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# ANTIPODAL SETS IN INFINITE DIMENSIONAL BANACH SPACES

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ABSTRACT. The following strengthening of the Elton-Odell theorem on the existence of a $(1+\epsilon)-$separated sequences in the unit sphere $S_X$ of an infinite dimensional Banach space $X$ is proved: There exists an infinite subset $S \subseteq S_X$ and a constant $d > 1$, satisfying the property that for every $x, y \in S$ with $x \neq y$ there exists $f \in B_{X^*}$ such that $d \leq f(x) - f(y)$ and $f(y) \leq f(z) \leq f(x)$, for all $z \in S$.

## 0. Introduction

A set $S$ in a Banach space $X$ is called $d-$separated $(d > 0)$ if $\|x - y\| \geq d \ \forall \ x \neq y \in S$. For infinite dimensional Banach spaces the parameter

$$K(X) = \sup\{d : \exists S \subseteq B_X, S \text{ infinite and } d\text{−separated}\}$$

is called Kottman's constant or separation constant of $X$ and by a well known theorem of Elton and Odell [8] is strictly greater than 1. In the present paper we study the parameter $K_a(X)$ for infinite dimensional Banach spaces, which was introduced on [17] as $\text{ant}(X)$. The definition of $K_a(X)$ is based on the notion of bounded and separated antipodal sets [17]. Bounded and separated antipodal sets were introduced as a strengthening of the classical concept of antipodal sets (see [5] and [18]) to include spaces of any dimension whereas the original definition was suitable for spaces of finite dimension [18]. We remind the reader the following definitions.

**Definition 0.1** ([18])**.** A subset of an $n-$dimensional real vector space $X$ is said to be antipodal if for every $x, y \in S$ with $x \neq y$ there exist distinct parallel support hyperplanes $P, Q$ such that $x \in P$ and $y \in Q$.

**Definition 0.2** ([17])**.** Let $(X, \|\cdot\|)$ be a normed space.

(a) A subset $S$ of $X$ is called antipodal if for every $x, y \in S$ with $x \neq y$ there exists $f \in X^*$ such that $f(x) < f(y)$ and $f(x) \leq f(z) \leq f(y) \ \forall z \in S$.

(b) An antipodal subset $S$ of $X$ is said to be bounded and separated, in short b.s.a subset, if there exist positive constants $c_1, c_2$ and $d$ such that
  (1) $\|x\| \leq c_1, \forall x \in S$ and
  (2) for every $x, y \in S$ with $x \neq y$ there exists $f \in X^*$ with $\|f\| \leq c_2$, such that $0 < d \leq f(y) - f(x)$ and $f(x) \leq f(z) \leq f(y) \ \forall z \in S$.

A subset $S$ of a normed space $X$, as above, will be called $(c_1, c_2, d)-$b.s.a subset of $X$.

(c) If $X$ is infinite dimensional we set

$$K_a(X) = \sup\{d : \exists \ S \subseteq B_X, S \text{ infinite } (1, 1, d) - \text{b.s.a set}\}.$$

It is clear that if $(X, \|\cdot\|)$ is finite dimensional then Definition 0.1 and Definition 0.2 (a) coincide.

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In relation with the Elton-Odell Theorem it would be interesting to know if every infinite dimensional Banach space $X$ contains an infinite bounded and separated antipodal subset with constants $c_1 = c_2 = 1$ and $d > 1$ or equivalently if $K_a(X) > 1$, for every infinite dimensional Banach space. Indeed it is obvious that in that case we would have a stronger version of the Elton-Odell Theorem. The above question was posed in [17] and our main aim is to provide an affirmative answer. For spaces that contain isomorphically $c_0$ or $l_p$ for some $1 \leq p < \infty$ the structural properties of these spaces suffice (Proposition 1.2) while for uniformly smooth spaces only the geometric properties of those spaces are needed (Proposition 1.4). The main tool in order to pass to more general classes of spaces is Theorem 1.7 whose proof is essentially based on the proof of Theorem 1 of [17]. Using Theorem 1.7 we prove that $K_a(X) > 1$ when $X$ is a reflexive Banach space (Corollary 1.8) or $X$ has a separable dual (Corollary 1.9). For the general case, apart from Theorem 1.7 the highly non trivial Theorem 4.1 of [9] is also needed. If $X$ is any (real) Banach space then $B_X$ (resp. $S_X$) denotes its closed unit ball (resp. unit sphere). The Banach-Mazur distance between two isomorphic Banach spaces $X$ and $Y$ is defined as

$$d(X, Y) = \inf\{\|T\| \left\|T^{-1}\right\| : T \text{ is an invertible operator from } X \text{ onto } Y\}.$$

## 1. Bounded and separated antipodal sets in infinite dimensions.

From here on we concern ourselves with infinite dimensional Banach spaces, except if stated otherwise. We start this section with some remarks concerning bounded and separated antipodal sets (see [17]).

**Remark 1.1.** Let $(X, \|\cdot\|)$ be a Banach space.

(1) Let $S$ be a bounded and separated antipodal subset of $X$. It is easy to see that if $\lambda > 0$, $S$ is also bounded and separated with constants $c_1, \lambda c_2, \lambda d$ then the same is valid for the set $\lambda S = \{\lambda x : x \in S\}$ with constants $\lambda c_1, c_2, \lambda d$. Thus a bounded and separated antipodal subset of $X$ can be defined as a subset $S$ of $B_X$ that satisfies the property: there exists $d > 0$ such that for every $x \neq y \in S$ there is $f \in B_{X^*}$ with $d \leq f(y) - f(x)$ and $f(x) \leq f(z) \leq f(y)$, for every $z \in S$.
(2) Let $S$ be a $\lambda-$equilateral ($\|x - y\| = \lambda$ for every $x \neq y \in S$) subset of $X$. Then the set $S$ is a $(M, 1, \lambda)-$b.s.a subset of $X$, where $M = \sup\{\|x\| : x \in S\}$.
(3) Let $\{(x_\gamma, x_\gamma^*) : \gamma \in \Gamma\}$ be a bounded biorthogonal system in $X$ with $M \geq \|x_\gamma\| \left\|x_\gamma^*\right\|$, for every $\gamma \in \Gamma$. We consider the biorthogonal system $\{(y_\gamma, y_\gamma^*) : \gamma \in \Gamma\}$ with $y_\gamma = \frac{x_\gamma}{\|x_\gamma\|}$ and $y_\gamma^* = \|x_\gamma\| x_\gamma^*$, $\gamma \in \Gamma$. Then the minimal system $\{y_\gamma : \gamma \in \Gamma\}$ is a $(1, M, 1)-$b.s.a subset of $X$.
(4) The Elton-Odell Theorem states that: If $\dim X = \infty$, then there exists a $(1 + \epsilon)-$separated sequence in $S_X$. Therefore $K(X) > 1$. Since it is apparent that $K(X) \leq 2$ we get that $1 < K(X) \leq 2$.
(5) Since every infinite dimensional Banach space $X$ contains an infinite Auerbach system, that is, a biorthogonal system $\{(x_n, x_n^*) : n \in \mathbb{N}\}$ such that $\|x_n\| = \|x_n^*\| = 1$, for $n \in \mathbb{N}$, (see [6] and [11] Th. 1.20) by (3) we get that $K_a(X) \geq 1$. We also have that $K_a(X) \leq K(X)$.
(6) Let $Y$ be a subspace of $X$. Then $K(Y) \leq K(X)$ and $K_a(Y) \leq K_a(X)$.

In the next Proposition, which strengthens Theorem 3 of [12], it is proved that if a Banach space $X$ contains isomorphically $c_0$ or $l_p$ for some $1 \leq p \leq \infty$, then $K_a(X) > 1$.

**Proposition 1.2.** Let $X$ be a Banach space. If $X$ contains isomorphically $c_0$ or $l_1$, then $K_a(X) = 2$. If $X$ contains $l_p$ for some $1 < p < \infty$, then $K_a(X) \geq 2^{1/p}$.

*Proof.* We start by observing that the set $\{\sum_{k=1}^{n} e_k - e_{n+1} : n \in \mathbb{N}\}$ is both normalized and 2-equilateral in $c_0$ and the set $\{e_n : n \in \mathbb{N}\}$ is normalized and $2^{1/p}-$equilateral in $l_p$, $1 \leq p < \infty$. We may assume that $(X, \|\cdot\|) \equiv c_0$ or $l_p$, $1 \leq p < \infty$ and it suffices to prove the conclusion for an equivalent norm $\|\cdot\|'$ on $X$. We tackle separately the cases $(X, \|\cdot\|) \equiv c_0$ and $(X, \|\cdot\|) \equiv l_p$, $1 < p < \infty$. The case $(X, \|\cdot\|) \equiv l_1$ can be proved with any of the two ways that will be presented, so it is excluded. Our proof is based essentially on the proof of James' Non-distortion Theorem and the remarks that follow its proof in [14] Prop. 2.e.3. Let $(X, \|\cdot\|) \equiv c_0$ or $l_p$, $1 \leq p < \infty$ and $\|\cdot\|'$ an equivalent norm on $X$. Then for every $\epsilon > 0$ there exists a block basic sequence $(x_k)$ of the canonical basis of $X$, $(e_n)$, such that

$$\|x_k\|' = 1,\ k \in \mathbb{N} \text{ and}$$
$$\left\|\sum_{k=1}^{\infty} a_k x_k\right\|' \geq \frac{1}{1+\epsilon}\,\|(a_k)\| \text{ for every } (a_k) \subseteq X.$$

Moreover, if $(X, \|\cdot\|) \equiv c_0$ or $l_1$ we have that

$$(1+\epsilon)\,\|(a_k)\| \geq \left\|\sum_{k=1}^{\infty} a_k x_k\right\|', \text{ for every } (a_k) \subseteq X.$$

We note that the space $Z = [x_k]$ is isomorphic to $X$ since the bases $(x_k)$ and $(e_k)$ are equivalent. With $(x_k^*)$ we denote the biorthogonal functionals of $(x_k)$.

Let now $(X, \|\cdot\|) \equiv l_p$, $1 < p < \infty$, $\epsilon > 0$ and $(x_k)$ a block basic sequence of $(e_n)$ such that

$$\|x_k\|' = 1,\ k \in \mathbb{N} \text{ and}$$
$$\left\|\sum_{k=1}^{\infty} a_k x_k\right\|' \geq \frac{1}{1+\epsilon}\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{1/p}, \text{ for every } (a_k) \subseteq X.$$

We will show that the set $S = \{x_k : k \in \mathbb{N}\}$ is a $(1, 1+\epsilon, 2^{1/p})-$b.s.a. subset of $Z$. Indeed, we have that,

$$\begin{aligned}\|x_k^*\|' &= \sup\left\{|a_k| : \left\|\sum_{n=1}^{\infty} a_n x_n\right\|' \leq 1\right\}\\ &\leq \sup\left\{|a_k| : \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \leq 1+\epsilon\right\},\ k \in \mathbb{N}\end{aligned}$$

and for $k, l \in \mathbb{N}$ with $k \neq l$,

$$\begin{aligned}\|x_k^* - x_l^*\|' &= \sup\left\{|a_k - a_l| : \left\|\sum_{n=1}^{\infty} a_n x_n\right\|' \leq 1\right\}\\ &\leq \sup\left\{|a_k - a_l| : \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \leq 1+\epsilon\right\}\\ &= (1+\epsilon)2^{1/q}, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.\end{aligned}$$

For $k, l \in \mathbb{N}$ with $k \neq l$ we set $g_{kl} = 2^{(1/p)-1}(x_k^* - x_l^*) \in Z^*$, then

$$\|g_{kl}\|' = 2^{(1/p)-1} \|x_k^* - x_l^*\| \leq 2^{(1/p)-1}(1+\epsilon)2^{1/q} = (1+\epsilon) \text{ and}$$

$$g_{kl}(x_k) = 2^{(1/p)-1}, \; g_{kl}(x_m) = 0, \; g_{kl}(x_l) = -2^{(1/p)-1}, \; m \notin \{k, l\}.$$

Thus $S$ is a bounded and separated antipodal subset of $Z$ with constants $c_1 = 1$, $c_2 = 1+\epsilon$, $d = g_{kl}(x_k) - g_{kl}(x_l) = 2 \cdot 2^{1/p-1} = 2^{1/p}$.

From (1) of Remark 1.1 now it is direct that $S$ is a $\left(1, 1, \frac{2^{1/p}}{1+\epsilon}\right)$ −b.s.a. subset of $Z$, so we get that $K_a(X) \geq 2^{1/p}$.

For the case $(X, \|\cdot\|) \equiv c_0$ we consider $\epsilon > 0$ and $(x_k)$ a block basic sequence of $(e_n)$ such that

$$\|x_k\|' = 1, \; k \in \mathbb{N} \text{ and}$$

$$(1+\epsilon)\|(a_k)\|_\infty \geq \left\|\sum_{k=1}^{\infty} a_k x_k\right\|' \geq \frac{1}{1+\epsilon}\|(a_k)\|_\infty, \text{ for every choice}$$

of scalars, $(a_k)$, tending to zero.

For $n \in \mathbb{N}$,

$$\left\|\sum_{k=1}^{n} x_k - x_{n+1}\right\|' \leq (1+\epsilon)\left\|\sum_{k=1}^{n} e_k - e_{n+1}\right\| = (1+\epsilon) \text{ and}$$

$$\begin{aligned} \|x_n^*\|' &= \sup\left\{|a_n| : \left\|\sum_{k=1}^{\infty} a_k x_k\right\|' \leq 1\right\} \\ &\leq \sup\left\{|a_n| : \left\|\sum_{k=1}^{\infty} a_k x_k\right\| \leq 1+\epsilon\right\} \leq (1+\epsilon). \end{aligned}$$

We set $y_n = \frac{1}{1+\epsilon}\left(\sum_{k=1}^{n} x_k - x_{n+1}\right)$ and $y_n^* = \frac{1}{1+\epsilon}x_n^*$. We will show that the set $S = \{y_n : n \in \mathbb{N}\}$ is a $\left(1, 1, \frac{2}{(1+\epsilon)^2}\right)$ −b.s.a subset of $(X, \|\cdot\|')$. Indeed for $m < n$ and $k \in \mathbb{N}$ it is easy to see that

$$-\frac{1}{(1+\epsilon)^2} = y_{m+1}^*(y_m) \leq y_{m+1}^*(y_k) \leq y_{m+1}^*(y_n) = \frac{1}{(1+\epsilon)^2}.$$

Since $\epsilon > 0$ was arbitrary the proof is complete. ∎

**Remark 1.3.** (1) From a classical result we know that $K(l_p) = 2^{1/p}$, $1 < p < \infty$, (see [4]). Thus $K_a(l_p) \leq K(l_p) = 2^{1/p}$. On the other hand the canonical basis $\{e_n : n \in \mathbb{N}\}$ of $l_p$ is a normalized $2^{1/p}$−equilateral subset of $l_p$, so $K_a(l_p) \geq 2^{1/p}$ and consequently $K_a(l_p) = 2^{1/p}$.

(2) If $1 < p < \infty$ and $X \cong l_p$ it is not valid that $K_a(X) = 2^{1/p}$. There exists a renorming $\|\cdot\|'$ of $l_2$ such that $K_a(l_2, \|\cdot\|') \geq \sqrt{3} > \sqrt{2}$ (see [15]). Also in Proposition 1.13 it is proved that for any Banach space $X$ there exists an equivalent norm such that $K_a(X, \|\cdot\|') = 2$.

**Proposition 1.4.** Let $X$ be a uniformly smooth Banach space. Then $K_a(X) > 1$.

*Proof.* Let $\{(x_i, x_i^*) : i \in \mathbb{N}\}$ be an Auerbach system in $X$. The space $X$ is uniformly smooth, so its dual space is uniformly convex. From the strict convexity of $X^*$ for $i \neq j$ and $s, t \in (0, 1)$ with $s + t = 1$ we have

$$\|s x_i^* - t x_j^*\| < 1. \tag{1.1}$$

For $i \neq j$, we set $\lambda_{ij} = \frac{1}{\left\| \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right\|}$. By (1.1) $\lambda_{ij} > 1$. Also

$$\left\| \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right\| \geq \left( \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right)(x_i) = \frac{1}{2}.$$

So $\lambda_{ij} \in (1, 2]$, $i \neq j$. We choose $M \in [\mathbb{N}]^\omega$ such that the following limit exists

$$\lim_{i<j\in M} \lambda_{ij}.$$

We will show that $\lim_{i<j} \lambda_{ij} > 1$. Let now $M = (i_k)_{k\in\mathbb{N}}$. We assume that $\lim_{i<j\in M} \lambda_{ij} = 1$. Then the sequences

$$\left(\left\|\lambda_{i_k i_{k+1}} x_{i_k}^*\right\|\right)_{k\in\mathbb{N}}, \ \left(\left\|\lambda_{i_k i_{k+1}} x_{i_{k+1}}^*\right\|\right)_{k\in\mathbb{N}}, \ \left(\left\|\frac{1}{2}\lambda_{i_k i_{k+1}}(x_{i_k}^* - x_{i_{k+1}}^*)\right\|\right)_{k\in\mathbb{N}}$$

all converge to 1. By the uniform convexity of $X^*$ we get that

$$\left\| x_{i_k}^* + x_{i_{k+1}}^* \right\| \to 0, \ k \to \infty \ [16].$$

On the other hand $\left( x_{i_k}^* + x_{i_{k+1}}^* \right)(x_{i_k}) = 1$, for $k \in \mathbb{N}$, a contradiction. Thus there exist $i_0 \in M$ and $c > 1$ such that $\lambda_{ij} \geq c$, for every $i_0 \leq i < j \in M$. For $i_0 \leq i < j \in M$, from the choice of $\lambda_{ij}$ we take that

$$\left\| \lambda_{ij} \left( \frac{1}{2}x_i^* - \frac{1}{2}x_i^* \right) \right\| = 1.$$

Further

$$\lambda_{ij} \left( \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right)(x_i - x_j) = \lambda_{ij} \left( \frac{1}{2} + \frac{1}{2} \right) = \lambda_{ij} \geq c > 1$$

and. of course,

$$\lambda_{ij} \left( \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right)(x_k) = 0, \ k \notin \{i, j\}.$$

Our proof is complete. ■

**Corollary 1.5.** Let $(X, \|\cdot\|)$ be a Banach space such that $(X^*, \|\cdot\|)$ is strictly convex (so $(X, \|\cdot\|)$ is smooth) and $\{(x_i, x_i^*) : i \in \mathbb{N}\}$ is an Auerbach system in $X$. Then the set $\{x_i : i \in \mathbb{N}\}$ is a bounded and separated antipodal set with constants $c_1 = c_2 = 1$ and $d = (1+)$ (that is for every $i \neq j \in \mathbb{N}$ $\exists f \in B_{X^*}$ such that $1 < f(x_i) - f(x_j)$ and $f(x_j) \leq f(x_k) \leq f(x_i)$, $k \in \mathbb{N}$).

*Proof.* As in the proof of Proposition 1.4 from the strict convexity of $X^*$ for $i \neq j \in \mathbb{N}$ we set $\lambda_{ij} = \frac{1}{\left\| \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right\|}$, $f = \lambda_{ij} \left( \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right)$, and we get that $\lambda_{ij} > 1$ and $\|f\| = 1$. Thus for $i \neq j \in \mathbb{N}$ and $k \in \mathbb{N}$ we have

$$f(x_i) - f(x_j) = \lambda_{ij} \left( \frac{1}{2}x_i^* - \frac{1}{2}x_j^* \right)(x_i - x_j) = \lambda_{ij} \left( \frac{1}{2} + \frac{1}{2} \right) = \lambda_{ij} > 1 \text{ and}$$

$$-\frac{\lambda_{ij}}{2} = f(x_j) \leq f(x_k) \leq f(x_i) = \frac{\lambda_{ij}}{2}.$$

Our proof is complete. ■

The next result concerns smooth Banach spaces of finite dimension.

**Proposition 1.6.** Let $X$ be a smooth Banach space with $\dim X = n$. Then $X$ contains a $(1, 1, d)-$b.s.a. set with $d > 1$ and cardinality $2n$.

*Proof.* Let $\{(x_i, x_i^*) : 1 \le i \le n\}$ be an Auerbach basis of $X$. The space $X$ is smooth consequently $X^*$ is strictly convex. Thus, since $\left\|\frac{1}{2}x_i^* \pm \frac{1}{2}x_j^*\right\| < 1$, for every $1 \le i \ne j \le n$, there exist $a, b > 1$ such that

$$\left\|a\left(\frac{1}{2}x_i^* + \frac{1}{2}x_j^*\right)\right\| = 1$$

and

$$\left\|b\left(\frac{1}{2}x_i^* - \frac{1}{2}x_j^*\right)\right\| = 1.$$

Equivalently for every $1 \le i \ne j \le n$ there exist $s, t > \frac{1}{2}$ such that $\left\|s(x_i^* + x_j^*)\right\| = 1$ and $\left\|t(x_i^* - x_j^*)\right\| = 1$.

We will show that the set $\{\pm x_i : 1 \le i \le n\}$ satisfies the conclusion. Let now $1 \le i \ne j \le n$, we set $\phi_1 = t\left(x_i^* - x_j^*\right)$ and $\phi_2 = s\left(x_i^* + x_j^*\right)$. Then

(1) $$x_i^*(x_i) = 1,\ x_i^*(-x_i) = -1,\ x_i^*(x_i + x_i) = 2 \text{ and } x_i^*(x_k) = 0,$$

for every $1 \le k \ne i \le n$. So $x_i$ and $-x_i$are separated by $x_i^*$.

(2) $$\phi_1(x) = \begin{cases} t, & x = x_i \\ -t, & x = x_j \\ -t, & x = -x_i \\ t, & x = -x_j \\ 0, & x = \pm x_k,\ k \notin \{i, j\} \\ t + t > 1, & x = x_i - x_j \end{cases}$$

So the pairs $x_i, x_j$ and $-x_i, -x_j$ are separated by $\phi_1$.

(3) $$\phi_2(x) = \begin{cases} -s, & x = -x_i \\ s, & x = x_j \\ s, & x = x_i \\ -s, & x = -x_j \\ 0, & x = \pm x_k,\ k \notin \{i, j\} \\ s + s > 1, & x = x_j - x_i \end{cases}$$

Consequently the pairs $-x_i, x_j$, and $x_i, -x_j$, are separated by $\phi_2$, so we are finished. ■

We mention here that for every uniformly smooth Banach space, $X$ and each $\{(x_i, x_i^*) : i \in \mathbb{N}\}$ Auerbach system in $X$ there exists an infinite subset $M$ of $\mathbb{N}$ such that the set $\{\pm x_i : i \in M\}$ is a $(1, 1, d)-$subset of $X$, with $d > 1$. Indeed as in the proof of Proposition 1.4 we can prove that there exist an infinite subset $M$ of $\mathbb{N}$ and $\mu_{ij} \in (1, 2]$, for $i \ne j \in M$ such that

$$\left\|\mu_{ij}\left(\frac{1}{2}x_i^* + \frac{1}{2}x_j^*\right)\right\| = 1$$

and

$$\mu_{ij} \ge r > 1, \text{ for } i \ne j \in M.$$

Now as in the proof of Proposition 1.6 we take that $x_i, -x_i$ are separated by $x_i^*$, the points $x_i, x_j$ are separated by $\lambda_{ij}\left(\frac{1}{2}x_i^* - \frac{1}{2}x_j^*\right)$ and $x_i, -x_j$ are separated by $\mu_{ij}\left(\frac{1}{2}x_i^* + \frac{1}{2}x_j^*\right)$, for $i \ne j \in M$. Analogously it can be proved that for every Banach space $X$ such that $(X^*, \|\cdot\|)$ is strictly convex and each Auerbach system in, $\{(x_i, x_i^*) : i \in \mathbb{N}\}$ in $X$ the set $\{\pm x_i : i \in \mathbb{N}\}$ is a bounded and separated antipodal subset of $X$ with constants $c_1 = c_2 = 1$ and $d = (1+)$.

It has been noted that the fact that, $K_a(X) > 1$, for every Banach space $X$ is a strengthening of the Elton-Odell Theorem. The proof of this fact in the case $X$ contains isomorphically $c_0$ or $l_p$ for some $1 \le p < \infty$ or $X$ is uniformly smooth is independent from the Elton-Odell Theorem. The last no longer remains true in our approach, for the general case or for the case where $X$ is reflexive or $X$ has a separable dual. For the last two cases we need some parts of the proof of the Elton-Odell Theorem. For the general case Theorem 4.1 of [9], which is an independent proof of the Elton-Odell Theorem, is needed. To utilize these results we are going to prove Theorem 1.7 below. In our proof we need Theorem 1 of [17] which we now state.

**Theorem** (1 of [17])**.** Let $(X, \|\cdot\|)$ be a Banach space. Then for every $\epsilon > 0$ there exists an equivalent norm $\|\cdot\|'$ on $X$ such that:

(1) $d((X, \|\cdot\|), (X, \|\cdot\|')) \le 1 + \epsilon$ (the Banach-Mazur distance) and
(2) $(X, \|\cdot\|')$ contains an infinite equilateral set.

**Theorem 1.7.** Let $X$ be a Banach space and $(x_n)$ a normalized weakly null and $d-$separated sequence in $X$ with $d > 0$. Then $K_a(X) \ge d$.

*Proof.* As $d-$separated, with $d > 0$, the sequence $(x_n)$, has no norm convergent subsequence. In addition $(x_n)$ is weakly null. From the method of the proof of Theorem 1 of [17] we have the following.

For every $\epsilon \in (0, 1)$, there exist a subsequence of $(x_n)$, still denoted by $(x_n)$, and an equivalent norm $\|\cdot\|'$ on $X$ such that

$$\frac{1}{1+\epsilon} \|x\| \le \|x\|' \le \frac{1}{(1-\epsilon)^2} \|x\|, \text{ for every } x \in X \text{ and} \tag{1.2}$$

$$\|sx_n + tx_m\|' = \lim_{\substack{k<l \\ k\to\infty}} \|sx_k + tx_l\|, \text{ for every } n \ne m \text{ and } s, t \in \mathbb{R}. \tag{1.3}$$

By (1.3), $\|x_n\|' = 1$, $n \ge 1$ and $\|x_n - x_m\|' = \lim_{\substack{k<l \\ k\to\infty}} \|x_k - x_l\| = \lambda \ge d$, $n \ne m$. Thus $(x_n)$ is a $\lambda-$equilateral subset of $S_X^{\|\cdot\|'}$ with $\lambda \ge d$. So from (2) of Remark 1.1 the set $\{x_n : n \in \mathbb{N}\}$ is a $(1, 1, \lambda)-$b.s.a subset of $(X, \|\cdot\|')$ (see also the proof of Proposition 2 of [17]). Now for $n \ne m \in \mathbb{N}$, we choose $f_{nm}$ in $S_{X^*}^{\|\cdot\|'}$ such that,

$$f_{nm}(x_n - x_m) = \|x_n - x_m\|' = \lambda \ge d \text{ and} \tag{1.4}$$

$$f_{nm}(x_m) \le f_{nm}(x_k) \le f_{nm}(x_n), \text{ for every } k \notin \{n, m\}. \tag{1.5}$$

By (1.2) we also have for the dual norms

$$\frac{1}{1+\epsilon} \|x^*\|' \le \|x^*\| \le \frac{1}{(1-\epsilon)^2} \|x^*\|', \text{ for every } x^* \in X^*. \tag{1.6}$$

Now we set $g_{nm} = (1-\epsilon)^2 f_{nm}$. (1.6) then gives $\|g_{nm}\| \le \|f_{nm}\|' = 1$, $n \ne m$. Moreover by (1.4)

$$g_{nm}(x_n - x_m) = (1-\epsilon)^2 \lambda \ge (1-\epsilon)^2 d, \ n \ne m.$$

Combining these inequalities with (1.5) we get that the set $\{x_n : n \in \mathbb{N}\}$ is a $(1, 1, (1-\epsilon)^2 d)-$b.s.a subset of $(X, \|\cdot\|)$. Since $\epsilon \in (0, 1)$ was arbitrary, $K_a(X) \ge d$. ■

By Theorem 1.7 our strategy to prove Corollaries 1.8 and 1.9 and Theorem 1.10 will be to produce a normalized weakly null sequence which is $d-$separated with $d > 1$. Also for the second and third of these results we assume, as we may by Proposition 1.2, that each Banach space does not contain isomorphically $c_0$ or $l_1$.

**Corollary 1.8.** Let $X$ be a reflexive Banach space. Then $K_a(X) > 1$.

*Proof.* Since the Banach space $X$ does not contain isomorphically $c_0$, from the proof of Elton-Odell's theorem we take that $X$ contains a normalized, basic and $d-$separated sequence $(x_n)$ with $d > 1$. Since the sequence $(x_n)$ is basic we have, by Lemma 1.6.1 of [1] and by the compactness of $(B_X, w)$, that 0 is the only weak cluster point of $(x_n)$ and hence $x_n \xrightarrow{w} 0$. ■

**Corollary 1.9.** Let $X$ be a Banach space with separable dual. Then $K_a(X) > 1$.

*Proof.* Since $X$ has a separable dual we can choose a normalized and weakly null shrinking basic sequence $(y_n)$ in $X$ (Prop. 1.b.13 [14] and Prop. 3.2.7 [1]). Further, by the remarks that follow Theorem 1.a.5 of [14] we may also assume that $\|P_n\| \leq 1 + 20^{-n}$, $n \in \mathbb{N}$, for the associated projections to the basic sequence $(y_n)$. Now again by the proof of the Elton-Odell Theorem (recall that $X$ does not contain isomorphically $c_0$) there exists a block basic sequence $(x_n)$ of $(y_n)$ and $d > 1$ such that $\|x_n - x_m\| > d$, for every $n \neq m \in \mathbb{N}$. Again by Proposition 3.2.7 of [1] the sequence $(x_n)$ is weakly null, so we are finished. ■

Now we pass to the proof of the general case.

**Theorem 1.10.** Let $X$ be a Banach space. Then $K_a(X) > 1$.

*Proof.* Since $X$ does not contain isomorphically $l_1$, by Rosenthal's $l_1-$Theorem [19], we may choose a basic, normalized and weakly null sequence $(x_n)$ in $X$. Now $X$ does not contain isomorphically $c_0$, so by Theorem 4.1 of [9] there exists a normalized weakly null block-basic sequence $(y_n)$ of $(x_n)$ with spreading model $(e_i)$ such that $\|e_1 - e_2\| > 1$. For the definition of spreading models we refer the reader to [3] and [1]. For our purposes the following property suffices:

$$\lim_{\substack{n<m \\ n\to\infty}} \|y_n - y_m\| = \|e_1 - e_2\| > 1.$$

This proof ends rather abruptly. Are you sure it's properly written? ■

It has been noted that $K_a(X) \leq K(X)$, for any Banach space $X$. It is unknown to us if there exists a Banach space $X$ such that $K_a(X) < K(X)$. What we do have is a partial answer in the case $X$ is reflexive, where $K_a(X) = K(X)$.

**Theorem 1.11.** Let $X$ be a reflexive Banach space. Then $K_a(X) = K(X)$.

*Proof.* Let $0 < \lambda < K(X)$ and $(x_n)$ a normalized $\lambda-$separated sequence in $X$. By the reflexivity of $X$ we may assume that $x_n \xrightarrow{w} x_0$, for some $x_0 \in B_X$. We consider now the semi-normalized, weakly null and $\lambda-$separated sequence $(y_n)$ with $y_n = x_n - x_0$, $n \in \mathbb{N}$. As in Theorem 1.7 for every $\epsilon \in (0,1)$ there exists an equivalent norm $\|\cdot\|'$ in $X$ such that,

$$\frac{1}{1+\epsilon}\|x\| \leq \|x\|' \leq \frac{1}{(1-\epsilon)^2}\|x\|, \text{ for } x \in X,$$

$$\|sy_n + ty_m\|' = \lim_{k<l}\|sy_k + ty_l\|, \text{ for } n \neq m, s, t \in \mathbb{R} \text{ and}$$

$$\frac{1}{1+\epsilon}\|x^*\|' \leq \|x^*\| \leq \frac{1}{(1-\epsilon)^2}\|x^*\|', \text{ for } x^* \in X^*.$$

Thus the sequence $(y_n)$ is $\|\cdot\|'-$equilateral, so the set $\{y_n : n \in \mathbb{N}\}$ is a $(c_1, 1, \lambda)-$b.s.a subset in $\|\cdot\|'$, with $c_1 = \sup\{\|y_n\| : n \in \mathbb{N}\}$. For $n \neq m \in \mathbb{N}$ we consider $f_{nm} \in B_{X^*}^{\|\cdot\|'}$ such that

$f_{nm}(y_m) \le f_{nm}(y_k) \le f_{nm}(y_n)$, for every $k \in \mathbb{N}$ and $f_{nm}(y_n) - f_{nm}(y_m) \ge \lambda$. Further we put $g_{nm} = (1-\epsilon)^2 f_{nm}$, for $n \ne m \in \mathbb{N}$. Then for $n \ne m \in \mathbb{N}$ we have

$$\|g_{nm}\| \le 1 \tag{1.7}$$

$$g_{nm}(y_m) \le g_{nm}(y_k) \le g_{nm}(y_n), \text{ for every } k \in \mathbb{N} \text{ and} \tag{1.8}$$

$$\begin{aligned} g_{nm}(x_n - x_m) &= g_{nm}((x_n - x_0) - (x_m - x_0)) \\ &= g_{nm}(y_n - y_m) \\ &= (1-\epsilon)^2 f_{nm}(y_n - y_m) \\ &\ge (1-\epsilon)^2 \lambda. \end{aligned}$$

Thus $K_a(X) \ge (1-\epsilon)^2\lambda$, for every $0 < \epsilon < 1$ and $\lambda < K(X)$, and consequently $K_a(X) = K(X)$. ■

**Remark 1.12.** Let us summarize some of the current known results concerning the parameters $K_a(X)$ and $K(X)$. Let X be a Banach space.

(1) $K_a(X) \le K(X) \le 2$.
(2) $K(Y) \le K(X)$ and $K_a(Y) \le K_a(X)$, where $Y$ is an infinite dimensional subspace of $X$.
(3) $1 < K(X)$ [8].
(4) $\sqrt[5]{4} \le K(X)$, if $X$ is non reflexive space [13].
(5) $1 < K_a(X)$ (Theorem 1.10).
(6) $K_a(X) = K(X)$, if $X$ is reflexive (Theorem 1.11).
Particularly, $K_a(X) = K(X) \ge 2^{1/p}$ if $X \cong l_p$, $1 < p < \infty$ and $K_a(X) = K(X) = 2$ if $X \cong c_0$ or $l_1$ (Proposition 1.2). Note that the case $1 < p < \infty$ of Proposition 1.2 is an easy consequence of Theorem 3 of [12] and of Theorem 1.11.

Let $X$ be a Banach space. With $[X]$ we denote the class of Banach spaces $Y$ such that $X \cong Y$ and we define a pseudo-metric $D$ on $[X]$ in the following way:

$$D(X,Y) = \inf\{\log \|T\| \left\|T^{-1}\right\| : X \cong^T Y\}.$$

In [12] Kottman defines for each Banach space $X$ the set

$$\overline{K(X)} = \{K(Y) : Y \cong X\}$$

and proves (Theorem 7) that there exists $b \in [1,2]$ such that $\overline{K(X)} = (b,2]$ or $[b,2]$. The same way we define the set

$$\overline{K_a(X)} = \{K_a(Y) : Y \cong X\}$$

and we prove a similar result.

**Proposition 1.13.** For each Banach space $X$ there exists $b \in [1,2]$ such that $\overline{K_a(X)} = (b,2]$ or $[b,2]$.

*Proof.* First we will show that $\max \overline{K_a(X)} = 2$. This can be done the same way that Kottman proved in Theorem 7 of [12] that $\max \overline{K(X)} = 2$ (essentially Kottman proves that $\max \overline{K_a(X)} = 2$). We briefly describe his argument. We consider an Auerbach system $\{(x_i, x_i^*) : i \in \mathbb{N}\}$ in $X$ and put $V = \text{conv}\{B_X \cup \{\pm 2x_i : i \in \mathbb{N}\}\}$. Further we consider the Minkowski functional of $V$, which here is an equivalent norm, $\|\cdot\|'$, on $X$ such that

$$\|x\| \le 2\|x\|' \le 2\|x\|, \; x \in X$$

(See also the proof of Theorem 3 of [17]). Now it is not difficult to prove that the set $\{\pm 2x_i : i \in \mathbb{N}\}$ is a normalized 2-equilateral in the norm $\|\cdot\|'$. Therefore $K_a(\|\cdot\|') = 2$.

We know that the topological space $([X], D)$ is connected [12], consequently it suffices to show that the function $K_a : [X] \to \mathbb{R}$ is continuous. Let $Y \in [X]$ and $\epsilon > 0$. We choose $\delta > 1$ such that

$$2 - \frac{2}{\delta} < \epsilon \text{ and } \delta^2 < K_a(Y) \ (K_a(Y) > 1, \text{ Theorem 1.10}). \tag{1.9}$$

Let also $Z \in [X]$ with $D(Y, Z) \leq \log \delta$. Then there exists a linear operator $T : Y \to Z$ with $\|y\|_Y \leq \|T(y)\|_Z \leq \delta \|y\|_Y$, for every $y \in Y$ (so we have in addition that $\|T^{-1}\| \leq 1$). We consider an arbitrary $s$ with $\delta < s < K_a(Y)$, $\{y_i : i \in \mathbb{N}\}$ a $(1, 1, s)-$b.s.a. subset of $Y$ and $f_{ij} \in B_{Y^*}$ such that $f_{ij}(y_j) \leq f_{ij}(y_k) \leq f_{ij}(y_i)$ and $f_{ij}(y_i) - f_{ij}(y_j) \geq s$, for every $i \neq j \in \mathbb{N}$ and $k \in \mathbb{N}$. Now we put

$$z_i = \frac{T(y_i)}{\delta} \text{ and } g_{ij} = f_{ij} \circ T^{-1}, \text{ for } i \neq j \in \mathbb{N}.$$

Then $\|z_i\| \leq 1$ and $\|g_{ij}\| \leq \|f_{ij}\| \left\|T^{-1}\right\| \leq 1$, for $i \neq j \in \mathbb{N}$. Also if $k \in \mathbb{N}$

$$\begin{aligned} g_{ij}(z_j) &= \left(f_{ij} \circ T^{-1}\right)\left(\frac{T(y_j)}{\delta}\right) = f_{ij}\left(\frac{y_j}{\delta}\right) \leq f_{ij}\left(\frac{y_k}{\delta}\right) \\ &= g_{ij}(z_k) \leq f_{ij}\left(\frac{y_i}{\delta}\right) = g_{ij}(z_i) \end{aligned}$$

and

$$\begin{aligned} g_{ij}(z_i - z_j) &= \left(f_{ij} \circ T^{-1}\right)\left(\frac{T(y_i)}{\delta} - \frac{T(y_j)}{\delta}\right) = \frac{1}{\delta}\left(f \circ T^{-1}\right)\left(T(y_i) - T(y_j)\right) \\ &= \frac{1}{\delta} f_{ij}(y_i - y_j) \geq \frac{s}{\delta} > 1. \end{aligned}$$

Since $s$ was an arbitrary point of $(\delta, K_a(Y))$ we get that $K_a(Z) \geq \frac{K_a(Y)}{\delta}$. We have that

$$K_a(Y) - K_a(Z) \leq K_a(Y) - \frac{K_a(Y)}{\delta} \leq 2 - \frac{2}{\delta} < \epsilon \text{ (by 1.9)}.$$

Again by (1.9) $K_a(Y) \geq \delta^2$, so $K_a(Z) \geq \frac{K_a(Y)}{\delta} > \delta$. By the last inequality we can repeat our proof exchanging the roles of $Y$ and $Z$, to take $K_a(Z) - K_a(Y) < \epsilon$ thus $|K_a(Z) - K_a(Y)| < \epsilon$, so we are finished. ■

**Remark 1.14.** Let $X$ be a Banach space.

(1) Let $Y$ be an infinite dimensional closed subspace of $X$. Since the sets $\overline{K(X)}$ and $\overline{K_a(X)}$ are intervals of the form $(b, 2]$ or $[b, 2]$ for some $b \in [1, 2]$, by (2) of Remark 1.12 we obtain that

$$\overline{K(X)} \subseteq \overline{K(Y)} \text{ and } \overline{K_a(X)} \subseteq \overline{K_a(Y)}.$$

(2) Proposition 1.2, Remark 1.3 and Remark 1.12 yield that:
  (a) If $X \cong l_p$, $1 < p < \infty$, then $\overline{K_a(X)} = \overline{K(X)} = [2^{1/p}, 2]$.
  (b) If $X \cong l_1$ or $X \cong c_0$, then $\overline{K_a(X)} = \overline{K(X)} = \{2\}$.
  (c) If $X$ is reflexive, then $\overline{K_a(X)} = \overline{K(X)}$.
  (d) If $X$ is non reflexive, then $\inf \overline{K(X)} \geq \sqrt[5]{4}$ [13].
  (e) The Elton-Odell Theorem [8] and Theorem 1.10 yield that

$$\overline{K(X)} \subseteq (1, 2] \text{ and } \overline{K_a(X)} \subseteq (1, 2].$$

We note that the following question seems to be open:

Does there exist a Banach space $X$ such that $\overline{K(X)} = (1, 2]$?

Should there exists such a space, it must be a space that is reflexive and does not contain isomorphically any of the spaces $l_p$, ( Remark 1.14 (2) (a) and (d)). So such a space would have the properties of the space of Tsirelson.

The study of bounded and separated antipodal sets with constants $c_1 = c_2 = 1$ and $d > 1$ is also interesting in finite dimensional spaces. In this case we would be interested in the cardinality of such sets. We note that the cardinality of a $(1, 1, d > 1)-$b.s.a set in a finite dimensional space $(X, \|\cdot\|)$ is also finite. By a result of Danzer and Grünbaum [5] the maximum cardinality of an antipodal set in $\mathbb{R}^n$ is $2^n$, so the cardinality of a $(1, 1, d > 1)-$b.s.a set in $(X, \|\cdot\|)$, where $\dim X = n$, cannot exceed $2^n$. On the other hand a $(1, 1, d > 1)-$b.s.a set in $X$ is $d-$separated, thus known estimations of the cardinality of normalized $d-$separated subsets of $X$ with $d > 1$, may play some role see [2] and [10]). In Proposition 1.6 we proved that that if $X$ is a $n-$dimensional smooth Banach space then $X$ contains a $(1, 1, d > 1)-$b.s.a set of cardinality $2n$. Note that results on $(1, 1, d > 1)-$b.s.a sets will appear elsewhere.

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