BULLETIN OF THE
HELLENIC MATHEMATICAL SOCIETY
Volume 63, 2019 (13–16)

A NOTE ON SINGLE VALUED PRIMITIVES

V. NESTORIDIS

Abstract. We prove that if a holomorphic function f on a domain $\Omega \subset \mathbb{C}$ admits single valued primitives on Ω of any order, then f extends holomorphically to the least simply connected domain $\widehat{\Omega} \subset \mathbb{C}$, containing Ω.

Dedicated to Professor Alain Bernard

1. Introduction

It is well known that any holomorphic function on a simply connected open set admits a single valued primitive of any order. On multiply connected open sets the previous result fails; for instance, on $\mathbb{C} \setminus \{0\}$ the function $\dfrac{1}{z}$ does not admit a single valued primitive, as $\log z$ is not single valued on $\mathbb{C} \setminus \{0\}$.

Suppose now, that Ω is a multiply connected open set and let f be a holomorphic function on Ω, admitting single valued primitives on Ω of any order. What can be said then?

If Ω is connected, then we will prove that f admits a holomorphic extension on $\widehat{\Omega}$, which is the least simply connected domain such that $\Omega \subset \widehat{\Omega} \subset \mathbb{C}$. The set $\widehat{\Omega}$ is equal to X^c, where X is the component of ∞ in the set $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$. Since Ω is connected, then we will see that $\widehat{\Omega}$ is also connected, as well. For a general open set $\Omega \subset \mathbb{C}$, the envelope $\widehat{\Omega}$ can be defined as the least simply connected open set, such that $\Omega \subset \widehat{\Omega} \subset \mathbb{C}$. Then, one can easily see that $\widehat{\Omega} = X^c$, where X is the component of ∞ in $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$. Suppose that $\Omega \subset \mathbb{C}$ is a connected open set and let G be the component in $\widehat{\Omega}$ containing Ω. Then, G is a simply connected open set such that $\Omega \subset G \subset \mathbb{C}$ and $G \subset \widehat{\Omega}$. Since, $\widehat{\Omega}$ is the least open set with the above property, it follows that $G = \widehat{\Omega}$. Therefore, for any domain $\Omega \subset \mathbb{C}$, its simply connected envelope $\widehat{\Omega}$ is also a domain. In this case, if f is a holomorphic function on Ω admitting simple valued primitives on Ω of any order we will show that f holomorphically extends on $\widehat{\Omega}$.

We note that if Ω is not connected the above fails. Such an example is the following: On the open set $\Omega = D \cup B$, where $D = \{z \in \mathbb{C} : |z| < 1\}$ and $B = \{z \in \mathbb{C} : |z| > 5\}$, we consider the function f, such that $f|_D \equiv 0$ and $f|_B \equiv 1$.

In the case where Ω is a domain, the proof relates to the fact that $\int_\gamma f(z)P(z)dz = 0$, for all polynomials P and all closed rectifiable curves γ in Ω, which is the hypothesis of a theorem of Privalov ([1], page 437). For a proof without using the theorem of Privalov see [2].

Date: Received 13 Nov. 2018. Accepted 19 Mar. 2019.

1991 *Mathematics Subject Classification.* Primary 30B40, Secondary 30E20.

Key words and phrases. primitive, simply connected envelope, single valued holomorphic extension, monodromy theorem.

2. The result

Theorem 2.1. *Let $\Omega \subset \mathbb{C}$ be a domain and f a holomorphic function on Ω. We assume that for every natural number $n \geq 1$ there exists a holomorphic function h_n on Ω such that its nth derivative satisfies $(h_n)^{(n)} = f$. Then f has a holomorphic extension on the simply connected envelope of Ω in $\mathbb{C}$; that is on the least simply connected domain $\widehat{\Omega} \subset \mathbb{C}$ such that $\Omega \subset \widehat{\Omega}$.*

Proof. Let us start with the following.

> **Claim:** If f has a primitive of order n, then $\int_\gamma P(\zeta)f(\zeta)d\zeta = 0$ if P is a polynomial with $\deg P \leq n-1$.

Indeed, if f has a single valued primitive, then the integral on a closed contour is zero and the same holds for any constant multiple of f. Thus, $\int_\gamma P(\zeta)f(\zeta)d\zeta = 0$ if $\deg P = 0$. So the claim holds for $n = 1$.

We assume by induction that if for a holomorphic function $h : \Omega \to \mathbb{C}$ there exists a single valued holomorphic function $G : \Omega \to \mathbb{C}$ such that $G^{(n)} = h$, then we shall show that for every rectifiable closed curve γ in Ω and for every polynomial P with $\deg P \leq n-1$ we have $\int_\gamma P(z)h(z)dz = 0$.

We will show that if for a holomorphic function $\omega : \Omega \to \mathbb{C}$, there exists a single valued holomorphic function $L : \Omega \to \mathbb{C}$ such that $L^{(n+1)} = \omega$ then we have $\int_\gamma Q(z)\omega(z)dz = 0$ for all rectifiable closed curves γ in Ω and all polynomials Q with $\deg Q \leq n$.

We have $(L^{(n)})' = L^{(n+1)} = \omega$. By integration by parts, we obtain

$$\int_\gamma z^n \omega(z)dz = \int_\gamma z^n dL^{(n)}(z) = -\int_\gamma L^{(n)}(z)dz^n = -n\int_\gamma z^{n-1}L^{(n)}(z)dz.$$

By the induction hypothesis for the function $h = L^{(n)}$ we have that $\int_\gamma z^{n-1}h(z)dz = 0$. Thus, we proved that $\int_\gamma z^n\omega(z)dz = 0$. For $0 \leq k \leq n-1$ we also have that $\int_\gamma z^k\omega(z)dz = 0$ by the induction hypothesis, because $\omega = (L')^{(n)}$ and $\deg(z^k) \leq n-1$. By linearity we get, $\int_\gamma Q(z)\omega(z)dz = 0$ for all polynomials Q with $\deg Q \leq n$. The proof of the inductive argument is complete.

Since f has single valued primitives on Ω of any order, it follows that $\int_\gamma P(z)f(z)dz = 0$ for all polynomials P.

Now, a theorem of Privalov ([1], page 137), implies the following: Let δ be a rectifiable Jordan curve in Ω and let G denote the interior of δ. Then, the Cauchy transform $F(z) = \dfrac{1}{2\pi i}\displaystyle\int_\delta \frac{f(\zeta)}{\zeta - z}d\zeta$, seen as a holomorphic function on G, has the property, that for almost every point A in δ (with respect to the arc-length measure) the non-tangential limit of $F(z)$, when $z \to A$, $z \in G$ exists and is equal to $f(A)$.

Consider a new Jordan curve $\delta' \subset \Omega$ surrounding δ, so that the doubly connected domain bounded by δ and δ' is included in Ω. Let $H(z) = \dfrac{1}{2\pi i}\displaystyle\int_{\delta'} \frac{f(\zeta)}{\zeta - z}d\zeta$; then H is holomorphic in the interior G' of δ' and $H(z) = F(z)$ on G. Thus, $H(z) = f(z)$ at almost every point in δ. By analytic continuation, the function H is a holomorphic extension on G' of the restriction of f on the doubly connected domain A bounded by δ and δ'. If $A \subset \Gamma \subset \Omega$ and Γ is a domain, then by analytic continuation H and f are compatible and define a single valued holomorphic function on $\Gamma \cup G'$. In particular, this holds for $\Gamma = \Omega$, since Ω is connected.

Let $\Phi \subset \overline{\Phi} \subset \Omega$ be a finitely connected domain bounded by a finite set of disjoint rectifiable Jordan curves. Then, according to the previous discussion, the restriction $f|_{\Phi}$ has a holomorphic extension on the simply connected envelope $\widehat{\Phi}$, which is the union of Φ with the bounded components of $\mathbb{C} \setminus \Phi$, which are finitely many.

It is well known that any domain $\Omega \subset \mathbb{C}$ has an exhaustion by compact sets K_n, $n = 1, 2, \ldots$, which are the closures of finitely connected domains K_n^{o}, bounded by finitely many disjoint rectifiable Jordan curves. Then, $\widehat{K_n^{\mathrm{o}}} = X_n^{\mathrm{c}}$, where X_n is the component of ∞ in the set $(\mathbb{C} \cup \{\infty\}) \setminus K_n^{\mathrm{o}}$.

Let Y denote the component of ∞ in $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$. Then, $Y^{\mathrm{c}} = \widehat{\Omega}$. By de Morgan rules, we have that $\bigcup\limits_n \widehat{K_n^{\mathrm{o}}} = \bigcup\limits_n X_n^{\mathrm{c}} = (\bigcap\limits_n X_n)^{\mathrm{c}}$. Since $K_n^{\mathrm{o}} \subset K_{n+1}^{\mathrm{o}}$, it follows that $X_n \supset X_{n+1}$ and $\{X_n\}_{n=1}^{\infty}$ is a decreasing sequence of connected compact sets. Therefore, $\bigcap\limits_n X_n = X$ is a connected compact set.

Since $X_n \supset Y$, it follows that $X = \bigcap\limits_n X_n \supset Y$ and X is a connected set containing ∞ and included in $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$. Since Y is the component of ∞ in $(\mathbb{C} \cup \{\infty\}) \setminus \Omega$, it follows that $X \subset Y$; hence $X = Y$ and $\widehat{\Omega} = \bigcup\limits_n \widehat{K_n^{\mathrm{o}}}$.

Since Ω is a domain, the envelope $\widehat{\Omega}$ is also a domain. We have proven that the restriction $f|_{K_n^{\mathrm{o}}}$ has a holomorphic extension f_n on $\widehat{K_n^{\mathrm{o}}}$. Since $K_n^{\mathrm{o}} \subset K_{n+1}^{\mathrm{o}}$, it follows $X_n \supset X_{n+1}$ and $\widehat{K_n^{\mathrm{o}}} = X_n^{\mathrm{c}} \subset X_{n+1}^{\mathrm{c}} = \widehat{K_{n+1}^{\mathrm{o}}}$. We will show that $f_{n+1}|_{K_n^{\mathrm{o}}} = f_n$. This holds by analytic continuation, because on the domain $\widehat{K_n^{\mathrm{o}}}$, both f_n and $f_{n+1}|_{K_n^{\mathrm{o}}}$ are holomorphic extensions of $f|_{K_n^{\mathrm{o}}}$. Thus, the sequence f_n, $n = 1, 2, \ldots$, defines a single valued holomorphic extension of f on $\widehat{\Omega}$. This completes the proof.

An alternative argument would be by monodromy [5], since $\widehat{\Omega}$ is simply connected. If σ is a curve in $\widehat{\Omega}$, starting at a point $w \in \Omega$, then σ is contained in K_m^{o} for some $m \in \mathbb{N}$ and the function f_m induces a continuation of the Taylor expansion of f with center w along σ. Since, $\widehat{\Omega} = \bigcup\limits_n \widehat{K_n^{\mathrm{o}}}$ is simply connected, by monodromy, the function f has a single valued holomorphic extension on $\widehat{\Omega}$.

For a proof of Theorem 2.1. without the use of the theorem of Privalov, see [2].

□

Remark 2.2. *Obviously, if f has a holomorphic extension F on the simply connected domain $\widehat{\Omega}$, then F has single valued primitives on $\widehat{\Omega}$ of any order. Since $\Omega \subset \widehat{\Omega}$, the converse of Theorem 2.1. also holds. In fact, the hypothesis and the conclusion of Theorem 2.1. are equivalent to the fact that $\int_\gamma P(z)f(z)dz = 0$ for all polynomials P and all rectifiable closed curves γ in the domain Ω. In addition, it is equivalent to consider only polygonal closed curves γ in Ω.*

Remark 2.3. *The equality $\int_\gamma P(z)f(z)dz = 0$ for all polynomials P relates to the assumption of the classical F. and M. Riesz theorem on the open unit disc $\mathbb{D}$, where $\gamma = \partial\mathbb{D}$ is the unit circle. In some sense, this condition can be rephrased as " f has single valued primitives of any order on $\partial\mathbb{D}$", because the dz - integration can be restricted to curves included in $\partial\mathbb{D}$. Extensions of the F. and M. Riesz theorem can be found in* [3] *and* [4].

Remark 2.4. *Let $n \in \{1, 2, \dots\}$ be fixed. One can verify that if $\int_\gamma P(z)f(z)dz = 0$ holds for all rectifiable closed curves γ in Ω and for all polynomials P with $\deg P \leq n-1$, then there exists a holomorphic function L on Ω such that $L^{(n)} = f$. Therefore, for each $n \in \{1, 2, \dots\}$, the holomorphic function f has a single valued primitive of order n, if and only if, $\int_\gamma P(z)f(z)dz = 0$ for all rectifiable closed curves γ in Ω and for all polynomials P with $\deg P \leq n-1$.*

References

[1] G. M. Goluzin, *Geometric theory of functions of a complex variable*, Translations of Mathematical Monographs, Vol. 26, AMS, Providence, Rhode Island 02904 (1969).
[2] V. Nestoridis, One valued primitives and the F. and M. Riesz theorem (arXiv: 1610.01365).
[3] E. Bishop, The structure of certain measures, Duke Math. J., V. 25, No. 2, 1958, 283 -289.
[4] E. Bishop, Boundary measures of analytic differentials, Duke Math. J., V. 27, No. 3, 1960, 331 - 340.
[5] W. Rudin, *Real and Complex Analysis*, 3rd edition, Mc Graw-Hill Inc., 1967.

Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis, 15784 Athens, Greece
E-mail address: vnestor@math.uoa.gr