Nonlinear double resonant periodic problems with the scalar p-Laplacian

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Abstract

We study a nonlinear elliptic problem driven by the p-Laplacian and with a double resonance at \( \pm \infty \). Following a variational approach based on the nonsmooth critical point theory and Morse theoretic techniques, we have multiplicity with at least three nontrivial solutions, when the double resonance occurs with respect to two successive eigenvalues of the negative p-Laplacian \( \lambda_m < \lambda_{m+1} \), \( m \neq 0 \).

Keywords: Spectrum of the scalar p-Laplacian, double resonance, critical groups, reduced homology sequence, Morse theory, multiplicity theorem.

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1 Introduction

In this paper we study the following nonlinear periodic problem:

\[
\begin{align*}
-(|u'(t)|^{p-2}u'(t))' &= f(t, u(t)) \text{ a.e. on } T = [0, b], \\
u(0) &= u(b), \quad u'(0) = u'(b), 
\end{align*}
\]

(1)

Scalar periodic problems with double resonance were investigated primarily for semilinear equations. In this direction we mention the works of Fabry–Fonda [7], Gasiński–Papageorgiou [10], Omari–Zanolin [13] and Su–Zhao [16]. Fabry–Fonda [7], Gasiński–Papageorgiou [10] and Su–Zhao [16], use Landesman–Lazer type conditions, while Omari–Zanolin [13] use certain nonresonance conditions involving the quotient \( \frac{2F(t,x)}{x^2} \) where \( F(t,x) = \int_0^x f(t,s)ds \) is the potential function corresponding to \( f(t,x) \). In Fabry–Fonda [7] and Omari–Zanolin [13] the approach is degree theoretic, while in Gasiński–Papageorgiou [10] and Su–Zhao [16] the authors use variational methods coupled with techniques from Morse theory. From the aforementioned works Fabry–Fonda [7] and Omari–Zanolin
prove only existence theorems, while Gasiński–Papageorgiou [10] and Sun–Zhao [16] have multiplicity results. Gasiński–Papageorgiou [10] produce four solutions, while Sun–Zhao [16] obtain two solutions. For equations driven by the periodic scalar $p$-Laplacian, to the best of our knowledge, there is only the work of Kyritsi–Papageorgiou [11], where the authors prove an existence theorem using conditions similar to those employed by Omari–Zanolin [13]. Combining variational methods based on the critical point theory with Morse theoretic techniques, we show that we have existence when the double resonance occurs at any spectral interval and we have multiplicity with at least three nontrivial solutions, when the double resonance occurs at any spectral interval distinct from the “principal” one $[\lambda_0 = 0, \lambda_1]$.

2 Multiplicity Theorem

The hypotheses on the reaction term $f(t, x)$ are:

1. $f(t, x)$ is a Carathéodory function s.t. for a.a. $t \in T$ $f(t, 0) = 0$ and

   \begin{align*}
   (i) \quad &|f(t, x)| \leq a(t)(1 + |x|^{p-1}) \text{ for a.a. } t \in T, \quad \forall x \in \mathbb{R}, \quad a \in L^1(T)_+; \\
   (ii) \quad &\exists m \geq 1 \text{ s.t.} \\
   &\bar{\lambda}_m \leq \liminf_{|x| \to \infty} \frac{f(t, x)}{|x|^{p-2}x} \leq \limsup_{|x| \to \infty} \frac{f(t, x)}{|x|^{p-2}x} \leq \bar{\lambda}_{m+1} \\
   &\text{uniformly for a.a. } t \in T, \text{ and} \\
   &\lim_{|x| \to \infty} \frac{f(t, x)x - pF(t, x)}{|x|^{p}} = +\infty \text{ uniformly for a.a. } t \in T;
   \\
   (iii) \quad &\exists \theta \in L^1(T), \theta(t) \leq 0 \text{ a.e. on } T, \theta \neq 0 \text{ such that} \\
   &\limsup_{x \to 0} \frac{pF(t, x)}{|x|^p} \leq \theta(t) \text{ uniformly for a.a. } t \in T,
   \\
   (iv) \quad &\text{for every } r > 0, \text{ there exists } \xi_r > 0 \text{ s.t.} f(t, x) + \xi_r |x|^{p-2}x \geq 0 \text{ for} \\
   &\text{a.a. } t \in T, \quad \forall x \in [-r, r].
   
   \end{align*}

We set $G_\pm(t, x) = \int_0^x g_\pm(t, s)ds$ and consider the $C^1$-functionals $\psi_\pm : W^{1,p}_{\text{per}}(0, b) \to \mathbb{R}$ defined by

$$
\psi_\pm(u) = \frac{1}{p} \|u\|_{p}^p + \frac{\xi}{p} \|u\|_{p}^p - \int_0^b G_\pm(t, u(t))dt \text{ for all } u \in W^{1,p}_{\text{per}}(0, b).
$$

Also, let $\varphi : W^{1,p}_{\text{per}}(0, b) \to \mathbb{R}$ be the energy functional for problem (1) defined by

$$
\varphi(u) = \frac{1}{p} \|u\|_{p}^p - \int_0^b F(t, u(t))dt \text{ for all } u \in W^{1,p}_{\text{per}}(0, b).
$$

We know that $\varphi \in C^1(W^{1,p}_{\text{per}}(0, b))$. 

PROPOSITION 2.1. If hypotheses \( H \) hold, then \( \psi_{\pm} \) satisfy the C-condition.

PROOF: We do the proof for \( \psi_{+} \), the proof for \( \psi_{-} \) being similar.

We consider a sequence \( \{u_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b) \) s.t.

\[
|\psi_{+}(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1
\]

and

\[
(1 + \|u_n\|)\psi'_{+}(u_n) \to 0 \text{ in } W^{1,p}_{\text{per}}(0,b)^* \text{ as } n \to \infty.
\]

From (3) we have

\[
| \langle A(u_n), h \rangle | > -\varepsilon \int_0^b |u_n|^{p-2}u_nhdt - \int_0^b g_+(t,u_n)hdt | \leq \varepsilon_n \|h\| \frac{1}{1 + \|u_n\|}
\]

for all \( h \in W^{1,p}_{\text{per}}(0,b) \), with \( \varepsilon_n \to 0^+ \).

In (4) we choose \( h = -u_n^- \in W^{1,p}_{\text{per}}(0,b) \). Then

\[
\|\psi_{-}(u_n)\|_p^p = 2, \|u_n^-\|_p^p \leq \varepsilon_n \text{ for all } n \geq 1,
\]

\[
\Rightarrow u_n^- \to 0 \text{ in } W^{1,p}_{\text{per}}(0,b).
\]

Claim: \( \{u_n^+\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b) \) is bounded.

We proceed by contradiction. So, suppose that \( \|u_n^+\| \to \infty \). We set \( y_n = \frac{u_n^+}{\|u_n^+\|}, n \geq 1 \). Then \( \|y_n\| = 1 \) for all \( n \geq 1 \) and so we may assume that

\[
y_n \xrightarrow{w} y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T).
\]

For (4) and (5) we have

\[
| \langle A(y_n), h \rangle | > -\varepsilon \int_0^b |y_n|^{p-2}y_nhdt - \int_0^b g_+(t,u_n^+)hdt | \leq \varepsilon'_n \|h\| \text{ with } \varepsilon'_n \to 0^+.
\]

Choose \( h = y_n^+ - y \in W^{1,p}_{\text{per}}(0,b) \), pass to the limit as \( n \to \infty \) and use (6).

We obtain

\[
\lim_{n \to \infty} | \langle A(y_n), y_n - y \rangle | = 0,
\]

\[
\Rightarrow y_n \to y \text{ in } W^{1,p}_{\text{per}}(0,b), \text{ hence } \|y\| = 1, y \geq 0.
\]

Note that because of \( H(2) \) and (6), we have that \( \{ \frac{g_+(u_n^+)}{\|u_n^+\|^{p-1}} \}_{n \geq 1} \subseteq L^1(T) \) is uniformly integrable. So, by virtue of the Dunford-Pettis theorem, we may assume that

\[
\frac{g_+(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \tilde{g}_+ \text{ in } L^1(T).
\]

Using hypothesis \( H(3) \) and reasoning as in the proof of Proposition 14 of Alizicović–Papageorgiou–Staicu [1], we show that

\[
\tilde{g}_+ = (\xi + \varepsilon) y^{p-1} \text{ with } \bar{\lambda}_m \leq \xi(t) \leq \bar{\lambda}_{m+1} \text{ a.e. on } T.
\]
So, if we return to (7), pass to the limit as \( n \to \infty \) and use (8), (9) and (10), then
\[
< A(y), h > = \int_0^b \xi y^{p-1} h dt \text{ for all } h \in W_{\text{per}}^1(0, b),
\]
\[
\Rightarrow A(y) = \xi y^{p-1},
\]
\[
\Rightarrow -((y'(t))^{p-2} y'(t))' = \xi(t)y(t)^{p-1} \text{ a.e. on } T,
\]
\[
y(0) = y(b), y'(0) = y'(b).
\]

Recall that \( \tilde{\lambda}_m \leq \xi(t) \leq \tilde{\lambda}_{m+1} \text{ a.e. on } T \). If \( \xi \neq \tilde{\lambda}_m \) and \( \xi \neq \tilde{\lambda}_{m+1} \), we have that \( y = 0 \), which contradicts (8). If \( \xi(t) = \tilde{\lambda}_m \text{ a.e. on } T \) or \( \xi(t) = \tilde{\lambda}_{m+1} \text{ a.e. on } T \) then from (11) and since \( m \geq 1 \), we infer that \( y \) must be nodal, which contradicts (8). Therefore \( \{ u^n_+ \}_{n \geq 1} \subseteq W_{\text{per}}^{1,p}(0, b) \) is bounded. This proves the Claim.

From (5) and the Claim we infer that \( \{ u_n \}_{n \geq 1} \subseteq W_{\text{per}}^{1,p}(0, b) \) is bounded. So, we may assume that \( u_n \to u \) in \( W_{\text{per}}^{1,p}(0, b) \) and \( u_n \to u \) in \( C(T) \). Hence, if in (4) we let \( h = u_n - u \) and pass to the limit as \( n \to \infty \), then
\[
\lim_{n \to \infty} < A(u_n), u_n - u > = 0,
\]
\[
\Rightarrow u_n \to u \text{ in } W_{\text{per}}^{1,p}(0, b)
\]

This proves that \( \psi_+ \) satisfies the C-condition. Similarly for \( \psi_- \).

\[\square\]

**Proposition 2.2.** If hypotheses H hold, then \( \varphi \) satisfy the C-condition.

**Proof:** We consider a sequence \( \{ u_n \}_{n \geq 1} \subseteq W_{\text{per}}^{1,p}(0, b) \) s.t.
\[
| \varphi(u_n) | \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1 \quad (12)
\]
and
\[
(1 + \| u_n \|) \varphi'(u_n) \to 0 \text{ in } W_{\text{per}}^{1,p}(0, b)^* \text{ as } n \to \infty. \quad (13)
\]

From (13) we have
\[
| < A(u_n), h > - \int_0^b f(t, u_n) h dt | \leq \frac{\varepsilon_n \| h \|}{1 + \| u_n \|} \quad (14)
\]
for all \( h \in W_{\text{per}}^{1,p}(0, b) \) with \( \varepsilon_n \to 0^+ \).

In (14) we choose \( h = u_n \) and obtain
\[
-\| u_n \|_p^p + \int_0^b f(t, u_n) u_n dt \leq \varepsilon_n \text{ for all } n \geq 1. \quad (15)
\]

On the other hand from (12), we have
\[
\| u_n' \|_p^p - \int_0^b p F'(t, u_n) dt \leq p M_2 \text{ for all } n \geq 1. \quad (16)
\]
Adding (15) and (16), we obtain
\[
\int_0^b \left[ f(t, u_n) u_n - pF(t, u_n) \right] dt \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \geq 1.
\] (17)

Claim: \( \{u_n\}_{n \geq 1} \subseteq W_{\text{per}}^{1,p}(0,b) \) is bounded.

We argue indirectly. So, suppose that \( \|u_n\| \to \infty \) and set \( y_n = \frac{u_n}{\|u_n\|} \) for all \( n \geq 1 \). Then \( \|y_n\| = 1 \) for all \( n \geq 1 \) and so we may assume that
\[
y_n \overset{w}{\to} y \text{ in } W_{\text{per}}^{1,p}(0,b) \text{ and } y_n \to y \text{ in } C(T) \text{ as } n \to \infty.
\] (18)

From (14) we have
\[
\begin{align*}
|\langle A(y_n), h \rangle - \int_0^b \frac{f(t, u_n)}{\|u_n\|^{p-1}} h dt| & \leq \frac{\varepsilon_n \|h\|}{(1 + \|u_n\|) \|u_n\|^{p-1}} \\
& \quad \text{for all } n \geq 1.
\end{align*}
\] (19)

It is clear from hypothesis \( H(\xi) \) that \( \left( \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \right)_{n \geq 1} \subseteq L^1(T) \) is uniformly integrable. Hence, if we set \( h = y_n - y \) and pass to the limit as \( n \to \infty \), then
\[
\lim_{n \to \infty} \langle A(y_n), y_n - y \rangle = 0,
\]
\[
\Rightarrow \quad y_n \to y \text{ in } W_{\text{per}}^{1,p}(0,b), \text{ hence } \|y\| = 1.
\] (20)

Since \( \left( \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \right)_{n \geq 1} \subseteq L^1(T) \) is uniformly integrable, by the Dunford–Pettis theorem, we may assume that
\[
\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \overset{w}{\to} \tilde{\theta} \text{ in } L^1(T)
\] (21)

with \( \tilde{\theta} = \xi |y|^{p-2}y, \tilde{\lambda}_m \leq \xi(t) \leq \tilde{\lambda}_{m+1} \) a.e. on \( T \) (see the proof of Proposition 2.1). Passing to the limit as \( n \to \infty \) in (19) and using (20) and (21), we obtain
\[
\begin{align*}
\langle A(y), h \rangle & = \int_0^b \xi |y|^{p-2}y dt \text{ for all } h \in W_{\text{per}}^{1,p}(0,b),
\end{align*}
\]
\[
\Rightarrow \quad A(y) = \xi |y|^{p-2}y,
\]
\[
\Rightarrow \quad -\langle (y'(t))^{p-2}y'(t)' \rangle = \xi(t) |y(t)|^{p-2}y(t) \text{ a.e. on } T,
\]
\[
y(0) = y(b), y'(0) = y'(b).
\] (22)

We know that \( \tilde{\lambda}_m \leq \xi(t) \leq \tilde{\lambda}_{m+1} \) a.e. on \( T \). First suppose that \( \xi \neq \tilde{\lambda}_m \) and \( \xi \neq \tilde{\lambda}_{m+1} \). Then from (22) we have that \( y = 0 \), which contradicts (20). So, we assume that \( \xi(t) = \tilde{\lambda}_m \) a.e. on \( T \) or \( \xi(t) = \tilde{\lambda}_{m+1} \) a.e. on \( T \). Then we have \( y(t) \neq 0 \) for a.a. \( t \in T \) (see Binding–Rynne [3]). Therefore \( |u_n(t)| \to \infty \) for a.a. \( t \in T \) and this by virtue of hypothesis \( H(\eta) \) implies
\[
f(t, u_n(t)) u_n(t) - pF(t, u_n(t)) \to +\infty \text{ for a.a. } t \in T,
\]
\[
\Rightarrow \quad \int_0^b \left[ f(t, u_n(t)) u_n(t) - pF(t, u_n(t)) \right] dt \to +\infty \text{ (by Fatou's lemma)}
\] (23)
Comparing (17) and (23), we reach a contradiction. This proves the Claim.

By virtue of the Claim, we may assume that \( u_n \to u \) in \( W^{1,p}_{\text{per}}(0,b) \) and \( u_n \to u \) in \( C(T) \). Using \( h = u_n - u \) in (14) and passing to the limit as \( n \to \infty \), we obtain
\[
\lim_{n \to \infty} < A(u_n), u_n - u > = 0,
\]

\[\Rightarrow u_n \to u \text{ in } W^{1,p}_{\text{per}}(0,b)\]

This proves the proposition.

PROPOSITION 2.3. If hypotheses \( H \) hold, then \( u = 0 \) is a local minimizer of \( \psi_{\pm} \) and of \( \varphi \).

PROOF: We do the proof for \( \psi_+ \), the proofs for \( \psi_- \) and \( \varphi \) being similar. By virtue of hypothesis \( H(iii) \), given \( \varepsilon > 0 \), we can find \( \delta = \delta(\varepsilon) > 0 \) s.t.
\[
F(x,t) \leq \frac{1}{p} (\theta(t) + \varepsilon)|t|^p \text{ for a.a. } t \in T, \text{ all } |x| \leq \delta.
\]

(24)

Let \( u \in \overline{C}(T) \) with \( ||u||_{C^1(T)} \leq \delta \). Then
\[
\psi_+(u) = \frac{1}{p} ||u''||^p_p + \frac{\varepsilon}{p} ||u||^p_p - \int_0^b G_+(t,u)dt 
\]

\[\geq \frac{1}{p} ||u''||^p_p - \int_0^b F(t,u^+)(t)dt
\]

\[\geq \frac{1}{p} ||u''||^p_p - \frac{1}{p} \int_0^b \theta |u|^p dt - \frac{\varepsilon}{p} ||u||^p \text{ (see (24))}
\]

\[\geq \frac{\xi_0 - \varepsilon}{p} ||u||^p
\]

(25)

Choosing \( \varepsilon \in (0,\xi_0) \) we infer that
\[
\psi_+(u) \geq 0 \text{ for all } u \in \overline{C}(T) \text{ with } ||u||_{C^1(T)} \leq \delta,
\]

\[\Rightarrow u = 0 \text{ is a local } \overline{C}(T)\text{-minimizer of } \psi_+,
\]

\[\Rightarrow u = 0 \text{ is a local } W^{1,p}_{\text{per}}(0,b)\text{-minimizer of } \psi_+
\]

(see Proposition 3.3 of Kyritsi–Papageorgiou [12]).

Similarly for the functionals \( \psi_- \) and \( \varphi \). \( \square \)

We may assume that \( u = 0 \) is an isolated critical point of \( \psi_- \). Indeed, otherwise we can find \( \{u_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b) \setminus \{0\} \) such that \( u_n \to 0 \) in \( W^{1,p}_{\text{per}}(0,b) \) and
\[
\psi'_+(u_n) = 0 \text{ for all } n \geq 1,
\]

\[\Rightarrow A(u_n) + \varepsilon |u_n|^{p-2}u_n = N_{g_+}(u_n) \text{ for all } n \geq 1,
\]

where \( N_{g_+}(u)(\cdot) = g_+(\cdot, u(\cdot)) \) for all \( u \in W^{1,p}_{\text{per}}(0,b) \).
Acting on (26) with \(-u_n^- \in W_{\text{per}}^{1,p}(0,b)\), we obtain \(u_n \geq 0\) for all \(n \geq 1\) and so (26) becomes

\[
A(u_n) = N_f(u_n) \text{ for all } n \geq 1,
\]

where \(N_f(u)(\cdot) = f(\cdot, u(\cdot)) \text{ for all } u \in W_{\text{per}}^{1,p}(0,b),\)

\[
\Rightarrow \ u_n \in C^2(T) \text{ is a solution of (1) for all } n \geq 1.
\]

Hence we have produced a whole sequence of distinct nontrivial (and in fact positive) solutions of (1) and so we are done.

Reasoning as in Aizicovici–Papageorgiou–Staicu [1] (see the proof of Proposition 29), we can find \(\rho_+ \in (0,1)\) small s.t.

\[
\psi_+(0) = 0 < mf[\psi_+(u) : ||u|| = \rho_+] = \eta_+.
\]

In a similar way, we show that we can find \(\rho_- \in (0,1)\) small s.t.

\[
\psi_-(0) = 0 < mf[\psi_-(u) : ||u|| = \rho_-] = \eta_-.
\]

Now we are ready to produce two constant sign solutions for problem (1).

\[\Box\]

**PROPOSITION 2.4.** If hypotheses \(H\) hold, then problem (1) has at least two constant sign solutions

\[u_0 \in \text{int} \mathcal{C}_+ \text{ and } u_0 \in -\text{int} \mathcal{C}_+ .\]

**PROOF:** Let \(\xi \in \mathbb{R}, \xi > 0\). Then

\[
\psi_+(\xi) = -\int_0^b F(t, \xi)dt
\]

From hypothesis \(H(\mathit{ii})\) it follows that

\[
\tilde{\lambda}_m \leq \liminf_{|\xi| \to \infty} \frac{pF(t, \xi)}{|\xi|^p} \leq \limsup_{|\xi| \to \infty} \frac{pF(t, \xi)}{|\xi|^p} \leq \tilde{\lambda}_{m+1} \text{ uniformly for a.a. } t \in T.
\]

(see, for example, Aizicovici–Papageorgiou–Staicu [1], Remark 26). Therefore

\[
\psi_+(\xi) \to -\infty \text{ as } \xi \to +\infty.
\]

From (27), (29) and Proposition 2.1, it follows that we can apply the mountain pass theorem, and obtain \(u_0 \in W_{\text{per}}^{1,p}(0,b)\) s.t.

\[
\psi_+(0) = 0 < \eta_+ \leq \psi_+(u_0)
\]

\[
\psi_+(u_0) = 0.
\]

From (30) we have \(u_0 \neq 0\). From (31) we have

\[
A(u_0) + \varepsilon |u_0|^{p-2}u_0 = N_{\mathcal{P}_+}(u_0).
\]
Acting on (32) with $-u_0 \in W_{per}^{1,p}(0,b)$, we show that $u_0 \geq 0$. So (32) becomes

$$A(u_0) = N_f(u_0),$$

$$\Rightarrow -\left(\frac{u_0'(t)}{p-2}u_0(t)\right)' = f(t, u_0(t)) \text{ a.e. on } T,$$

$$v_0(0) = v_0(b), \quad v_0'(0) = v_0'(b),$$

$$\Rightarrow u_0 \in \tilde{C}_+ \setminus \{0\} \text{ solves problem (1)}. \quad (33)$$

Let $r = \|u_0\|_\infty$. Then by virtue of hypothesis $H(iii)$, we can find $\xi_r > 0$ s.t.

$$f(t, u_0(t)) + \xi_r u_0(t)^{p-1} \geq 0 \text{ a.e. on } T,$$

$$\Rightarrow \left(\frac{u_0'(t)}{p-2}u_0(t)\right)' \leq \xi_r u_0(t)^{p-1} \text{ a.e. on } T \text{ (see (33))},$$

$$\Rightarrow u_0 \in \text{int} \tilde{C}_+ \text{ (see Vazquez [17])}.$$

Similarly, working this time with $\psi_-$ and using (28), we obtain a second constant sign solution $v_0 \in -\text{int} \tilde{C}_+$.

Next using Morse theory, we will produce a third nontrivial solution for problem (1). To this end, first we prove the following auxiliary result which will be helpful in computing certain critical groups at infinity. Our result extends Lemma 2.4 of Perera–Schechter [15] which is formulated in Hilbert spaces, with stronger hypotheses and for functionals satisfying the $PS$-condition.

**Lemma 2.5.** If $X$ is a Banach, $(\tau, u) \rightarrow h_\tau(u)$ belongs in $C^1([0,1] \times X)$ and is bounded, there is $R > 0$ s.t. for all $\tau \in [0,1]$ $K_{h_\tau} \subseteq B_R$, the maps $u \rightarrow \partial_\tau h_\tau(u)$ and $u \rightarrow h_\tau'(u)$ are both locally Lipschitz, $h_0$ and $h_1$ satisfy the $C$-condition and there exist $\beta \in \mathbb{R}$ and $\delta > 0$ s.t.

$$h_\tau(u) \leq \beta \Rightarrow (1 + \|u\|)\|h_\tau'(u)\|_* \geq \delta \text{ for all } \tau \in [0,1],$$

then $C_k(h_0, \infty) = C_k(h_1, \infty)$ for all $k \geq 0$.

**Proof:** Since by hypothesis $h \in C^1([0,1] \times X)$, we know that its admits a pseudogradient vector field $\partial = (v_0, v_1) : [0,1] \times (X \setminus B_R) \rightarrow [0,1] \times X$. Moreover, taking into account the construction of the pseudogradient vector field, we know that we can have $v_0 = \partial_\tau h_\tau$. Also by definition $(\tau, u) \rightarrow v_\tau(u)$ is locally Lipschitz and in fact for every $\tau \in [0,1]$, $v_\tau(\cdot)$ is a pseudogradient vector field for the function $h_\tau(\cdot)$ (see Chang [4], p.19 and Papageorgiou–Kyritsi [14], p.237). Then for every $(\tau, u) \in [0,1] \times (X \setminus B_R)$ we have

$$< h'_\tau(u), v_\tau(u) > \geq \|h'_\tau(u)\|_*^2 \quad (34)$$

The map $X \setminus B_R \ni u \rightarrow \frac{\partial_\tau h_\tau(u)}{\|h'_\tau(u)\|_*} \in X$ is well defined and locally Lipschitz. Since $(\tau, u) \rightarrow h_\tau(u)$ is bounded, we can find $\eta \leq \beta$ s.t.

$$\eta \in \{h_\tau(\cdot) : \tau \in [0,1], \|u\| \leq R\}.$$
We choose \( \eta < \beta \) s.t. \( h_0^\eta \neq 0 \) or \( h_0^\eta \neq \emptyset \) (if no such \( \eta \) can be found, then \( C_k(h_0, \infty) = C_k(h_1, \infty) = \delta_{k,0} \mathbb{Z} \) for all \( k \geq 0 \) and so we are done). To fix things, we assume that \( h_0^\eta \neq \emptyset \) and let \( y \in h_0^\eta \). We consider the following Cauchy problem:

\[
\frac{d\xi}{d\tau} = w_\tau(\xi), \quad \tau \in [0, 1], \xi(0) = y. \tag{35}
\]

Since \( w_\tau(u) \) is locally Lipschitz, this Cauchy problem has a unique local flow (see, for example, Gasinski–Papageorgiou [9], p.618). We have

\[
\begin{aligned}
\frac{d}{d\tau} h_\tau(\xi) &= \langle h_\tau'(\xi), \frac{d\xi}{d\tau} \rangle + \partial_+ h_\tau(\xi) \\
&= \langle h_\tau'(\xi), h_\tau(\xi) \rangle + \partial_+ h_\tau(\xi) \quad (\text{see (35)}) \\
&\leq -|\partial_+ h_\tau(\xi)| + \partial_+ h_\tau(\xi) \quad (\text{see (34)}) \\
&\leq 0,
\end{aligned}
\]

\( \Rightarrow \) \( \tau \to h_\tau(\xi(\tau, y)) \) is nonincreasing,

\( \Rightarrow \) \( h_\tau(\xi(\tau, y)) \leq h_0(\xi(0, y)) = h_0(y) \leq \eta \leq \beta, \)

\( \Rightarrow \) \( (1 + \|\xi(\tau, y)\|) \|h_\tau'(\xi(\tau, y))\| \geq \delta \) (by hypothesis),

\( \Rightarrow \) \( h_\tau'(\xi(\tau, y)) \neq 0 \).

This shows that in fact the flow \( \xi(\cdot, y) \) is global.

Then \( \xi(1, \cdot) \) is a homeomorphism between \( h_0^\eta \) and a subset of \( h_0^\eta \). Reversing the time \( t \to 1 - t \), we establish that \( h_0^\eta \) is homeomorphic to a subset of \( h_0^\eta \).

Therefore \( h_0^\eta \) and \( h_0^\eta \) are homotopy equivalent and so

\[
H_k(X, h_0^\eta) = H_k(X, h_1^\eta) \quad \text{for all } k \geq 0,
\]

\( \Rightarrow \) \( C_k(h_0, \infty) = C_k(h_1, \infty) \) for all \( k \geq 0 \).

Using this lemma we can have some useful information concerning the critical groups at infinity of \( \varphi \).

**PROPOSITION 2.6.** If hypotheses \( H \) hold, then \( C_{m+1}(\varphi, \infty) \neq 0 \) (m \geq 1 as in hypothesis \( H(\mathrm{ii}) \)).

**PROOF:** Let \( \mu \in (\lambda_m, \lambda_{m+1}) \) and consider the \( C^1 \)-functional \( \chi : W^{1,p}_{\text{per}}(0, b) \to \mathbb{R} \) define by

\[
\chi(u) = \frac{1}{p} \|u\|_p^p - \frac{\mu}{p} \|u\|_p^p \quad \text{for all } u \in W^{1,p}_{\text{per}}(0, b).
\]

We consider the homotopy

\[
h(\tau, u) = (1 - \tau)\varphi(u) + \tau \chi(u) \quad \text{for all } (\tau, u) \in [0, 1] \times W^{1,p}_{\text{per}}(0, b).
\]

Clearly we may assume that \( K_\varphi \) is finite (otherwise we already have infinitely many distinct nontrivial solutions of (1) and so we are done). Note that \( h(0, \cdot) = \)
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\( \varphi \) satisfies the C-condition (see Proposition 2.2) and \( h(1, \cdot) = \chi \) also satisfies the C-condition since \( \mu \in (\bar{\lambda}_m, \bar{\lambda}_{m+1}) \).

**Claim:** There exist \( \beta \in \mathbb{R} \) and \( \delta > 0 \) s.t.

\[
h(\tau, u) \leq \beta \Rightarrow (1 + \|u\|) \|h_n'(\tau, u)\|_\ast \geq \delta \text{ for all } \tau \in [0, 1].
\]

We argue by contradiction. So, suppose that the claim is not true. Since \( h \) is bounded, we can find \( (\tau_n)_{n \geq 1} \subseteq [0, 1] \) and \( (u_n)_{n \geq 1} \subseteq \mathcal{W}^1_{p, \text{per}}(0, b) \) s.t.

\[
\tau_n \to \tau, \|u_n\| \to \infty, h(\tau_n, u_n) \to -\infty \text{ and } (1 + \|u_n\|) \|h_n'(\tau_n, u_n)\| \to 0.
\]

By virtue of the last convergence in (36), we have

\[
|<A(u_n), h>- (1 - \tau_n) \int_0^b f(t, u_n) hdt - \tau_n \mu \int_0^b |u_n|^{p-2} u_n hdt| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \tag{37}
\]

for all \( h \in \mathcal{W}^1_{p, \text{per}}(0, b) \) with \( \epsilon_n \to 0^+ \).

Let \( y_n = \frac{u_n}{\|u_n\|} \). Then \( \|y_n\| = 1 \) for all \( n \geq 1 \) and so we may assume that

\[
y_n \overset{w}{\rightharpoonup} y \text{ in } \mathcal{W}^1_{p, \text{per}}(0, b) \text{ and } y_n \to y \text{ in } C(T). \tag{38}
\]

From (37) we have

\[
|<A(y_n), h>- (1 - \tau_n) \int_0^b f(t, u_n) hdt - \tau_n \mu \int_0^b |y_n|^{p-2} y_n hdt| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } n \geq 1. \tag{39}
\]

Recall (see (21)) that

\[
\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{p-1}} \overset{w}{\rightharpoonup} \tilde{\theta} = \xi |y|^{p-2} y \text{ in } L^1(T) \text{ with } \bar{\lambda}_m \leq \xi(t) \leq \bar{\lambda}_{m+1} \text{ a.e. on } T. \tag{40}
\]

In (39), we choose \( h = y_n - y \) and pass to the limit as \( n \to \infty \). Using (38), we obtain

\[
\lim_{n \to \infty} |<A(y_n), y_n - y| = 0,
\]

\[
\Rightarrow y_n \to y \text{ in } \mathcal{W}^1_{p, \text{per}}(0, b) \text{ and so } \|y\| = 1. \tag{41}
\]

Hence, if in (39) we pass to the limit as \( n \to \infty \) and use (40) and (41), then

\[
< A(y), h > = (1 - \tau) \int_0^b \xi |y|^{p-2} y hdt + \tau \mu \int_0^b |y|^{p-2} y hdt
\]

for all \( h \in \mathcal{W}^1_{p, \text{per}}(0, b) \),

\[
\Rightarrow A(y) = \xi r y \text{ with } \xi_r = (1 - \tau) \xi + \tau \mu,
\]

\[
\Rightarrow -(|y'(t)|^{p-2} y'(t))' = \xi_r(t) |y(t)|^{p-2} y(t) \text{ a.e. on } T,
\]

\[
y(0) = y(b), y'(0) = y'(b). \tag{42}
\]
Note that \( \tilde{\lambda}_m \leq \xi_{\tau}(t) \leq \tilde{\lambda}_{m+1} \) a.e. on \( T \).
If \( \tau \in (0, 1] \), then
\[
\xi_{\tau} \neq \tilde{\lambda}_m \text{ and } \xi_{\tau} \neq \tilde{\lambda}_{m+1},
\]
\[
\Rightarrow y = 0 \text{ (see (42)), which contradicts (41).}
\]

So, suppose \( \tau = 0 \). Then \( \xi_0 = \xi \) and we proceed as in the proof of Proposition 2.2 to reach a contradiction, using hypothesis \( H(ii) \) and the third convergence in (30). This proves the Claim.

Then we can apply Lemma 2.5 and we have
\[
C_k(\varphi, \infty) = C_k(\chi, \infty) \text{ for all } k \geq 0.
\]  \( \tag{43} \)

Since \( \mu \in (\tilde{\lambda}_m, \tilde{\lambda}_{m+1}) \), \( u = 0 \) is the only critical point of \( \chi \). Hence
\[
C_k(\chi, \infty) = C_k(\chi, 0) \text{ for all } k \geq 0.
\]  \( \tag{44} \)

Let \( r > 0 \) and set \( E_0 = \{ u \in W^{1,p}_{\text{per}}(0, b) : \|u'\|^p_p < \mu \|u\|^p_p, \|u\| = r \} \) and \( D = \{ u \in W^{1,p}_{\text{per}}(0, b) : \|u'\|^p_p \geq \mu \|u\|^p_p \} \). Evidently \( E_0 \cap D = \emptyset \). Also \( \partial D_r = \{ u \in W^{1,p}_{\text{per}}(0, b) : \|u\| = r \} \) is a Banach \( C^1 \)-manifold, hence locally contractible. Since \( E_0 \) is an open subset of \( \partial D_r \), \( E_0 \) is locally contractible. Similarly \( W^{1,p}_{\text{per}}(0, b) \setminus D \) is locally contractible. Note that since \( \mu \in (\tilde{\lambda}_m, \tilde{\lambda}_{m+1}) \), we have \( i(E_0) = m + 1 \), where \( i \) denotes the index introduced by Fadell–Rabinowitz [8]. Similarly \( i(W^{1,p}_{\text{per}}(0, b) \setminus D) = m + 1 \). Invoking Theorem 3.6 of Cingolani–Degiovanni [6], we know that there exists \( C \subseteq W^{1,p}_{\text{per}}(0, b) \) compact s.t. the pair \( (E_0 \cup C, E_0) \) and \( D \) homologically link in dimension \( m + 1 \) and so \( C_{m+1}(\chi, 0) \neq 0 \) (see Chang [4], p.89). From (43) and (44) we conclude that \( C_{m+1}(\varphi, \infty) \neq 0 \).

\[ \square \]

Next we compute the critical groups at infinity of \( \psi_\pm \).

**Proposition 2.7.** If hypotheses \( H \) hold, then \( C_k(\psi_+, \infty) = C_k(\psi_-, \infty) = 0 \) for all \( k \geq 0 \).

**Proof:** We do the proof for \( \psi_+ \), the proof for \( \psi_- \) being similar.

Let \( \mu \in (\tilde{\lambda}_m, \tilde{\lambda}_{m+1}) \) and consider the \( C^1 \)-functional \( \sigma_+ : W^{1,p}_{\text{per}}(0, b) \to \mathbb{R} \) defined by
\[
\sigma_+(u) = \frac{1}{p} \|Du\|^p_p + \frac{\varepsilon}{p} \|u\|^p_p - \frac{\mu + \varepsilon}{p} \|u^+\|^p_p
\]
for all \( u \in W^{1,p}_{\text{per}}(0, b) \), with \( \varepsilon \in (0, \tilde{\lambda}_2) \).

We consider the homotopy \( h_+ : [0, 1] \times W^{1,p}_{\text{per}}(0, b) \to \mathbb{R} \) defined by
\[
h_+(\tau, u) = (1 - \tau)\psi_+(u) + \tau \sigma_+(u) \text{ for all } (\tau, u) \in [0, 1] \times W^{1,p}_{\text{per}}(0, b).
\]

As before, without any loss of generality, we assume that \( K_{\psi_+} \) is finite.

**Claim:** There exist \( \beta \in \mathbb{R} \) and \( \delta > 0 \) s.t.
\[
h_+(\tau, u) \leq \beta \Rightarrow (1 + \|u\|) \|h_+(\tau, u)\|_{\ast} \geq \delta \text{ for all } \tau \in [0, 1].
\]
As before, we argue by contradiction. So, suppose we can find \( \{\tau_n\}_{n \geq 1} \subseteq [0, 1] \) and \( \{u_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b) \) s.t.

\[
\tau_n \to \tau \in [0, 1], \quad \|u_n\| \to \infty, h_+(\tau_n, u_n) \to -\infty \quad \text{and} \quad \left(1 + \|u_n\|\right)h'_+(\tau_n, u_n) \to 0.
\]  

(45)

From the last convergence in (45), we have

\[
|< A(u_n), h > + \varepsilon \int_0^b |u_n|^{p-2}u_n h dt - (1 - \tau_n) \int_0^b g_+(t, u_n) h dt - \tau_n (\mu + \varepsilon) \int_0^b (u_n^+)^{p-2} u_n^+ h dt| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}
\]

(46)

for all \( h \in W^{1,p}_{\text{per}}(0,b) \) with \( \varepsilon_n \to 0^+ \).

In (46) we choose \( h = -u_n^- \in W^{1,p}_{\text{per}}(0,b) \) and

\[
\left\| (u_n^-)' \right\|^p_p + \varepsilon \|u_n^-\|^p_p \leq \varepsilon_n \quad \text{for all} \ n \geq 1,
\]

\[
\Rightarrow \quad u_n^- \to 0 \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ as } n \to \infty.
\]

(47)

From (45) (second convergence) and (47) it follows that \( \|u_n^+\| \to \infty \). We set \( y_n = \frac{u_n^+}{\|u_n^+\|} \quad n \geq 1 \). Then \( \|y_n\| = 1 \) for all \( n \geq 1 \) and so we may assume that

\[
y_n \rightharpoonup y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } y_n \to y \text{ in } C(T).
\]

(48)

From (46) and (47), we have

\[
\left| < A(y_n), h > + \varepsilon \int_0^b y_n^{p-1} h dt - (1 - \tau_n) \int_0^b \frac{g_+(t, u_n^+)}{\|u_n^+\|^{p-1}} h dt - \tau_n (\mu + \varepsilon) \int_0^b y_n^{p-1} h dt \right| \leq \varepsilon_n \|h\|
\]

(49)

with \( \varepsilon_n \to 0^+ \).

In (49) we choose \( h = y_n - y \). Passing to the limit as \( n \to \infty \) and using (48) we obtain

\[
\lim_{n \to \infty} < A(y_n), y_n - y > = 0,
\]

\[
\Rightarrow \quad y_n \to y \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and so } \|y\| = 1, y \geq 0.
\]

(50)

Recall that

\[
\frac{g_+ (\cdot, u_n^+ (\cdot))}{\|u_n^+\|^{p-1}} \rightharpoonup \bar{\theta}_+ = (\xi + \varepsilon) y^{p-1} \text{ in } L^1(T) \text{ and } \bar{\lambda}_m \leq \xi(t) \leq \bar{\lambda}_{m+1} \text{ for a.a. } t \in T.
\]

(51)

Therefore, if in (49) we pass to the limit as \( n \to \infty \) and use (50) and (51), then
as for all \( h \in W_{\text{per}}^{1,p}(0,b) \)

with \( \xi_{\tau} = (1 - \tau)\xi + \tau \nu \),

\[ A(y) = \xi_{\tau}y^{p-1}, \]

\[ -\left( |y^{\prime}(t)|^{p-2}y^{\prime}(t) \right)^{\prime} = \xi_{\tau}y(t)^{p-1} \text{ a.e. on } T, \]

\[ y(0) = y(b), \quad y^{\prime}(0) = y^{\prime}(b). \]  

(52)

We know that \( \hat{\xi}_{m} \leq \xi_{\tau} \leq \hat{\xi}_{m+1} \) a.e. on \( T \). If \( \tau \in (0,1] \), then \( \xi_{\tau} \neq \hat{\xi}_{m} \), \( \xi_{\tau} \neq \hat{\xi}_{m+1} \) and so by virtue of (52), we have \( y = 0 \), which contradicts (50).

The same is true if \( \tau = 0 \) and \( \xi_{0} \neq \hat{\xi}_{m} \), \( \xi_{0} \neq \hat{\xi}_{m+1} \). Finally, if \( \tau = 0 \) and \( \xi_{0} = \hat{\xi}_{m} \), or \( \xi_{0} = \hat{\xi}_{m+1} \) a.e. on \( T \), then from (52) and since \( m \geq 1 \), \( y(\cdot) \) must be nodal again a contradiction (see (50)). This proves the claim.

The claim permits the use of Lemma 2.5 and we have

\[ C_{k}(\psi_{+}, \infty) = C_{k}(\sigma_{+}, \infty) \text{ for all } k \geq 0. \]

(53)

Since \( \mu \in (\hat{\xi}_{m}, \hat{\xi}_{m+1}) \), \( u = 0 \) is the only critical point of \( \sigma_{+} \) and so

\[ C_{k}(\sigma_{+}, \infty) = C_{k}(\sigma_{+}, 0) \text{ for all } k \geq 0. \]

(54)

Let \( \eta \in L^{\infty}(\Omega), \eta \geq 0, \eta \neq 0 \) and consider the homotopy \( \hat{h}_{+} : [0,1] \times W_{\text{per}}^{1,p}(0,b) \rightarrow \mathbb{R} \) defined by

\[ \hat{h}_{+}(\tau, u) = \sigma_{+}(u) - \tau \eta u \text{ for all } (\tau, u) \in [0,1] \times W_{\text{per}}^{1,p}(0,b). \]

We claim that

\[ (\hat{h}_{+})^{\prime}(\tau, u) \neq 0 \text{ for all } \tau \in [0,1], u \neq 0. \]

(55)

Suppose that (55) is not true. We can find \( \tau \in [0,1] \) and \( u \neq 0 \) s.t.

\[ (\hat{h}_{+})^{\prime}(\tau, u) = 0, \]

\[ A(u) + \varepsilon|u|^{p-2}u = (\mu + \varepsilon)(u)^{p-1} + \tau \eta. \]

(56)

On (56) we act with \(-u^{-} \in W_{\text{per}}^{1,p}(0,b) \) and obtain \( ||(u^{-})^{\prime}||_{p}^{p} + \varepsilon||u^{-}||_{p}^{p} = 0, \) i.e., \( u \geq 0 \). So, (56) becomes

\[ A(u) = \mu u^{p-1} + \tau \eta, u \geq 0, u \neq 0. \]

(57)

First suppose that \( \tau = 0 \). Then

\[ A(u) = \mu u^{p-1} \text{ (see (57))}, \]

\[ -\left( |u^{\prime}(t)|^{p-2}u^{\prime}(t) \right)^{\prime} = \mu u(t)^{p-1} \text{ a.e. on } T \]

\[ u(0) = u(b), u^{\prime}(0) = u^{\prime}(b), \]

\[ u \text{ must be nodal (recall } m \geq 1), \text{ which contradicts (57).} \]
So, we assume that $\tau \in (0,1]$. Then
\[
A(u) = \mu u^{p-1} + \tau \eta,
\Rightarrow
\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = \mu u(t)^{p-1} + \tau \eta(t) \text{ a.e. on } T,
\]
\[
u(0) = u(b), \nu'(0) = u'(b),
\]
We have $u \in C_+ \setminus \{0\}$ and $\left(\left|u'(t)\right|^{p-2}u'(t)\right)' \leq 0$ a.e. on $T$. It follows that $u \in \text{int}C_+$ (see Vazquez [17]).
Let $y \in \bar{C}_+$ and consider
\[
R(y, u)(t) = \left|y'(t)\right|^{p} - \left|u'(t)\right|^{p-2}u'(t)\left(\frac{y^p}{u^p}\right)'(t)
\]
From the generalized Picone identity of Allegretto–Huang [2], we have
\[
0 \leq \int_0^b R(y, u)(t) dt
= \|y\|_{\infty}^p - \int_0^b \left|u'(t)\right|^{p-2}u'(t)\left(\frac{y^p}{u^p}\right)'(t) dt \text{ (by integration by parts)}
= \|y\|_{\infty}^p - \int_0^b (\mu u^p + \tau \eta) dt \text{ (see (58))}
\leq \|y\|_{\infty}^p - \mu \|y\|_{\infty}^p \text{ (recall } \eta \geq 0).
\]
We choose $y = \bar{u}_0 \in \text{int}C_+$. Then
\[
0 \leq -\mu \|\bar{u}_0\|_{\infty}^p < 0, \text{ a contradiction.}
\]
This proves that (55) holds. Then the homotopy invariance property of critical groups (see Chang [5], p. 334) implies that
\[
C_k(\sigma_+, 0) = C_k(\bar{\sigma}_+, 0) \text{ for all } k \geq 0,
\]
where $\bar{\sigma}_+(u) = \sigma_+(u) - \eta u$ for all $u \in W_1^1(0, b)$. From the previous argument, we know that $\bar{\sigma}_+$ has no critical points. Then
\[
C_k(\bar{\sigma}_+, 0) = 0 \text{ for all } k \geq 0,
\Rightarrow
C_k(\psi_+, \infty) \text{ for all } k \geq 0 \text{ (see (59), (54) and (53)).}
\]
Similarly, we show that $C_k(\psi_-, \infty) = 0$ for all $k \geq 0$. \hfill $\Box$

Having this proposition, we can have a precise computation of the critical groups of $\varphi$ at $u_0 \in \text{int}C_+$ and $v_0 \in -\text{int}C_+$. Recall that $v_0, v_0$ are the two constant sign solutions of (1) obtained in Proposition 2.4.

**Proposition 2.8.** If hypotheses $H$ hold and $u_0 \in \text{int}C_+$ and $v_0 \in -\text{int}C_+$ are the two constant sign solutions of (1) obtained in Proposition 2.4, then $C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. 

PROOF: We do the proof for $u_0$, the proof for $u_0$ being similar.

First note that we may assume that $\{0, u_0\}$ are the only critical points of $\psi_+$ (otherwise, we already have one more solution $u_0 \in \text{int}C_+$ of (1) distinct from $\{0, u_0, v_0\}$; note that $K_{\psi_+} \subseteq C_+$).

Let $\eta < 0 < \xi < \bar{\eta}_+$ (see (27)) and consider the following triple of sets

$$
\psi_+^\eta \subseteq \psi_+^\xi \subseteq W^{1,p}_{\text{per}}(0,b).
$$

For this triple, we consider the long exact sequence of homology groups

$$
\cdots \to H_k(W^{1,p}_{\text{per}}(0,b), \psi_+^\eta) \xrightarrow{i_*} H_k(W^{1,p}_{\text{per}}(0,b), \psi_+^\xi) \xrightarrow{\partial_*} H_{k-1}(\psi_+^\xi, \psi_+^\eta) \to \cdots
$$

(60)

By $i_*$ we denote the group homomorphism induced by the inclusion $(W^{1,p}_{\text{per}}(0,b), \psi_+^\eta) \hookrightarrow (W^{1,p}_{\text{per}}(0,b), \psi_+^\xi)$ and $\partial_*$ is the boundary homomorphism. From the rank theorem, we have

$$
\text{rank} H_k(W^{1,p}_{\text{per}}(0,b), \psi_+^\xi) = \text{rank}(\ker \partial_*) + \text{rank}(\text{im} \partial_*) \quad \text{(see (60))},
$$

$$
= \text{rank}(\text{im} i_*) + \text{rank}(\text{im} \partial_*). \quad \text{(from the exactness of (60))}.
$$

(61)

Recalling that $\{0, u_0\}$ are the only critical points of $\psi_+$ and since

$$
\eta < 0 = \psi_+(0) < \bar{\eta}_+ \leq \psi_+(u_0),
$$

we have

$$
\text{rank} H_k(W^{1,p}_{\text{per}}(0,b), \psi_+^\eta) = C_k(\psi_+, \infty) = 0 \quad \text{for all } k \geq 0 \quad \text{(see Proposition 2.7)},
$$

$$
\Rightarrow \text{im} i_* = \{0\}. \quad \text{(62)}
$$

Also $H_{k-1}(\psi_+^\xi, \psi_+^\eta) = C_{k-1}(\psi_+, 0) = \delta_{k-1} Z$ for all $k \geq 0$ (see Proposition 2.3). Therefore

$$
\text{rank}(\text{im} \partial_*) \leq 1. \quad \text{(63)}
$$

Finally since $0 < \xi < \bar{\eta}_+ \leq \psi_+(u_0)$, we have

$$
H_k(W^{1,p}_{\text{per}}(0,b), \psi_+^\xi) = C_k(\psi_+, u_0) \quad \text{for all } k \geq 0 . \quad \text{(64)}
$$

So, if in (61), we use (62), (63), (64), then

$$
\text{rank} C_1(\psi_+, u_0) \leq 1. \quad \text{(65)}
$$

From the proof of Proposition 2.4, we know that $u_0$ is a critical point of $\psi_+$ of mountain pass type. Hence $C_1(\psi_+, u_0) \neq 0$ (see Chang [4], p.89). Combining this with (65) we infer that

$$
C_k(\psi_+, u_0) = \delta_{k+1} Z \quad \text{for all } k \geq 0. \quad \text{(66)}
$$
Consider the homotopy $h_+ : [0,1] \times W^{1,p}_{\text{per}}(0,b) \to \mathbb{R}$ defined by

$$h_+(\tau, u) = (1 - \tau)\varphi(u) + \tau\psi_+(u) \text{ for all } (\tau, u) \in [0,1] \times W^{1,p}_{\text{per}}(0,b).$$

Claim: We may assume that we can find $\rho \in (0,1)$ small s.t. $u_0$ is the only critical point for all $\tau \in [0,1]$ of $h_+(\tau, \cdot)$ in $B_\rho(u_0) = \{ u \in W^{1,p}_{\text{per}}(0,b) : \| u - u_0 \| = \rho \}$.

Suppose we can find $\{\tau_n\}_{n \geq 1} \subseteq [0,1]$ and $\{\bar{u}_n\}_{n \geq 1} \subseteq W^{1,p}_{\text{per}}(0,b)$ s.t.

$$\tau_n \to \tau \in [0,1], \bar{u}_n \to u_0 \text{ in } W^{1,p}_{\text{per}}(0,b) \text{ and } (h_+)'(\tau_n, \bar{u}_n) = 0 \text{ for all } n \geq 1. \quad (67)$$

We have

$$A(\bar{u}_n) + \tau_n|\bar{u}_n|^{p-2}\bar{u}_n = (1 - \tau_n)N_{\varphi_{\tau_n}}(\bar{u}_n) + \tau_n N_f(\bar{u}_n) \text{ for all } n \geq 1,$$

$$\Rightarrow -|\bar{u}_n'(t)|^{p-2}\bar{u}_n'(t) = f(t, \bar{u}_n(t)) + (1 - \tau_n)f(t, -\bar{u}_n(t))$$

$$+ \tau_n|\bar{u}_n|^{p-1} \text{ a.e. on } T,$$

$$\bar{u}_n(0) = \bar{u}_n(b), \bar{u}_n'(0) = \bar{u}_n'(b). \quad (68)$$

From (68), arguing as in the proof of Proposition 3.3 of Kyritsi–Papageorgiou [12], we establish that $\{\bar{u}_n\}_{n \geq 1} \subseteq C^1(T)$ is relatively compact. Therefore we have

$$\bar{u}_n \to u_0 \text{ in } C^1(T) \text{ (see (67)).} \quad (69)$$

Recall that $u_0 \in \text{int}\bar{C}_+$. So, we can find $n_0 \geq 1$ s.t.

$$\Rightarrow \bar{u}_n \in \text{int}\bar{C}_+ \text{ for all } n \geq n_0 \text{ (see (69)),}$$

$$\Rightarrow -|\bar{u}_n'(t)|^{p-2}\bar{u}_n'(t) = f(t, \bar{u}_n(t)) \text{ a.e. on } T,$$

$$\bar{u}_n(0) = \bar{u}_n(b), \bar{u}_n'(0) = \bar{u}_n'(b),$$

$$\Rightarrow \{\bar{u}_n\}_{n \geq 1} \subseteq \text{int}\bar{C}_+ \text{ are nontrivial solutions of (1) and so we are done.}$$

This proves the Claim.

Then the Claim and the homotopy invariance property of the critical groups (see Chang [5], p.334), we have

$$C_k(\varphi, u_0) = C_k(\psi_+, u_0) \text{ for all } k \geq 0,$$

$$\Rightarrow C_k(\varphi, u_0) = \delta_{k,1} \mathbb{Z} \text{ for all } k \geq 0 \text{ (see (68))}.$$  

In the similar fashion, using this time $\psi_-$, we show that $C_k(\varphi, v_0) = \delta_{k,1} \mathbb{Z}$ for all $k \geq 0$.

Now we can state the multiplicity theorem for problem (1) under double resonance conditions.

**THEOREM 2.9.** If hypotheses H hold, then problem (1) has at least three nontrivial solutions

$$u_0 \in \text{int}\bar{C}_+, v_0 \in -\text{int}\bar{C}_+ \text{ and } y_0 \in C^1(T).$$
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**Proof:** From Proposition 2.4, we already have two nontrivial constant sing solutions of (1)

$$u_0 \in \text{int} \mathcal{C}_+ \text{ and } v_0 \in -\text{int} \mathcal{C}_+.$$  

From Proposition 2.8, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}Z \text{ for all } k \geq 0.$$  

(70)

Also, by virtue of Proposition 2.3, we have

$$C_k(\varphi, 0) = \delta_{k,0}Z \text{ for all } k \geq 0.$$  

(71)

Recall that $C_{m+1}(\varphi, \infty) \neq 0$ (see Proposition 2.6). This implies that there exists $y_0 \in K_\varphi$ s.t.

$$C_{m+1}(\varphi, y_0) \neq 0, m \geq 2.$$  

(72)

Comparing (72) with (70) and (71), we infer that $y_0 \neq \{0, u_0, v_0\}$. Also $y_0 \in C^1(T)$ and solves problem (1).

So our work here shows that multiplicity (producing at least three nontrivial solutions) can happen when we have double resonance at any spectral interval beyond the “principal” one $[\lambda_0 = 0, \lambda_1]$.

References


