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CONTINUITY OF WEIGHTED ESTIMATES FOR SUBLINEAR OPERATORS

MICHAEL PAPADIMITRAKIS AND NIKOLAOS PATTAKOS

Abstract. In this note we prove that if a sublinear operator T satisfies a certain weighted estimate in the $L^p(w)$ space for all $w \in A_p$, $1 < p < +\infty$, then

$$\lim_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)} = \|T\|_{L^p(w_0)\to L^p(w_0)},$$

where d_* is the metric defined in [6] and w_0 is a fixed A_p weight. This generalizes a previous result on the same subject, obtained in [6], for linear operators.

1. Introduction and notation

We are going to work with positive $L^1_{loc}(\mathbb{R}^n)$ functions w (called weights), that satisfy the following condition for some $1 < p < +\infty$:

$$[w]_{A_p} := \sup_Q \Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}dx\Big)^{p-1} < +\infty.$$

The number $[w]_{A_p}$ is called the A_p characteristic of the weight w and we say that $w \in A_p$. The supremum is taken over all cubes Q of $\mathbb{R}^n$. These weights were first studied by Muckenhoupt in [4] and in [3] the interested reader may find a very nice exposition of this weighted theory and its applications.

In [6] the authors defined a metric d_* in the set of A_p weights. For two weights $u, v \in A_p$ we define

$$d_*(u,v) := \|\log u - \log v\|_*,$$

(modulo positive multiplicative constants) where for a function f in $L^1_{loc}(\mathbb{R}^n)$ we define the $BMO(\mathbb{R}^n)$ norm (modulo constants) as

$$\|f\|_* := \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx.$$

The notation f_Q is used to denote the average value of the function f over the cube Q (we will also use the notation $\langle f\rangle_Q$). In addition, the authors proved that if a linear operator T satisfies the weighted estimate

$$\|T\|_{L^p(w)\to L^p(w)} \le F([w]_{A_p}),$$

for all $w \in A_p$, where F is a positive increasing function, then for any fixed weight $w_0 \in A_p$ we have

$$\lim_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)} = \|T\|_{L^p(w_0)\to L^p(w_0)},$$

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which means that the operator norm of T on the $L^p(w)$ space is a continuous function of the weight w with respect to the d_* metric. In this note we are going to extend this result for sublinear operators T, namely for operators that satisfy $|T(f_1+f_2)| \leq |T(f_1)|+|T(f_2)|$ and $|T(kf)| = |k||T(f)|$ for all functions f, f_1, f_2 and scalar k. Our main result is the following:

Theorem 1. *Suppose that for some $1 < p < +\infty$, a sublinear operator T satisfies the inequality*

$$\|T\|_{L^p(w)\to L^p(w)} \leq F([w]_{A_p}),$$

for all $w \in A_p$, where F is a positive increasing function. Fix an A_p weight w_0. Then

$$\lim_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)} = \|T\|_{L^p(w_0)\to L^p(w_0)}.$$

Let us mention that the method used in [6] can not be used for sublinear operators.

Remark 2. In [1] Buckley showed that the Hardy-Littlewood maximal operator, defined as

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ that contain the point x, satisfies the estimate

$$\|M\|_{L^p(w)\to L^p(w)} \leq c[w]_{A_p}^{\frac{1}{p-1}},$$

for $1 < p < +\infty$, and all weights $w \in A_p$, where the constant $c > 0$ is independent of the weight w. This means that the assumptions of Theorem 1 hold for M.

Remark 3. Estimates like the one presented in Theorem 1 can be used in many different areas of mathematics. For example, in [2] such a continuity result was used for the study of PDE with random coefficients and in [5] the sharp asymptotic behavior of the $L^2(w)$ norm of the Riesz projection P_+, with respect to the $[w]_{A_2}$ characteristic, comes into play in the study of Schauder bases.

We present the proof of Theorem 1 in the next section.

2. Proof of theorem 1

The main tool for the proof is the inequality (proved in [6])

$$\|T\|_{L^p(u)\to L^p(u)} \leq \|T\|_{L^p(v)\to L^p(v)}(1 + c_{[v]_{A_p}} d_*(u,v)), \tag{1}$$

that holds for all A_p weights $u, v \in A_p$ that are sufficiently close in the d_* metric, and for sublinear operators T that satisfy the assumptions of our Theorem. The positive constant $c_{[v]_{A_p}}$ that appears in the inequality depends on the dimension n, on p, on the function F and on the A_p characteristic of the weight v. Since the quantities n, p, F are fixed we only write the subscript $c_{[v]_{A_p}}$ to emphasize this dependence on the characteristic.

Proof. We apply inequality (1) with $u = w$ and $v = w_0$ to obtain

$$\|T\|_{L^p(w)\to L^p(w)} \leq \|T\|_{L^p(w_0)\to L^p(w_0)}(1 + c_{[w_0]_{A_p}} d_*(w,w_0)).$$

By letting $d_*(w,w_0)$ go to 0 we get

$$\limsup_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)} \leq \|T\|_{L^p(w_0)\to L^p(w_0)}.$$

Now it suffices to prove the inequality

$$\|T\|_{L^p(w_0)\to L^p(w_0)} \leq \liminf_{d_*(w,w_0)\to 0} \|T\|_{L^p(w)\to L^p(w)},$$

in order to finish the proof. For this reason we use inequality (1) with $u = w_0$ and $v = w$

$$\|T\|_{L^p(w_0)\to L^p(w_0)} \leq \|T\|_{L^p(w)\to L^p(w)}(1 + c_{[w]_{A_p}} d_*(w, w_0)).$$

At this point if we know that the constant $c_{[w]_{A_p}}$ remains bounded as the distance $d_*(w, w_0)$ goes to 0 we are done.

For this reason we assume that $d_*(w, w_0) = \delta$ is very close to 0. Then the function $\frac{w}{w_0}$ is an A_p weight with A_p characteristic very close to 1 (see [3] Corollary II-3.10-ii and Corollary IV-2.18 and the remarks following the last corollary). How close depends only on δ, not on w. Thus, if R is large enough, the weight $(\frac{w}{w_0})^R \in A_p$, with A_p characteristic independent of w (again see [3], Corollary II-3.10-ii and Corollary IV-2.18). Note that from the classical A_p theory, for sufficiently small $\epsilon > 0$, we have $w_0^{1+\epsilon} \in A_p$. Choose the numbers R, ϵ such that we have the relation $\frac{1}{R} + \frac{1}{1+\epsilon} = 1$, i.e. such that R and $R' = 1 + \epsilon$ are conjugate exponents. Then, by applying Hölder's inequality twice we have the following

$$\begin{aligned}
\langle w\rangle_Q \langle w^{-\frac{1}{p-1}}\rangle_Q^{p-1} &= \Big\langle \frac{w}{w_0} w_0\Big\rangle_Q \Big\langle \Big(\frac{w}{w_0}\Big)^{-\frac{1}{p-1}} w_0^{-\frac{1}{p-1}}\Big\rangle_Q^{p-1} \\
&\leq \Big\langle \Big(\frac{w}{w_0}\Big)^R\Big\rangle_Q^{\frac{1}{R}} \Big\langle w_0^{R'}\Big\rangle_Q^{\frac{1}{R'}} \Big\langle \Big(\frac{w}{w_0}\Big)^{-\frac{1}{p-1}\cdot R}\Big\rangle_Q^{\frac{p-1}{R}} \Big\langle w_0^{-\frac{1}{p-1}\cdot R'}\Big\rangle_Q^{\frac{p-1}{R'}} \\
&= \Big\langle \Big(\frac{w}{w_0}\Big)^R\Big\rangle_Q^{\frac{1}{R}} \Big\langle \Big(\Big(\frac{w}{w_0}\Big)^R\Big)^{-\frac{1}{p-1}}\Big\rangle_Q^{\frac{p-1}{R}} \Big\langle w_0^{R'}\Big\rangle_Q^{\frac{1}{R'}} \Big\langle (w_0^{R'})^{-\frac{1}{p-1}}\Big\rangle_Q^{\frac{p-1}{R'}} \\
&\leq \Big[\Big(\frac{w}{w_0}\Big)^R\Big]_{A_p}^{\frac{1}{R}} [w_0^{1+\epsilon}]_{A_p}^{\frac{1}{R'}} \leq C,
\end{aligned}$$

where C is a constant independent of the weight w. Therefore, $[w]_{A_p} \leq C$.

The last step is to remember how we obtain the constant $c_{[w]_{A_p}}$ that appears in inequality (1). The authors in [6] used the Riesz-Thorin interpolation theorem with change in measure and then expressed one of the terms that appears in their calculations as a Taylor series (see the proof of Theorem 3.1 in [6] and in particular the chain of inequalities at the last paragraph of page 506 there). The constant $c_{[w]_{A_p}}$ appears at exactly this point and it is not difficult to see that it depends continuously on $[w]_{A_p}$. Since this characteristic is bounded for w close to w_0 in the d_* metric we have that $c_{[w]_{A_p}}$ is bounded as well. This completes the proof. □

A consequence of the proof is the following remark.

Remark 4. Fix a weight $w_0 \in A_p$ and a sufficiently small positive number δ. There is a positive constant C that depends on $[w_0]_{A_p}$ and δ such that for all weights w with $d_*(w, w_0) < \delta$ we have $[w]_{A_p} \leq C$. In addition, from the inequality (see the proof of Theorem 1)

$$[w]_{A_p} \leq \Big[\Big(\frac{w}{w_0}\Big)^R\Big]_{A_p}^{\frac{1}{R}} [w_0^{1+\epsilon}]_{A_p}^{\frac{1}{R'}}, \tag{2}$$

and the Lebesgue dominated convergence theorem (by letting $R \to +\infty$ and remembering that the A_p constant of the weight $(\frac{w}{w_0})^R$ is independent of R) we obtain

$$\limsup_{d_*(w,w_0)\to 0} [w]_{A_p} \leq [w_0]_{A_p}.$$

In order to get the remaining inequality

$$[w_0]_{A_p} \leq \liminf_{d_*(w,w_0)\to 0} [w]_{A_p},$$

we rewrite (2) as

$$[w_0]_{A_p} \leq \left[\left(\frac{w_0}{w}\right)^R\right]_{A_p}^{\frac{1}{R}} [w^{1+\epsilon}]_{A_p}^{\frac{1}{R'}},$$

and we proceed in the same way as before. In this case the number ϵ depends on $[w]_{A_p}$. But we already know that for w close to w_0 in the d_* metric the A_p characteristic of w is bounded from above. This means that we are allowed to choose the same number ϵ for all weights w that are sufficiently close to w_0 and we are done. Therefore, the A_p characteristic of a weight $w \in A_p$ is a continuous function of the weight with respect to the d_* metric.

References

[1] Stephen M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities.* Trans. Amer. Math. Soc., volume 340, number 1, November 1993.

[2] J.G. Conlon and T. Spencer, *A strong central limit theorem for a class of random surfaces.* Communications in Mathematical Physics, Jan. 2014, Volume 325, issue 1, pp 1-15.

[3] J. Garcia-Cuerva and J. Rubio De Francia, *Weighted norm inequalities and related topics.* North Holland Math. Stud. 116, North Holland, Amsterdam 1985.

[4] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function.* Trans. Amer. Math. Soc. 165 (1972), 207-226.

[5] M. Nielsen and H. Sikic, *On stability of Schauder bases of integer translates.* Journal of Functional Analysis, volume 226, issue 4, 15 Feb. 2014, pages 2281-2293.

[6] N. Pattakos and A. Volberg, *The Muckenhoupt A_∞ class as a metric space and continuity of weighted estimates.* Math. Res. Lett. 19 (2012), no. 02, 499-510.

Department of Mathematics and Applied Mathematics, University of Crete, Voutes Campus, 70013 Heraklion, Greece.

E-mail address: mpapadim@uoc.gr

Department of Mathematics, Institute for Analysis, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany.

E-mail address: nikolaos.pattakos@gmail.com