeleton into a family of *spheres* and *tori*, where the spheres are boundaries of 3-cubes and each torus is the topological Cartesian product of two 4-cycles. These tori are minimum torus-subcomplexes with no strictly smaller subcomplex homeomorphic to a torus. The tori we get are 2-complexes with 16 faces and 1-skeleton $C_4 \square C_4$, where "$\square$" denotes Cartesian product of graphs (as in [4]). In fact, decomposition into only spheres is possible [5].

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A 2-complex $\mathcal{C}$ is **even** if each edge in $\mathcal{C}$ is an edge of a positive even number of 2-faces of $\mathcal{C}$. We say that two sub-complexes $\mathcal{C}', \mathcal{C}''$ of a 2-complex $\mathcal{C}$ are **face-disjoint** if no 2-face in one belongs to the other (though of course they may have vertices and edges in common). A complex $\mathcal{C}'$ is a **surface** if $|\mathcal{C}'|$ is homeomorphic to a surface (i.e., 2-dimensional sphere, torus, projective plane, etc.) We ask when the 2-skeleton of a polytope P has a face-disjoint decomposition into surfaces and we show that this holds for d-dimensional Platonic polyhedra whenever the 2-skeletons are even.

If a 2-complex $\mathcal{C}$ can be decomposed into a union of pairwise face-disjoint surfaces, then $\mathcal{C}$ is even. However, $\mathcal{C}$ can be even but not be a face-disjoint union of surfaces. For instance, there can be pinch points, or the surface could intersect itself like the Klein bottle in 3-space.

Section 2 below has formal definitions. Theorems covering each of the three infinite cases are in the third section, which also includes an analysis of the sporadic cases. Section 4 discusses the case of non-even complexes, the comparison of complexity for Platonic polytopes, and the role of divisibility in partitioning the 2-cells.

2. Background

For graph theory definitions and results, see [6]; for topology see [11]. For polytopes and complexes we mostly adopt the terminology of Ziegler [12].

A **polytope** is the convex hull of a finite set of points in $\mathbb{R}^n$. (Equivalently, a polytope P is a bounded intersection of a finite family of closed half-spaces.) A **face** of P is P or any subpolytope which lies in one of the bounding hyperplanes.

The **dimension** of a polytope P is the smallest integer d for which P is contained in a d-dimensional affine subspace of $\mathbb{R}^n$ for some n. A d-dimensional polytope is called a **d-polytope**. A **(polytopal) complex** is a finite collection $\mathcal{C}$ of polytopes in $\mathbb{R}^n$ such that for $P, Q \in \mathcal{C}$, (i) all faces of P are in $\mathcal{C}$ and (ii) $P \cap Q$ is a face of both P and Q. The *complex determined by a polytope* is the set of its faces.

For $-1 \leq k \leq d$, let $P^{(k)}$ and $\mathcal{C}^{(k)}$ denote the sets of all k-faces of a d-polytope or complex, where by definition the unique face of dimension -1 is the empty set, while the 0- and 1-dimensional faces are called **vertices** and **edges**, respectively. The $(d-1)$-dimensional faces of a d-polytope are called its **facets** or **cells**. A **subcomplex** of $\mathcal{C}$ is a complex $\mathcal{D} \subseteq \mathcal{C}$ such that for every r, $\mathcal{D}^{(r)} \subseteq \mathcal{C}^{(r)}$. The k-skeleton is a subcomplex for each k and the $d-1$-skeleton of a d-polytope is homeomorphic to the boundary sphere.

The **dimension of a complex** $\mathcal{C}$ is the largest k such that $\mathcal{C}^{(k)} \neq \emptyset$. The k**-skeleton** $\mathcal{C}^k$ of $\mathcal{C}$ is the subcomplex determined by the faces of dimension $\leq k$,

$$\mathcal{C}^k := \bigcup_{r \leq k} \mathcal{C}^{(r)}$$

We show that the 2-skeleton of any even Platonic polytope can be decomposed into a family of face-disjoint subcomplexes, each homeomorphic to a closed surface.

3. Splitting the 2-skeleton into surfaces

This section has four parts - for the three infinite classes and the sporadic cases.

3.1. **Simplexes.** Let $\mathbf{e}_1 = (1, 0, \ldots, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0)$, $\ldots$, $\mathbf{e}_d = (0, 0, \ldots, 1)$ be the standard basis of $\mathbb{R}^d$. The $\mathbf{(d-1)}$**-dimensional simplex** is the convex hull of the d points $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d\} \subseteq \mathbb{R}^d$. We denote the d-dimensional simplex as $\Delta_d \subseteq \mathbb{R}^{d+1}$. The n-faces of Δ_d are precisely the convex hulls of the sets $\{\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots \mathbf{e}_{i_{n+1}}\}$, where $\{i_1, i_2, \dots i_{n+1}\} \subseteq \{1, 2, \dots d+1\}$.

Thus for any $n \in \{0, 1, 2, \dots, d\}$, the simplex Δ_d has $\binom{d+1}{n+1}$ n-faces. In particular the 2-skeleton Δ_d^2 has $d+1$ vertices, $\binom{d+1}{2}$ edges and $\binom{d+1}{3}$ faces. We can draw a picture of Δ_d by drawing a regular $(d+1)$-gon and connecting all chords, as in Figure 1.

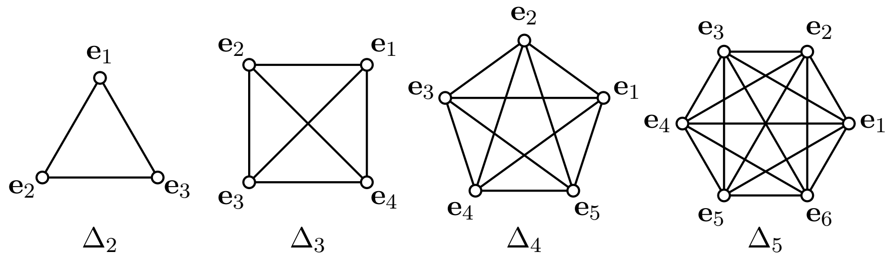

FIGURE 1. Two-, three-, four- and five-dimensional simplexes.

Observe that each edge of Δ_d is on the boundary of $d-1$ of Δ_d's triangular faces. Thus the 2-skeleton Δ_d^2 is *even* only when d is *odd*, so d must be odd if Δ_d^2 is to be decomposed into face-disjoint surfaces. For odd d, here is a construction for splitting Δ_d^2 into face-disjoint spheres: Draw Δ_d as a regular $(d+1)$-gon. Label its vertices in a counterclockwise sequence $\mathbf{e}_2$, $\mathbf{e}_4$, $\mathbf{e}_6$, $\dots$ $\mathbf{e}_{d+1}$, followed by $\mathbf{e}_1$, $\mathbf{e}_3$, $\mathbf{e}_5$, $\dots$ $\mathbf{e}_d$, so that $\mathbf{e}_{2i}$ is always the antipode (relative to the drawing) of $\mathbf{e}_{2i-1}$ (Figure 2, left).

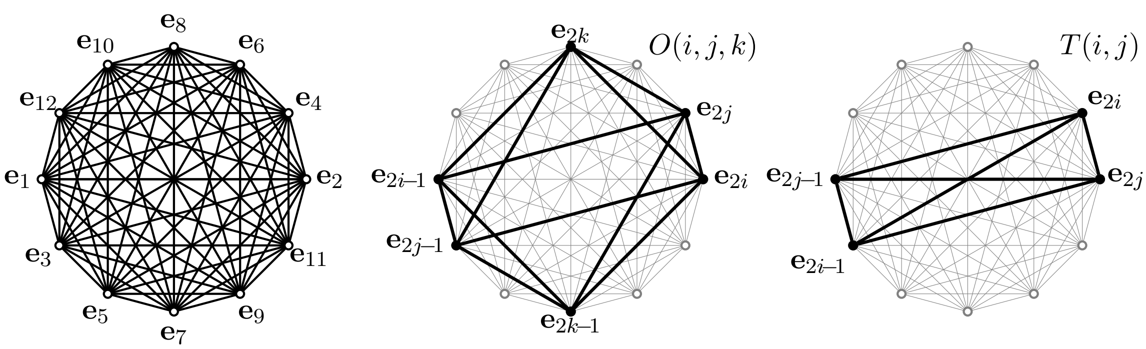

FIGURE 2. Left: Δ_d. Middle: $O(i,j,k)$. Right $T(i,j)$.

For each triple $\{2i, 2j, 2k\} \subseteq \{2, 4, 6, \dots, d+1\}$, the simplex Δ_d has eight triangle faces whose vertices are triples from $\{\mathbf{e}_{2i}, \mathbf{e}_{2j}, \mathbf{e}_{2k}, \mathbf{e}_{2i-1}, \mathbf{e}_{2j-1}, \mathbf{e}_{2k-1}\}$ but for which no triangle edge connects antipodes in the drawing. These eight triangles are shown in the middle drawing in Figure 2. Observe that they constitute the faces of *a three-dimensional octahedron boundary*, which we shall denote as $O(i,j,k)$.

For each $\{2i, 2j\} \subseteq \{2, 4, 6, \dots, d+1\}$, the convex hull of $\{\mathbf{e}_{2i}, \mathbf{e}_{2j}, \mathbf{e}_{2i-1}, \mathbf{e}_{2j-1}\}$, is *a three-dimensional tetrahedron, whose boundary we denote by* $T(i,j)$, and consists of four triangular faces in the set $\Delta_d^{(2)}$ (see Figure 2, right). Thus we have $\binom{(d+1)/2}{3}$ octahedron boundaries $O(i,j,k)$ and $\binom{(d+1)/2}{2}$ tetrahedron boundaries $T(i,j)$. And one may easily check that the decomposition is numerically feasible

$$8\binom{(d+1)/2}{3} + 4\binom{(d+1)/2}{2} = \binom{d+1}{3}.$$

Notice that no face of an $O(i,j,k)$ is a face of a $T(\ell,m)$, because each triangle of $T(\ell,m)$ has either the edge $\mathbf{e}_{2\ell}\mathbf{e}_{2\ell-1}$ or the edge $\mathbf{e}_{2m}\mathbf{e}_{2m-1}$ joining antipodes, but no triangle from $O(i,j,k)$ has such such edges. Further we claim any two distinct tetrahedrons $T(i,j)$ and $T(\ell,m)$ are face-disjoint. Indeed if k belongs to $\{i,j\} - \{\ell,m\}$, then any triangle of $T(i,j)$ contains either vertex $\mathbf{e}_{2k}$ or $\mathbf{e}_{2k-1}$, but neither of these vertices belong to any triangle of $T(\ell,m)$. It is equally easy to confirm that the octahedra $O(i,j,k)$ and $O(\ell,m,n)$ are pairwise face-disjoint. Just take $p \in \{i,j,k\} - \{\ell,m,n\}$ and note that every triangle of $O(\ell,m,n)$ has $\mathbf{e}_{2p}$ or $\mathbf{e}_{2p-1}$ as a vertex, yet no triangle of $O(i,j,k)$ has such a vertex.

Theorem 3.1. *For $d \geq 3$ odd, the 2-skeleton of the d-dimensional simplex Δ_d has a pairwise face-disjoint decomposition into $\binom{(d+1)/2}{3}$ octahedron boundaries and $\binom{(d+1)/2}{2}$ tetrahedron boundaries.*

Proof. We just need to check that the $O(i,j,k)$ and $T(i,j)$ use up all the triangles of Δ_d. But this is clear: For any triangle T of Δ_d, either one edge of T joins some $\mathbf{e}_{2\ell}$ to its antipode $\mathbf{e}_{2\ell-1}$, or no edge of T joins two antipodes. In the first case T is a triangle of $T(\ell,j)$, with j determined by the third vertex of T. In the second case T belongs to some $O(i,j,k)$. □

The decomposition for $d = 5$ was also found by Overbay [9].

Although it is topologically the boundary of a three-dimensional octahedron, $O(i,j,k)$ is not contained in any three-dimensional affine subspace of $\mathbb{R}^{d+1}$. (An equatorial square of $O(i,j,k)$, such as $\mathbf{e}_{2i}\mathbf{e}_{2j}\mathbf{e}_{2i-1}\mathbf{e}_{2j-1}$ is the boundary of a disk formed by two equilateral triangles $\mathbf{e}_{2i}\mathbf{e}_{2j}\mathbf{e}_{2i-1}$ and $\mathbf{e}_{2i}\mathbf{e}_{2j-1}\mathbf{e}_{2i-1}$ glued and folded along their common edge $\mathbf{e}_{2i}\mathbf{e}_{2i-1}$ that joins antipodes in the drawing. This folded square lies in the affine three-dimensional space containing the four points $\mathbf{e}_{2i}$, $\mathbf{e}_{2j}$, $\mathbf{e}_{2i-1}$ and $\mathbf{e}_{2j-1}$. Then $O(i,j,k)$ is the double cone over this folded square, with apexes $\mathbf{e}_{2k}$ and $\mathbf{e}_{2k-1}$.)

3.2. **Cross-Polytopes.** Let $\mathbf{e}_1$, $\mathbf{e}_2$, ... , $\mathbf{e}_d$ be the standard basis of $\mathbb{R}^d$. The d-dimensional **cross-polytope** (or *hyperoctahedron*) O_d is the convex hull of the $2d$ points $\pm\mathbf{e}_1, \pm\mathbf{e}_2, \ldots, \pm\mathbf{e}_d$ in $\mathbb{R}^d$. Figure 3 shows O_2, O_3 and O_4.

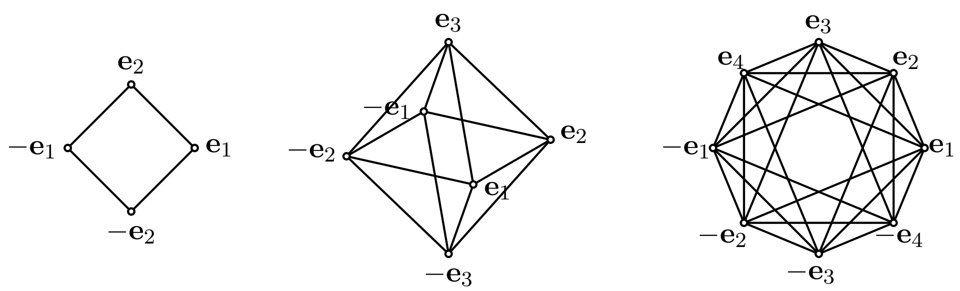

FIGURE 3. Two-, three- and four-dimensional cross-polytopes.

For $0 \leq n < d$, the n-faces of O_d are n-dimensional simplexes: For any choice of

$$\{m_1, m_2, \ldots m_{n+1}\} \subseteq \{1, 2, \ldots, d\}$$

and signs $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$, ..., $\epsilon_{n+1} = \pm 1$, there is an n-face (n-dimensional simplex) of O_d that is the convex hull of the $n+1$ affinely independent points $\{\epsilon_1 \mathbf{e}_{m_1},\ \epsilon_2 \mathbf{e}_{m_2},\ \ldots,\ \epsilon_{n+1}\mathbf{e}_{m_{n+1}}\}$. Thus O_d has $2^{n+1}\binom{d}{n+1}$ n-faces (for $0 \le n < d$), each of which is an n-dimensional simplex.

In particular, O_d has $4\binom{d}{2}$ edges. Any edge joins $\pm\mathbf{e}_i$ to $\pm\mathbf{e}_j$, for $i \neq j$. Likewise, O_d has $8\binom{d}{3}$ 2-faces, all triangles. They can be generated as follows: First select three distinct integers i, j, k from $\{1, 2, \ldots, d\}$. To this selection there corresponds eight triangular 2-faces of O_d, namely those eight whose vertices are

$$\begin{array}{cccc} \{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\}, & \{-\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\}, & \{\mathbf{e}_i, -\mathbf{e}_j, \mathbf{e}_k\}, & \{-\mathbf{e}_i, -\mathbf{e}_j, \mathbf{e}_k\}, \\ \{\mathbf{e}_i, \mathbf{e}_j, -\mathbf{e}_k\}, & \{-\mathbf{e}_i, \mathbf{e}_j, -\mathbf{e}_k\}, & \{\mathbf{e}_i, -\mathbf{e}_j, -\mathbf{e}_k\}, & \{-\mathbf{e}_i, -\mathbf{e}_j, -\mathbf{e}_k\}. \end{array}$$

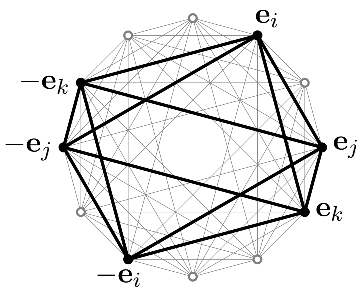

FIGURE 4. The 3-dimensional octahedron $O(i, j, k)$ (bold lines).

These eight triangles form the boundary of a 3-dimensional octahedron. We denote this boundary as $O(i, j, k)$. (See Figure 4.) Each triangle in the $O(i, j, k)$ is a triangle of O_d, but $O(i, j, k)$ is **not** the boundary 3-face of O_d, for each 3-face is simplex.

Theorem 3.2. *For $d \ge 3$ the 2-skeleton $O_d^{(2)}$ of the d-dimensional-cross polytope is face-disjoint union of the boundaries of $\binom{d}{3}$ 3-dimensional octahedrons, that is,*

$$O_d^{(2)} = \bigcup \Big\{ O(i, j, k) \mid \{i, j, k\} \subseteq \{1, 2, \ldots d\} \Big\}.$$

Proof. There are $\binom{d}{3}$ octahedra $O(i, j, k)$, each accounting for eight triangular faces of O_d. Further, no two distinct $O(i, j, k)$ and $O(i', j', k')$ share a triangular face, because the only way that could happen is if $\{i, j, k\} = \{i', j', k'\}$, and then $O(i, j, k) = O(i', j', k')$. Therefore the $\binom{d}{3}$ octahedra $O(i, j, k)$ account for all $8\binom{d}{3}$ triangular faces of $O_d^{(2)}$. □

3.3. **Cubes.** A decomposition for cubes was derived in a different way in [7], using [3], which gives additional types of decompositions into spheres and tori, allowing fewer but longer tori. The argument here is self contained. See the discussion section.

The d-dimensional cube Q_d is the convex hull of the 2^d points $\{(\pm 1, \pm 1, \ldots, \pm 1)\} \subseteq \mathbb{R}^d$; i.e., $Q_d = [-1, 1]^d := [-1, 1] \times [-1, 1] \times \cdots \times [-1, 1] \subseteq \mathbb{R}^d$, and $Q_d = Q_{d-1} \times [-1, 1]$.

Any edge of Q_d is a subset $\{x_1\} \times \{x_2\} \times \cdots \times [-1, 1] \times \cdots \times \{x_d\} \subseteq Q_d$, where the jth factor is the interval $[-1, 1]$ and any other factor is a singleton $\{x_i\} \subseteq \{-1, 1\}$. The 2-faces of Q_d are the subsets $\{x_1\} \times \cdots \times [-1, 1] \times \cdots \times [-1, 1] \times \cdots \times \{x_d\} \subseteq Q_d$, where all but the ith and jth factor is a singleton. Consequently Q_d has $d2^{d-1}$ edges and $\binom{d}{2}2^{d-2}$ 2-faces, all of which are squares. By similar reasoning, Q_d has $\binom{d}{n}2^{d-n}$ n-faces for any $0 \le n \le d$, and all of them are n-cubes.

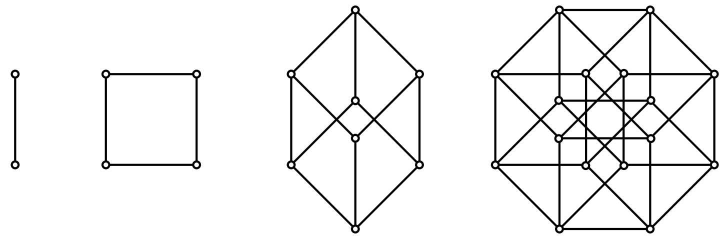

FIGURE 5. Cubes Q_1, Q_2, Q_3 and Q_4.

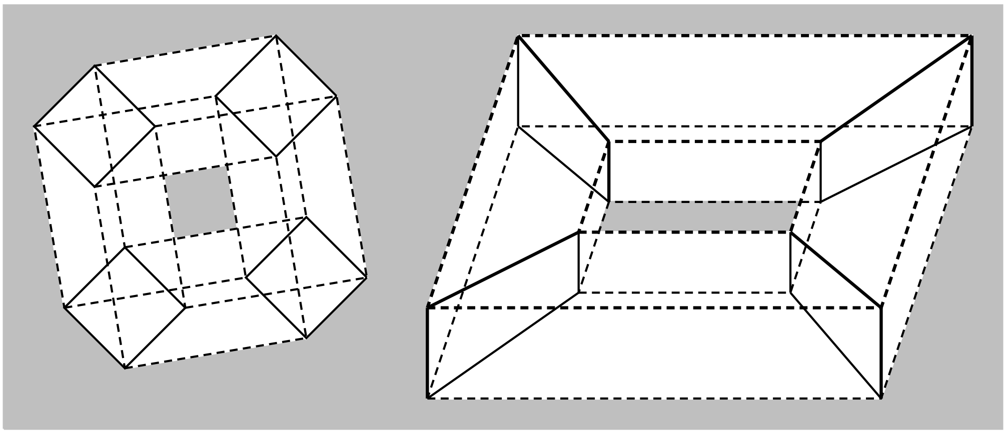

FIGURE 6. The torus $D \times \partial(D)$ in Q_4. Its boundary $T = \partial(D) \times \partial(D) \subseteq D \times D$ is the set of squares of $Q_4^{(2)}$ that have edges of both colors.

The previous paragraph implies that any edge of Q_d is on $d-1$ faces, so $Q_d^{(2)}$ is even if and only if d is odd. As Q_3^2 is the boundary of $[-1,1]^3$, it *is* a sphere. We now describe a decomposition of Q_5^2 that generalizes to any odd-dimensional cube.

Let D be the square disk $D = [-1,1] \times [-1,1]$, whose boundary $\partial(D) = C_4$ is homeomorphic to S^1. Now, $Q_4 = D \times D$ and $Q_5 = D \times D \times [-1,1]$.

The solid torus $D \times \partial(D) \subseteq Q_4$ has boundary $T = \partial(D) \times \partial(D) = C_4 \times C_4 \subseteq Q_4$, which is a genus-1 torus surface made up of 16 square faces. One can visualize this by coloring the edges of $\partial(D) \times \{(\pm1, \pm1)\} \subseteq Q_4$ bold and coloring the edges of $\{(\pm1, \pm1)\} \times \partial(D)$ dashed, thereby partitioning the edges of Q_4 into four disjoint bold 4-cycles and four disjoint dashed 4-cycles. The square faces of T are precisely the 16 faces of Q_4 with edges of both colors (bold and dashed). There remain 8 monochromatic squares in Q_4 that are not faces of T. Their boundaries partition the edges of Q_4 into 8 edge-disjoint monochromatic 4-cycles. See Figure 6.

In the 5-cube $Q_5 = D \times D \times [-1,1]$ there are two vertex-disjoint torus surfaces $T_- := T \times \{-1\} \subseteq Q_5^{(2)}$ and $T_+ := T \times \{1\} \subseteq Q_5^{(2)}$. For each vertex $x = (\pm1, \pm1)$ of D, the 5-cube Q_5 has a 3-cube 3-face $D \times \{x\} \times [-1,1]$, whose base is one of the four bold squares in Q_4. Also for each such vertex x of D, the 5-cube Q_5 has a 3-cube 3-face $\{x\} \times D \times [-1,1]$, whose base is one of the four dashed squares in Q_4. Then T_- and T_+, with the eight spheres $\partial\big(D \times \{x\} \times [-1,1]\big)$ and $\partial\big(\{x\} \times D \times [-1,1]\big)$ are a face-disjoint decomposition of Q_5^2 into 8 spheres and 2 tori.

A straightforward extension of this decomposition gives the following result, which is proved using different decompositions in [7]. See the Appendix.

Theorem 3.3. *The* 2*-skeleton of any odd-dimensional cube* Q_{2n+1} *is the face-disjoint union of* $\binom{n}{2}2^{2n-3}$ *torus boundaries and* $n4^{n-1}$ *cube boundaries (spheres).*

Sphere decompositions are possible for some hypercubes [5].

3.4. **Sporadic examples.** There exist exactly five Platonic polytopes not in any of the three infinite families.

The dodecahedron has a 4-dimensional counterpart, called the 120-cell, with 120 dodecahedron cells. The icosahedron has a 4-dimensional counterpart, the 600-cell, which has 600 tetrahedron cells. Last, in dimension 4, there is the 24-cell, having 24 cells that are 3-dimensional octahedra.

The 2-skeleta of the 3-dimensional dodecahedron and icosahedron are each their own decomposition into a single sphere. Neither the 120-cell nor the 600-cell has an even 2-skeleton, for they have respectively 3 and 5 faces per edge. Further, the 24-cell is also not even, as it has 3 faces per edge. For details see [2].

4. Discussion

In [4] it was asked if the 2-skeleton of every even n-complex has a decomposition as a face-disjoint union of surfaces. We have shown that this holds for Platonic polytopes. Splitting 2-complexes into surfaces is an enrichment of splitting graphs into cycles, and so represents another step in the barely begun program of generalizing graph theory to complexes. For a much earlier step, see, e.g., Sós, Erdős, and Brown [10].

In this section, we briefly consider some remaining questions. (1) What happens if a 2-complex is not even? (2) How is the decomposition process related to other measures of the *informational* complexity of the Platonic polytopes and their skeleta? (3) What is the role of divisibility in splitting?

4.1. **Splitting non-even complexes.** If the complex itself is not even, then there is no way to split it into closed surfaces, but allowing also surfaces with boundary, results from the even case may be applied. For instance, Q_d^2 is obtained from two disjoint copies of Q_{d-1}^2 by including all the *flange* 2-cells of the form $e \times Q_1$ for $e \in Q_{d-1}^{(1)}$. When $d \geq 4$ is even, then $d-1$ is odd. Selecting any 1-factor F of Q_{d-1}^1, the graph obtained by deleting F is regular of degree $d-2$ and hence (by Euler's theorem) a pairwise-edge-disjoint union of cycles.

Thus, the flange 2-cells can be placed into a variable number of cylinders (cycle $\times Q_1$) and exactly 2^{d-2} pairwise-vertex-disjoint disks. By a theorem of El-Zanati and Vanden Eynden [3] and arguing as in [7], the number of cylinders can be made as small as $(d-1)/2$ and as large as $(d-1)2^{d-3}$.

More generally, surfaces with boundary can split *any* 2-complex. Indeed, take a genus embedding of the 1-skeleton, and extend it to include any 2-cells of the genus surface which correspond to 2-cells in the complex. Any remaining 2-cells each go on a separate disk. For example, the genus embedding of Q_5^1 in S_5 has 40 square regions by Euler's formula. But $Q_5^{(2)}$ has 80 members, so one needs 40 disks, in addition to the closed orientable surface with 5 handles, for this decomposition.

4.2. **Comparing the complexity of Platonic polytopes.** One may compare the *relative informational complexities* of the various Platonic polytopes but simplicity depends on the lens used.

Restricting attention to the 1-skeleton, one may look first at embeddability of a graph *in* the simplex, cube, and cross-polytope, i.e., in the graphs K_n, Q_d, and $O^1_r := K_{2r} - 2K_r$, respectively. For K_n and O^1_r, the story is trivial but for cubes it is *not* simple, e.g., [8]. So cubes are *most* complex in terms of *embedding in.*

Embeddability *of* the 1-skeleton $\mathcal{P}^1$ of a Platonic polytope in orientable surfaces is given by its **genus** $\gamma(\mathcal{P}^1)$, which is the least number of handles among all such surfaces in which $\mathcal{P}^1$ can be embedded, and here the cube is *least* complex.

Indeed, complete graphs have an underlying 12-fold nature for their genus. The upper bound is easy and follows from Euler's formula but proving that $\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$, $n \geq 3$, requires quite distinct methods to construct embeddings of K_n for the different cases mod-12. Similarly, the 1-skeleton of the n-cross-polytope has genus $\lceil \frac{(n-1)(n-3)}{3} \rceil$, $n \geq 3$, and again a multi-case argument is required. See, e.g., [1].

In contrast, for $d \geq 3$, the cube graph Q_d has genus $1 + (d-4)2^{d-3}$ [6, p. 117]. Using $Q_d = Q_{d-1} \Box Q_1$, the easy recursive construction works for all d.

However, if we restrict our attention to the 2-*skeleton* of the Platonic polytopes (not just their underlying graphs), then our three theorems show that from the standpoint of surface decomposition the cross-polytope has simplest 2-skeleton, while the simplex is next simplest, and the cube appears to be the most complex.

4.3. **Divisibility.** Factorization, where one divides into isomorphic subcomplexes, requires divisibility; the latter condition is sufficient for cubes [5] but not for simplexes.

Indeed, S. Overbay [9] proved that Δ^2_5 has no factorization into tetrahedra boundaries although 4 divides $|\Delta^{(2)}_d|$ for $d \geq 3$ odd. Are there cases other than the cross-polytope where factorization is possible? What about *isometric* factorizations?

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Appendix

Proof of Theorem 3.3. Put $Q_{2n+1} = D \times D \times \cdots \times D \times [-1,1]$, that is, as n factors of D followed by a factor $[-1,1]$. For each choice of indices $1 \le i < j \le n$ there is a family of $4^{n-2} \cdot 2$ completely disjoint solid tori

$$\{x_1\} \times \cdots \times D \times \cdots \times \partial(D) \times \cdots \times \{x_n\} \times \{\epsilon\} \subseteq Q_{2n+1} \tag{1}$$

with D in the ith factor, $\partial(D)$ in the jth factor, each $x_i = (\pm1, \pm1)$ a vertex of D, and $\epsilon = \pm1$. Denote the family (1) of solid tori as $\mathcal{T}_{i,j}$, so $|\mathcal{T}_{i,j}| = 4^{n-2} \cdot 2$ and the tori in $\mathcal{T}_{i,j}$ have pairwise empty intersection. Any face of a torus $T \in \mathcal{T}_{i,j}$ has form

$$\{x_1\} \times \cdots \times E_i \times \cdots \times E_j \times \cdots \times \{x_n\} \times \{\epsilon\}, \tag{2}$$

where the E_i and E_j in the ith and jth factors are edges of $\partial(D)$. From this we infer that if $\{i,j\} \ne \{k,\ell\}$, then no torus in $\mathcal{T}_{i,j}$ shares a face with a torus in $\mathcal{T}_{k,\ell}$. Therefore we have a family

$$\mathcal{T} = \bigcup_{1 \le i < j \le n} \mathcal{T}_{i,j}$$

of $\binom{n}{2} 4^{n-2} \cdot 2 = \binom{n}{2} 2^{2n-3}$ pairwise face-disjoint tori in the 2-skeleton of Q_{2n+1}.

Also, for any $1 \le i \le n$, the 2-skeleton contains a family of 3-cubes of the following form, where $x_i \in \{(\pm1, \pm1)\}$, and all but the ith and $(n+1)$th factors are singletons:

$$\{x_1\} \times \cdots \times \; D \; \times \cdots \times \{x_n\} \times [-1,1] \subseteq Q_{2n+1}. \tag{3}$$

Denote this family of cubes as $\mathcal{S}_i$, so $|\mathcal{S}_i| = 4^{n-1}$ and any two cubes in $\mathcal{S}_i$ have empty intersection. Any face of a cube in $\mathcal{S}_i$ has one of the two forms

$$\begin{array}{l} \{x_1\} \times \cdots \times \; E \; \times \cdots \times \{x_n\} \times [-1,1] \quad \text{or} \\ \{x_1\} \times \cdots \times \; D \; \times \cdots \times \{x_n\} \times \{\epsilon\} \end{array}$$

where E is an edge of D (and the E and D occur in the ith factor, for $1 \le i \le n$). Comparing this to (2) we see that no cube in $\mathcal{S}_i$ shares a face with a torus in $\mathcal{T}$. Furthermore, notice that if $i \ne j$ then no cube in $\mathcal{S}_i$ shares a face with a cube in $\mathcal{S}_j$. Now form the family

$$\mathcal{S} = \bigcup_{1 \le i \le n} \mathcal{S}_i$$

of $n4^{n-1}$ pairwise face-disjoint cubes. We have created a collection $\mathcal{T} \cup \mathcal{S}$ of pairwise face-disjoint tori and cubes in the 2-skeleton of Q_{2n+1}.

By the above discussion the 2-skeleton contains $\binom{n}{2} 2^{2n-3}$ torus boundaries and $n4^{n-1}$ cube boundaries, and they are pairwise face-disjoint. We just need to show that they use up all the squares in $Q^{(2)}_{2n+1}$. Since the tori have 16 squares apiece, and each sphere has 6 squares, we have accounted for the following number of squares:

$$16\binom{n}{2} 2^{2n-3} + 6n4^{n-1} \;=\; \left(4\binom{n}{2} + 3n\right) 2^{2n-1} \;=\; \frac{(2n+1)2n}{2} 2^{2n-1} \;=\; \binom{2n+1}{2} 2^{(2n+1)-2}$$

which is indeed the number of squares in Q_{2n+1}. Hence, all are accounted for.

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