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TRAVELING KINKS IN AN INFINITE ARRAY OF WEAKLY COUPLED PENDULA

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ABSTRACT. We prove the existence of heteroclinic traveling waves (kinks) in an infinite array of weakly coupled pendula. Our approach is to apply a perturbation argument from the anti-continuum limit.

INTRODUCTION AND MAIN RESULT

The model. The following infinite system of linearly coupled ODEs describes the motion of an array of pendula each of which is coupled to its nearest neighbors by a torsional spring with a coupling coefficient k:

$$\ddot{v}_n - k(v_{n+1} - 2v_n + v_{n-1}) + \sin(v_n) = 0, \quad n \in \mathbb{Z}. \tag{1}$$

More precisely, v_i represents the angle formed by the ith pendulum with the vertical axis (assuming physical units have been scaled appropriately). We refer the reader to [KT, Le, Sc] for more details on the physical background of the problem.

The system (1) is known as the *discrete sine-Gordon equation* and also serves as a model of arrays of Josephson junctions [IS], or as a dynamical Frenkel-Kontorova model of electrons in a crystal lattice [BK].

Traveling wave solutions. We shall construct solutions of the above equation in the form of traveling waves (cf. [CMS]). To this end, we let $z = n$, write $v_n(t) = v(z, t)$, and seek a solution of the form $v(z, t) = v(\xi) = v(z - ct)$ satisfying the equation

$$c^2 \frac{d^2 v}{dz^2} - k[v(z + 1) - 2v(z) + v(z - 1)] + \sin(v) = 0. \tag{2}$$

Known results. In [S] explicit kink solutions to the above equation were given for a certain nonlinearity f (in place of the sine) which satisfies (H1) below. Periodic traveling wave solutions to (1) have been shown to exist recently in [FR1, FR2, SL] in the strong coupling regime (i.e. when $k \gg 1$). This was achieved in [FR1] using techniques from dynamical systems; in [FR2] using variational and topological techniques; in [SL] using a fixed point argument. However, as is pointed out in [FR1], kink solutions to (2) connecting $-\pi$ to π should not exist in this regime. Nevertheless, such solutions were constructed variationally in [KZ] provided that c is sufficiently large. The question of persistence of kink solutions in the continuum limit of (1) and (3) was

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discussed in [ACR, DKY, IP, OPB, SZE]. We also refer to [FR3, KKCR] for further results on the existence of localized structures in long range interaction lattices (stationary or traveling).

The main result. In the current paper, for given $c \neq 0$, we study heteroclinic waves for sufficiently small k, which from now on we will call ε. We will also consider a more general class of nonlinearities f satisfying (H1) below, covering both the discrete sine-Gordon equation as well as the important ϕ^4 model [BeK]:

$$\ddot{v}_n - k(v_{n+1} - 2v_n + v_{n-1}) + 2(v_n - v_n^3) = 0, \quad n \in \mathbb{Z}, \tag{3}$$

see Remark 1 below. Moreover, motivated from [BCC], we will allow infinite range and not just nearest neighbor or finite length interaction, although those are included as special cases.

The equation we will be dealing with is

$$u'' - \epsilon \sum_{k=-\infty}^{\infty} a_k u(z-k) + f(u) = 0, \quad z \in \mathbb{R}, \tag{4}$$

together with the conditions

$$\lim_{z \to -\infty} u(z) = 0, \quad \lim_{z \to \infty} u(z) = 1. \tag{5}$$

Here $\epsilon \geq 0$ is small, and we assume that

(H1) $f \in C^2(\mathbb{R}),\ f(0) = f(1) = 0,\ \ f'(0),\ f'(1) < 0;$

$$F(1) = 0,\ F(u) \neq 0 \ \ \forall u \in (0,1)$$

where $F(u) = \int_0^u f(s)ds$.

(H2) $\sum_{k=-\infty}^{\infty} a_k = 0,\ a_0 < 0,\ a_k = a_{-k},$ and $\sum_{k \geq 1} |a_k| k^2 < \infty$.

When $\epsilon = 0$, that is in the so called anti-continuum limit [MA], equation (4) becomes

$$u'' + f(u) = 0. \tag{6}$$

Under the hypotheses (H1), the above equation has a heteroclinic solution u_0 satisfying (5) (see [Ar]).

Our result is

Theorem 1. *If $\epsilon > 0$ is sufficiently small, then there exists a solution u_ϵ of (4) such that*

$$||u_\epsilon - u_0||_{H^2(\mathbb{R})} \leq C\epsilon$$

($C > 0$ is a constant independent of ϵ).

Remark 1. *The choice of the roots of f to be 0 and 1 is made for convenience purposes only and causes no loss of generality. For instance, the traveling kink problem (2) can be embedded in our framework by plainly letting*

$$u = \frac{v + \pi}{2\pi} \quad \textit{and} \quad f(u) = \frac{1}{2\pi}\sin(2\pi u - \pi).$$

Similarly, the corresponding change of variables for (3) is

$$u = \frac{v+1}{2} \quad \textit{and} \quad f(u) = (2u-1) - (2u-1)^3,$$

(keep in mind the first assumption in (H2)).

Method of proof. To prove this we adapt a technique from an earlier paper of ours (cf. [AFFS], but see also [dPK] for a related idea). We use two important properties:

(i) Nondegeneracy of u_0 The operator obtained by linearizing the left-hand side of (6) at u_0 has 0 as a simple isolated eigenvalue, the remaining spectrum being in the open left half-plane ([He], Section 5.4).

(ii) Hamiltonian form of the problem The equation (4) arises when seeking traveling waves to the lattice equation

$$\ddot{u}_n - \epsilon \sum_{k=-\infty}^{\infty} a_k u_{n-k} + f(u_n) = 0, \quad n \in \mathbb{Z},$$

which comes from the Hamiltonian on $\ell^2 \times \ell^2$ defined by

$$H(\mathbf{p}, \mathbf{u}) = \sum_n \left(\frac{1}{2} p_n^2 + \epsilon \sum_m a_{n-m} (u_n - u_m)^2 + F(u_n) \right),$$

where $\mathbf{p} = (\dot{u}_n) \in \ell^2$ and $\mathbf{u} = (u_n) \in \ell^2$.

As is explained in [SZ], the persistence of heteroclinic or homoclinic orbits of Hamiltonian systems under Hamiltonian perturbations is a delicate issue, as the Melnikov integral vanishes. An analogous difficulty also arises in the lattice setting at hand, as we will discuss in Section 2.

Outline of the paper. In Section 1 we present the proof of Theorem 1, in Section 2 we make some comments on the application of the standard Lyapunov-Schmidt reduction, and in the Appendices we prove some technical lemmas.

Notation. In what follows, $||\cdot||_{L^2}$, $||\cdot||_{L^\infty}$, and $||\cdot||_{H^i}$, $(i = 1, 2)$ denote the norms of the spaces $L^2(\mathbb{R})$, $L^\infty(\mathbb{R})$, and $H^i(\mathbb{R})$, respectively. Also,

$$(\phi, \psi) \equiv \int_{\mathbb{R}} \phi \psi dz; \qquad \phi \perp \psi \ \Leftrightarrow \ (\phi, \psi) = 0.$$

Unless specified otherwise C/c denotes a large/small positive constant independent of $\epsilon > 0$ whose value will change from line to line. In many cases we will not explicitly write the obvious dependence of functions on ϵ.

1. Proof of Theorem 1

1.1. Properties of the linear operator Δ.

Consider the linear operator Δ defined via

$$\Delta u := \sum_{k=-\infty}^{\infty} a_k u(z - k).$$

Then one can verify that if $u, v \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R})$, then

$$||\Delta u||_{L^2} \leq C ||u||_{L^2} \quad (C \text{ independent of } u), \tag{7}$$

and

$$(\Delta u, v) = (u, \Delta v).$$

We also have the following lemma whose proof will be given in Appendix A.

Lemma 1. *If $u \in C^1(\mathbb{R})$, $u' \in L^2(\mathbb{R})$;*

$$\lim_{z\to-\infty} u(z) = u(-\infty) \in \mathbb{R} \quad \textit{and} \quad \lim_{z\to\infty} u(z) = u(\infty) \in \mathbb{R}, \tag{8}$$

then

$$\Delta u \in L^2(\mathbb{R}) \quad \textit{and} \quad (\Delta u, u') = 0.$$

1.2. **Properties of the heteroclinic u_0.** It is well known (see [Ar], [HK]) that u_0 is the unique (up to translation) solution of (6), (5). Furthermore $u_0' > 0$, u_0 approaches its limits exponentially and

$$u_0'(z),\ |u_0''(z)|,\ |u_0'''(z)| \le Ce^{-c|z|}, \quad z \in \mathbb{R}.$$

The linear operator L^0 with $D(L^0) = H^2(\mathbb{R})$ and

$$L^0\phi = \phi'' + f'(u_0(z))\phi$$

is self-adjoint in $L^2(\mathbb{R})$ and $\sigma(-L^0) \subseteq \{0\} \cup [c,\infty)$ with 0 a simple eigenvalue corresponding to u_0'. (Note that since $f'(0),\ f'(1) < 0$ then $\sigma_{ess}(-L^0) \subseteq [c,\infty)$; thus the only thing left to prove is that 0 is the principal eigenvalue which follows from $u_0' > 0$.)

These properties imply the following important proposition.

Proposition 1. *Let $g \in L^2(\mathbb{R})$ with $g \perp u_0'$, then there exists a unique $\phi \in H^2(\mathbb{R})$ with $\phi \perp u_0'$ such that*

$$L^0\phi = g.$$

Furthermore, we have

$$||\phi||_{H^2} \le C||g||_{L^2}$$

with C independent of g.

1.3. **The perturbation argument.** We search for a solution of (4) in the form $u_\epsilon = u_0 + \phi_\epsilon$ with $\phi_\epsilon \in H^2(\mathbb{R})$ and $\phi_\epsilon \perp u_0'$. Then, the fluctuation ϕ_ϵ must satisfy

$$L^0\phi = \epsilon\Delta\phi + N(\phi) + \epsilon\Delta u_0 = E(\phi)$$

with

$$N(\phi) = -f(u_0 + \phi) + f(u_0) + f_u(u_0)\phi.$$

We note that $(E(\phi_\epsilon), u_0') = 0$ holds. However, if iterations of the form $L^0\phi_{n+1} = E(\phi_n)$ were to be performed, for capturing the desired ϕ_ϵ in the limit $n \to \infty$, the iteration ϕ_n may not satisfy this orthogonality condition which is necessary for solving for $\phi_{n+1} \in H^2(\mathbb{R})$. To deal with this issue, at each step of the iteration we will project $E(\phi_n)$ to $\{u_0'\}^\perp$ and then solve for the corresponding ϕ_{n+1}.

We thus define a mapping $T : H^2(\mathbb{R}) \cap \{u_0'\}^\perp \to H^2(\mathbb{R}) \cap \{u_0'\}^\perp$ via $T(\phi) = \psi$ where

$$L^0\psi = -b(\phi)u_0' + \epsilon\Delta\phi + N(\phi) + \epsilon\Delta u_0 \tag{9}$$

and

$$b(\phi) = \frac{1}{||u_0'||_{L^2}^2}\left(\epsilon\Delta\phi + N(\phi) + \epsilon\Delta u_0, u_0'\right). \tag{10}$$

Note that T is well defined via Proposition 1 since the right hand side of (9) is orthogonal to u_0'. (Note also that by Lemma 1 we have $\Delta u_0 \in L^2(\mathbb{R})$.)

Let

$$B_\epsilon = \{\phi \in H^2(\mathbb{R}) \cap \{u_0'\}^\perp :\ ||\phi||_{H^2} \le M\epsilon\}$$

with M a positive constant independent of $\epsilon > 0$ to be determined later. We will show that there exists a large $M > 0$ such that, provided $\epsilon > 0$ is sufficiently small, T maps B_ϵ into itself and is a contraction. Let $\phi \in B_\epsilon$, then via (9), (10) and Proposition 1,

$$
(11) \qquad \begin{aligned}
||\psi||_{H^2} &\leq C\left(|b(\phi)| + \epsilon||\Delta\phi||_{L^2} + ||N(\phi)||_{L^2} + \epsilon||\Delta u_0||_{L^2}\right) \\
&\stackrel{(10)}{\leq} C\left(\epsilon||\Delta\phi||_{L^2} + ||N(\phi)||_{L^2} + \epsilon||\Delta u_0||_{L^2}\right).
\end{aligned}
$$

The first and third term will be estimated from (7) and Lemma 1 respectively. To estimate the nonlinear term $N(\phi)$, we first recall the embedding $||\phi||_{L^\infty} \leq C||\phi||_{H^1}$ for every $\phi \in H^1(\mathbb{R})$. Hence, setting $C = \sup_{|s|\leq 2} |f''(s)|$, we have

$$
(12) \qquad |N(\phi)| \leq CM\epsilon|\phi| \quad \text{and} \quad |N(\phi_1) - N(\phi_2)| \leq CM\epsilon|\phi_1 - \phi_2|
$$

pointwise for all $\phi, \phi_1, \phi_2 \in B_\epsilon$. Thus, (11) yields

$$
\begin{aligned}
||\psi||_{H^2} &\leq C\left(\epsilon||\phi||_{L^2} + M\epsilon||\phi||_{L^2} + \epsilon\right) \\
&\leq C\left(M\epsilon + M^2\epsilon + 1\right)\epsilon.
\end{aligned}
$$

Choosing a large M (say $M = 2C$), then $||\psi||_{H^2} \leq M\epsilon$ provided $\epsilon > 0$ is sufficiently small, i.e. $T : B_\epsilon \to B_\epsilon$. Similarly we can show that T is a contraction in the H^2 norm.

Since B_ϵ is closed with respect to this norm, the Banach fixed point theorem gives us a fixed point $\phi_* \in B_\epsilon$ of T. Then

$$
(13) \qquad u_\epsilon = u_0 + \phi_*^\epsilon
$$

satisfies

$$
(14) \qquad u'' - \epsilon\Delta u + f(u) = -b_\epsilon u_0'
$$

for some $b_\epsilon \in \mathbb{R}$ ($b_\epsilon = b(\phi_*^\epsilon)$). Multiplying (14) by $u' = u_\epsilon'$ and integrating over $\mathbb{R}$ yields

$$
\int_{\mathbb{R}} u'u''dz - \epsilon(\Delta u, u') + \int_{\mathbb{R}} f(u)u'dz = -b_\epsilon \int_{\mathbb{R}} u_0'u'dz.
$$

Since $\phi_* \in H^2(\mathbb{R})$, we have that $u(-\infty) = 0$, $u(\infty) = 1$ and $u'(\pm\infty) = 0$. The left hand side of the above equation is 0 (see Lemma 1 and recall that $F(1) = 0$) and we get that

$$
b_\epsilon \int_{\mathbb{R}} u_0'u'dz = 0.
$$

This implies that $b_\epsilon = 0$ since

$$
\begin{aligned}
\int_{\mathbb{R}} u_0'u'dz &= \int_{\mathbb{R}} u_0'(u_0' + \phi_*')dz \geq \int_{\mathbb{R}} u_0'^2dz - ||u_0'||_{L^2}||\phi_*'||_{L^2} \geq \\
&\geq \int_{\mathbb{R}} u_0'^2dz - C\epsilon > 0,
\end{aligned}
$$

provided $\epsilon > 0$ is sufficiently small.

Therefore u given by (13) is a solution of (4) satisfying the estimate of Theorem 1, thereby completing the proof.

Remark 2. *If $f(u+1/2)$ is odd, as would be the case for applications to (2) and (3), then the proof of Theorem 1 becomes much simpler. Indeed, we can apply our fixed point argument restricted to the space of functions such that $\phi - 1/2$ is odd, without the need to introduce the projection operator. Keep in mind that $u_0 - \frac{1}{2}$ would be odd and u_0' would be even. So, the orthogonality condition $(E(\phi), u_0') = 0$ would be automatically satisfied.*

Remark 3. *If the $\epsilon = 0$ equation (4) has a unique even homoclinic solution u_0 (see* [BL] *for necessary and sufficient conditions on f), then the proof of persistence for ϵ small is considerably simplified by seeking $u_\epsilon = u_0 + \phi$ with $\phi \in H^2(\mathbb{R})$ even. Note that given an even $g \in L^2(\mathbb{R})$, there exists a unique even $\phi \in H^2(\mathbb{R})$ such that $\phi'' + f'(u_0)\phi = g$. Moreover $||\phi||_{H^2} \le C||g||_{L^2}$ for some $C > 0$ independent of g. This problem was briefly discussed at the end of* [B].

2. Some remarks on the standard Lyapunov-Schmidt approach

In this section we make some remarks on a difficulty that arises when trying to prove Theorem 1 using the standard Lyapunov-Schmidt reduction.

We have seen that u_0 satisfies (4) up to an order of ϵ. We begin by refining this approximation so that $u_{ap} = u_0 + \epsilon u_1$ satisfies (4) up to an order of ϵ^2. We choose $u_1 \in H^2(\mathbb{R})$, $u_1 \perp u_0'$ such that

$$u_1'' + f'(u_0)u_1 = \Delta u_0 \tag{15}$$

(this is possible via Lemma 1 and Proposition 1). Then, a simple calculation gives

$$-G(\epsilon) := u_{ap}'' - \epsilon\Delta u_{ap} + f(u_{ap}) = -\epsilon^2\Delta u_1 - N(\epsilon u_1)$$

and thus from (12):

$$||G(\epsilon)||_{L^2} \le C\epsilon^2. \tag{16}$$

We seek a solution of (4) in the form $u_\epsilon = u_{ap} + \psi_\epsilon$ with $\psi_\epsilon \in H^2(\mathbb{R})$. Then ψ_ϵ must satisfy

$$L^\epsilon\psi = N_{ap}(\psi) + G(\epsilon) \tag{17}$$

where $L^\epsilon\psi = \psi'' + f'(u_{ap})\psi - \epsilon\Delta\psi$ and $N_{ap}(\psi) = -f(u_{ap}+\psi) + f(u_{ap}) + f'(u_{ap})\psi$.

Since $\Delta : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and $||u_{ap} - u_0||_{L^\infty} \le C\epsilon$, L^ϵ is a regular $O(\epsilon)$ perturbation of L^0. From the special form of the perturbation, however, the simple eigenvalue 0 of L^0 is perturbed to an $O(\epsilon^2)$ simple eigenvalue of L^ϵ (this is the source of the difficulty). We point out that such small eigenvalues would not have been present if we were in the symmetric setting of Remark 2. More precisely we have the following Proposition whose proof we postpone to Appendix B.

Proposition 2. *If $\epsilon \ge 0$ is sufficiently small then $\sigma(-L^\epsilon) \subset \{\lambda_1(\epsilon)\} \cup [c, \infty)$ with $\lambda_1(\epsilon)$ simple corresponding to $\phi_1(\epsilon) \in H^2(\mathbb{R})$ with $||\phi_1(\epsilon)||_{H^2} = 1$. Moreover $\lambda_1(\epsilon)$, $\phi_1(\epsilon)$ depend smoothly on ϵ up to $\epsilon = 0$ and*

$$\begin{aligned} &\lambda_1(\epsilon) = O(\epsilon^2) \\ &\phi_1(\epsilon) = \frac{u_0'}{||u_0'||_{H^2}} + O(\epsilon) \qquad (\textit{here } ||O(\epsilon)||_{H^2} \le C\epsilon). \end{aligned} \tag{18}$$

Define the orthogonal projection P onto the span of ϕ_1 by

$$P\psi = (\psi, \phi_1(\epsilon))\frac{\phi_1(\epsilon)}{||\phi_1(\epsilon)||_{L^2}^2}.$$

According to this projection we have

$$H^2(\mathbb{R}) = \text{span}\{\phi_1\} \oplus X_1, \quad L^2(\mathbb{R}) = \text{span}\{\phi_1\} \oplus Y_1,$$

where X_1, Y_1 are respectively the kernel of P in $H^2(\mathbb{R})$ and $L^2(\mathbb{R})$. By decomposing ψ as $\psi = a\phi_1(\epsilon) + v$ ($a \in \mathbb{R}, v \in X_1$), one finds that (17) is equivalent to

$$\begin{aligned} L^\epsilon v = &\quad (I-P)\{N_{ap}(a\phi_1(\epsilon)+v) + G(\epsilon)\} \\ -a\lambda_1(\epsilon)\phi_1(\epsilon) = &\quad P\{N_{ap}(a\phi_1(\epsilon)+v) + G(\epsilon)\}. \end{aligned} \tag{19}$$

Applying Proposition 2, using (16), and the Banach fixed point theorem, we can uniquely solve $(19)_{(i)}$ for $v = v^*(a,\epsilon)$ in a neighborhood of $(a,v) = (0,0)$. This solution depends smoothly on a, ϵ and satisfies $||v^*(a,\epsilon)||_{H^2} = O(a^2+\epsilon^2)$ if $|a|, \epsilon \geq 0$ small. Using this in $(19)_{(ii)}$ and taking the L^2 inner product with $\phi_1(\epsilon)$ yields

$$B(a,\epsilon) := -a\lambda_1(\epsilon)||\phi_1(\epsilon)||^2_{L^2} - (N_{ap}(a\phi_1(\epsilon)+v^*) + G(\epsilon), \phi_1(\epsilon)) = 0 \tag{20}$$

i.e.

$$B(a,\epsilon) = -a\lambda_1(\epsilon)||\phi_1(\epsilon)||^2_{L^2} + \frac{1}{2}\left(f''(u_{ap})\phi_1^2, \phi_1\right)a^2 - (G(\epsilon),\phi_1) + O(a^\mu\epsilon^\nu) = 0$$

as $a, \epsilon \to 0$ with $\mu + \nu \geq 3$. If $\lambda_1(\epsilon) = d\epsilon + O(\epsilon^2)$ with $d \neq 0$ (indep. of ϵ), then we could apply the implicit function theorem to $\epsilon^{-2}B(\epsilon\tilde{a},\epsilon) = 0$ and find an $a_* = O(\epsilon)$ satisfying (20). However, since by $(18)_{(i)}$ we have $d = 0$, this analysis breaks down.

Appendix A. Proof of Lemma 1

$u \in L^\infty(\mathbb{R})$ and (H2) imply that $\Delta u \in L^\infty(\mathbb{R})$ and

$$\Delta u(z) = \sum_{k=-\infty}^{\infty} a_k[u(z-k) - u(z)] = \sum_{k=-\infty}^{\infty} a_k \int_0^{-k} u'(z+t)dt, \quad z \in \mathbb{R}.$$

Then, by the Cauchy-Schwarz inequality, we get

$$(\Delta u(z))^2 \leq \left(\sum_{k=-\infty}^{\infty}|a_k|\right)\sum_{k=-\infty}^{\infty}|a_k|\left(\int_0^{-k}u'(z+t)dt\right)^2 \overset{(H2)}{\leq} C\sum_{k=-\infty}^{\infty}|a_k|(-k)\int_0^{-k}u'^2(z+t)dt.$$

So,

$$\int_{-\infty}^{\infty}(\Delta u(z))^2\,dz \leq C\int_{-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty}|a_k|(-k)\int_0^{-k}u'^2(z+t)dt\right)dz =$$

$$= C\sum_{k=-\infty}^{\infty}|a_k|(-k)\int_{-\infty}^{\infty}\int_0^{-k}u'^2(z+t)dtdz = C\sum_{k=-\infty}^{\infty}|a_k|(-k)\int_0^{-k}\int_{-\infty}^{\infty}u'^2(z+t)dzdt =$$

$$= C\sum_{k=-\infty}^{\infty}|a_k|k^2||u'||^2_{L^2} \overset{(H2)}{\leq} C||u'||^2_{L^2} < \infty.$$

Thus, $\Delta u \in L^2(\mathbb{R})$.

We have

$$(\Delta u, u') = \int_{-\infty}^{\infty} u'(z) \sum_{k=-\infty}^{\infty} a_k[u(z-k) - u(z)]dz =$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} u'(z)[u(z-k) - u(z)]dz = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} u'(z+k)[u(z) - u(z+k)]dz =$$

$$\overset{a_k = a_{-k}}{=} \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} u'(z-k)[u(z) - u(z-k)]dz.$$

Thus,

$$2(\Delta u, u') = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} \frac{d}{dz} \left\{ u(z)u(z-k) - \frac{u^2(z)}{2} - \frac{u^2(z-k)}{2} \right\} dz \overset{(8)}{=} 0.$$

Appendix B. Proof of Proposition 2

Since zero is a simple eigenvalue of $-L^0$, it follows from regular perturbation theory (cf. Sec. 14.3 of [CH]) that it perturbs smoothly to a simple eigenvalue $\lambda(\epsilon)$ of $-L^\epsilon$. The corresponding eigenfunction $\phi(\epsilon)$ with $||\phi(\epsilon)||_{H^2} = 1$ also depends smoothly (in the H^2 norm) on $\epsilon \geq 0$ small and $\phi(0) = \frac{u_0'}{||u_0'||_{H^2}}$. It is easy to show that $\lambda(\epsilon)$ is the principal eigenvalue of $-L^\epsilon$; we denote it by $\lambda_1(\epsilon)$ and the corresponding H^2 normalized eigenfunction by $\phi_1(\epsilon)$. (Recall that $(-L^0\phi, \phi) \geq c||\phi||_{L^2}^2, \ \forall \phi \in H^2(\mathbb{R}), \ \phi \perp u_0'$, and $\left|\left|\phi_1(\epsilon) - \frac{u_0'}{||u_0'||_{H^2}}\right|\right|_{H^2} \leq C\epsilon$, to obtain $(-L^\epsilon \phi, \phi) \geq c||\phi||_{L^2}^2, \ \forall \phi \in H^2(\mathbb{R}), \ \phi \perp \phi_1(\epsilon)$.) We have

$$\phi_1'' + f'(u_{ap})\phi_1 - \epsilon \Delta \phi_1 = -\lambda_1 \phi_1$$

and

$$-\lambda_1(\phi_1, u_0') = \left(\phi_1, u_0''' + f'(u_{ap})u_0' - \epsilon \Delta u_0'\right) = \left(\phi_1, [f'(u_0 + \epsilon u_1) - f'(u_0)]u_0' - \epsilon \Delta u_0'\right).$$

Since $\phi_1(\epsilon) \overset{H^2}{\to} \frac{u_0'}{||u_0'||_{H^2}}$ as $\epsilon \to 0$, we get

$$-\lim_{\epsilon \to 0} \frac{\lambda_1(\epsilon)}{\epsilon} = \frac{1}{||u_0'||_{L^2}^2} \left(u_0', f''(u_0)u_1 u_0' - \Delta u_0'\right).$$

Differentiating (15) yields

$$u_1''' + f''(u_0)u_0' u_1 + f'(u_0)u_1' = \Delta u_0'$$

i.e.

$$\left(u_0', f''(u_0)u_1 u_0' - \Delta u_0'\right) = -\left(u_0', u_1''' + f'(u_0)u_1'\right) = -\left(u_0''' + f'(u_0)u_0', u_1'\right) = 0.$$

This and the smoothness of $\lambda_1(\epsilon)$ gives us $(18)_{(i)}$. From the smoothness of $\phi_1(\epsilon)$ (in the H^2 norm) we have $(18)_{(ii)}$.

References

[ACR] A.A. Aigner, A.R. Champneys, V.M. Rothos, *A new barrier to the existence of moving kinks in Frenkel-Kontorova lattices* Physica D 186, 148–170, (2003).

[AFFS] N.D. Alikakos, P.C. Fife, G. Fusco, C. Sourdis, *Analysis of the heteroclinic connection in a singularly perturbed system arising from the study of crystalline grain boundaries*, Interfaces Free Bound. 8, no. 2, 159–183, (2006).

[Ar] V. Arnold, *Ordinary Differential Equations*, MIT Press, Cambridge, MA, (1973).

[BCC] P. W. Bates, X. Chen, A. Chmaj, *Traveling waves of bistable dynamics on a lattice*, SIAM J. Math. Anal. 35, no. 2, 520–546, (2003).

[B] P.W. Bates, *On some nonlocal evolution equations arising in materials science*, in Nonlinear dynamics and evolution equations, Amer. Math. Soc., vol 48 of Fields Inst. Commun, 13–52 (2006).

[BeK] T.I. Belova, A.E. Kudryavtsev, Phys. Usp. 40, 359 (1997).

[BL] H. Berestycki, P.-L. Lions, *Nonlinear scalar field equations I: Existence of a ground state*, Arch. Rational Mech. Anal. 82, no. 4, 313–345, (1983).

[BK] O.M. Braun, Y.S. Kivshar, *The Frenkel-Kontorova Model*, Concepts, Methods, and Applications, Springer-Verlag, 2004.

[CH] S.-N. Chow, J. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, (1982).

[CMS] S.-N. Chow, J. Mallet-Paret, W. Shen, *Traveling waves in lattice dynamical systems*, J. Differential Equations 149, no. 2, 248–291, (1998).

[dPK] M. del Pino and M. Kowalczyk, *Renormalized energy of interacting Ginzburg-Landau vortex filaments*, J. Lond. Math. Soc. (2) 77, no. 3, 647–665, (2008).

[DKY] S.V. Dmitriev, P.G. Kevrekidis, N. Yoshikawa, *Discrete Klein-Gordon models with static kinks free of the Peierls-Nabarro potential*, J. Phys. A.: Math. Gen. 38, 7617–7627, (2005).

[FR1] M. Feckan, V.M. Rothos, *Kink-like periodic travelling waves for lattice equations with on-site and intersite potentials*, Dyn. PDEs 2, 357–370, (2005).

[FR2] M. Feckan, V.M. Rothos, *Travelling waves in Hamiltonian systems on 2D lattices with nearest neighbour interactions*, Nonlinearity 20, 319–341, (2007).

[FR3] M. Feckan, V.M. Rothos, *Travelling waves of discrete nonlinear Schrödinger equations with nonlocal interactions*, Applicable Analysis 89, 1387–1411, (2010).

[HK] J. Hale, H. Kocak, *Dynamics and Bifurcations*, Springer-Verlag, (1991).

[He] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics Vol. 840, Springer-Verlag, (1981).

[IS] Y. Imry, L. Schulman, *Qualitative theory of the nonlinear behavior of coupled Josephson junctions*, J. Appl. Phys. 49, 749–758, (1978).

[IP] G. Iooss, D.E. Pelinovsky, *Normal form for travelling kinks in discrete Klein-Gordon lattices*, Physica D 216, 327–345, (2006).

[KKCR] V. Koukouloyannis, P.G. Kevrekidis, J. Cuevas, V. Rothos, *Multibreathers in Klein-Gordon chains with interactions beyond nearest neighbors*, Physica D 242, 16–29, (2013).

[KZ] C-F. Kreiner, J. Zimmer, *Travelling wave solutions for the discrete sine-Gordon equation with nonlinear pair interaction*, Nonlinear Analysis 70, 3146–3158, (2009).

[KT] O. Kresse, L. Truskinovsky, *Mobility of lattice defects: discrete and continuum approaches*, J. Mech. Phys. Solids 51, no. 7, 1305–1332, (2003).

[Le] M. Levi, *Dynamics of discrete Frenkel-Kontorova models*, in: Analysis, et Cetera: Research Papers Published in Honor of Jürgen Moser's 60th Birthday, Academic Press Inc., 1990, 471–494.

[MA] R. MacKay, S. Aubry, *Proof of existence of breathers for time-reversible or hamiltonian networks of weakly coupled oscillators*, Nonlinearity 7, 1623–1643, (1994).

[OPB] O.F. Oxtoby, D.E. Pelinovsky, I.V. Barashenkov, *Travelling kinks in discrete ϕ^4 models*, Nonlinearity 19, 217–235, (2006).

[SL] A. Saadatpour, M. Levi, *Traveling waves in chains of pendula*, Physica D 244, 68–73, (2013).

[SZE] A.V. Savin, Y. Zolotaryuk, J.C. Eilbeck, *Moving kinks and nanopterons in the nonlinear Klein-Gordon lattice*, Physica D 138, 267–281, (2000).

[S] V. H. Schmidt, *Exact solution in the discrete case for solitons propagating in a chain of harmonically coupled particles lying in double-minimum potential wells*, Phys. Rev. B 20, 4397–4405, (1979) .

[Sc] A.C. Scott, *Nonlinear Science*, Oxford University Press, (2003).

[SZ] J. Shatah, C. Zeng, *Orbits homoclinic to centre manifolds of conservative PDEs*, Nonlinearity 16, 591–614, (2003).

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