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SOME OPEN PROBLEMS ON MULTIPLE ERGODIC AVERAGES

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ABSTRACT. We survey some recent developments and give a list of open problems regarding multiple recurrence and convergence phenomena of $\mathbb{Z}^d$ actions in ergodic theory and related applications in combinatorics and number theory.

1. INTRODUCTION

1.1. Objective. Ergodic theory has a long history of interaction with other mathematical fields and in particular with combinatorics and number theory. The seminal work of H. Furstenberg [151], where an ergodic proof of the theorem of Szemerédi [276] on arithmetic progressions was given, linked problems in ergodic theory, combinatorics, and number theory, and provided an ideal ground for cross-fertilization. In combinatorics, it has produced several far reaching extensions of the theorem of Szemerédi some of which still have no proof that avoids ergodic theory. In number theory, it provided some key ideas and tools in the proof of Green and Tao [167] that the primes contain arbitrarily long arithmetic progressions and subsequent extensions of this result. On the other direction, the field of ergodic theory has tremendously benefited as well, since the problems of combinatorial and number-theoretic nature gave a boost to the in depth study of several interesting recurrence and convergence phenomena that would have otherwise been ignored.

The connecting link between combinatorics and ergodic theory is that regularity properties of sets of integers with positive density correspond to multiple recurrence properties of measure preserving systems. One establishes these recurrence properties by analyzing the limiting behavior of some closely related multiple ergodic averages. The study of these ergodic averages has developed into a central part of ergodic theory and has generated tools and ideas that have found applications in areas outside ergodic theory. The purpose of these notes is to survey some of these developments and give a list of open problems on three closely related topics:

(1) The limiting behavior of multiple ergodic averages.
(2) Multiple recurrence properties of measure preserving systems.
(3) Problems of arithmetic nature regarding, for example, the existence of patterns on sets of integers with positive upper density[1].

The list of problems is greatly influenced by my personal interests and is by no means meant to be a comprehensive list of open problems in the area widely known as ergodic Ramsey theory. Almost exclusively, problems related to $\mathbb{Z}^d$ actions are considered and even within this confined class there are a few important topics not touched upon. For material and a list of problems that

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[1]The *upper density* $\bar{d}(E)$ of a set $E \subset \mathbb{Z}^d$ is defined by $\bar{d}(E) := \limsup_{N\to\infty} \frac{|E\cap[-N,N]^d|}{|[-N,N]^d|}$.

goes beyond the scope of this set of notes we refer the reader to the survey articles [37, 40, 41] and the references therein.

A first version of these notes was posted in 2011. Since then, several of the recorded problems have been partially or completely solved and new techniques and problems have surfaced. This updated version is meant to address these developments.

1.2. **The general framework.** We are given a measure space $(X, \mathcal{X}, \mu)$ with $\mu(X) = 1$, and invertible measure preserving transformations $T_1, \ldots, T_\ell \colon X \to X$ that commute, that is, satisfy $T_iT_j = T_jT_i$ for all $i, j \in \{1, \ldots, \ell\}$. In all applications we are interested in we can further assume that the measure space is *Lebesgue*, that is, X can be given the structure of a Polish space (i.e. metrizable, separable, complete) such that $\mathcal{X}$ is its Borel σ-algebra. We call $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ a *measure preserving system*, or simply, a *system*. We are also given bounded measurable functions $f_1, \ldots, f_\ell \colon X \to \mathbb{C}$, and sequences $a_1, \ldots, a_\ell \colon \mathbb{N} \to \mathbb{Z}$.

The first family of problems we are interested in concerns the study of the limiting behavior of the so called multiple ergodic averages

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a_1(n)} f_1 \cdots T_\ell^{a_\ell(n)} f_\ell \tag{1}$$

where $Tf := f \circ T$ and $T^k := T \circ \cdots \circ T$. One would like to know whether these averages converge as $N \to \infty$ (in $L^2(\mu)$ or pointwise), find some structured factors that control their limiting behavior (called characteristic factors), and if possible, find a formula, or a usable way to extract information, for the limiting function. When $\ell = 1$, such problems have been studied extensively and in several cases solved even for pointwise convergence (see the survey paper [265] for a variety of related results). Our main concern here is to study the averages (1) when $\ell \geq 2$. To get manageable problems, one typically restricts the class of eligible sequences and usually assumes that they are polynomial sequences, sequences arising from smooth functions, sequences related to the prime numbers, or random sequences of integers. We typically also assume that the transformations commute, or, to get started, that they are all equal. On the other hand, because of the nature of the implications in combinatorics that we are interested in, it is not desirable to impose assumptions on the structure of each individual measure preserving transformation. In several cases, the steps taken in order to attack such problems include (*i*) elementary uniformity estimates, (*ii*) ergodic structural results, and (*iii*) analysis of systems with special structure (often algebraic). We discuss these steps in more detail in subsequent sections.

The second family of problems concerns the study of expressions of the form

$$\mu(A \cap T_1^{a_1(n)} A \cap \cdots \cap T_\ell^{a_\ell(n)} A) \tag{2}$$

where $A \in \mathcal{X}$ has positive measure. One wants to know whether such expressions are positive for some $n \in \mathbb{N}$, or even better, for lots of $n \in \mathbb{N}$ (for instance on the average), and if possible, get some explicit lower bound that depends only on the measure of the set A and on ℓ (in some cases this is going to be of the form $(\mu(A))^{\ell+1}$). Such multiple recurrence results are often obtained by carrying out an in depth analysis of the limiting behavior of the averages (1). Usually they are not hard if an explicit formula of the limiting function is known, but they can be very tricky in the absence of such a formula, even when we work with very special systems of algebraic nature.

Concerning the third family of problems, and restricting ourselves to subsets of $\mathbb{Z}$, one is interested to know, for example, whether every set of integers with positive upper density contains patterns of the form

$$m,\ m + a_1(n), \dots,\ m + a_\ell(n)$$

for some (or lots of) $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Using a correspondence principle of H. Furstenberg (see Section 2.1), one can translate such statements to multiple recurrence statements in ergodic theory; an equivalent problem is then to show that the expressions (2) are positive for some $n \in \mathbb{N}$ when all the measure preserving transformations $T_1, \dots, T_\ell$ are equal. Similar questions can be asked on higher dimensions and concern patterns that can be found on subsets of $\mathbb{Z}^d$ with positive upper density. Such questions correspond to multiple recurrence statements when the transformations $T_1, \dots, T_\ell$ commute. This approach was initiated forty years ago by H. Furstenberg who gave an alternative proof of Szemerédi's theorem using ergodic theory in his foundational article [151]. Subsequently, H. Furstenberg and Y. Katznelson gave the first proof of the multidimensional Szemerédi theorem [158] and the density Hales-Jewett theorem [160], and V. Bergelson and A. Leibman proved a polynomial extension of Szemerédi's theorem [52] (currently no proof that avoids ergodic theory is known for this result). And the story does not end there, in the last two decades new powerful tools in ergodic theory were developed and used, and are currently being used, in order to prove several other deep results in density Ramsey theory. The reader will find a variety of such applications in subsequent sections and the extended bibliography section. Several additional applications can be found in the survey articles [37, 40, 41] and the references therein.

An additional motivation for studying such problems has to do with potential implications in number theory, in particular, there is a connection to problems of finding patterns in the set of the prime numbers. Knowing that every set of integers with positive upper density contains patterns of a certain sort could be an important first step towards proving an analogous result for the set of primes. This idea originates from work of B. Green and T. Tao [167], where it was used to show that the primes contain arbitrarily long arithmetic progressions. It was also subsequently used by T. Tao and T. Ziegler [284] to show that the primes contain arbitrarily long polynomial progressions.

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2. Some useful tools and observations

2.1. **Furstenberg correspondence principle.** In order to reformulate statements in combinatorics as multiple recurrence statements in ergodic theory, we use the following correspondence principle of Furstenberg [151, 152] (the formulation given is from [36]):

Furstenberg Correspondence Principle. *Let $d, \ell \in \mathbb{N}$, $E \subset \mathbb{Z}^d$ be a set of integers, and $\mathbf{v}_1, \dots, \mathbf{v}_\ell \in \mathbb{Z}^d$. Then there exist a system $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ and a set $A \in \mathcal{X}$, with $\mu(A) = \bar{d}(E)$, and such that*

$$\bar{d}(E \cap (E - n_1\mathbf{v}_1) \cap \cdots \cap (E - n_\ell\mathbf{v}_\ell)) \geq \mu(A \cap T_1^{n_1} A \cap \cdots \cap T^{n_\ell} A), \tag{3}$$

for every $n_1, \dots, n_\ell \in \mathbb{Z}$. Furthermore, if $\mathbf{v}_1 = \cdots = \mathbf{v}_\ell$, one can take $T_1 = \cdots = T_\ell$.

Let $a_1, \dots, a_\ell \colon \mathbb{N} \to \mathbb{Z}$ be a collection of sequences. Using the previous principle, we deduce that in order to show that every set of integers with positive upper density contains patterns of the form $m, m + a_1(n), \ldots, m + a_\ell(n)$ for some $m, n \in \mathbb{N}$, it suffices to show that for every system $(X, \mathcal{X}, \mu, T)$ and set $A \in \mathcal{X}$ with $\mu(A) > 0$ one has

$$\mu(A \cap T^{a_1(n)} A \cap \cdots \cap T_\ell^{a_\ell(n)} A) > 0$$

for some $n \in \mathbb{N}$. It is often more convenient to verify that a functional variant of this multiple recurrence property holds on average, namely, that for every $f \in L^\infty(\mu)$ that is non-negative and not identically zero one has

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int f \cdot T^{a_1(n)} f \cdot \ldots \cdot T^{a_\ell(n)} f \, d\mu > 0. \tag{4}$$

Thus, one is reduced to study the limiting behavior (in $L^2(\mu)$) of the averages

$$\frac{1}{N} \sum_{n=1}^{N} T^{a_1(n)} f \cdots T^{a_\ell(n)} f$$

in a depth that is sufficient to verify the positiveness property (4).

2.2. **Characteristic factors.** A notion that underlies the study of the limiting behavior of several multiple ergodic averages is that of the *characteristic factors*. Implicit use of this notion was already made by H. Furstenberg in [151], but the term "characteristic factor" was coined in a paper of H. Furstenberg and B. Weiss [163].

Given a system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$, we say that the sub-$\sigma$-algebras $\mathcal{X}_1, \ldots, \mathcal{X}_\ell$ of $\mathcal{X}$ are *characteristic factors* for the averages

$$A_N(f_1, \ldots, f_\ell) := \frac{1}{N} \sum_{n=1}^{N} T_1^{a_1(n)} f_1 \cdots T_\ell^{a_\ell(n)} f_\ell \tag{5}$$

if the following two conditions hold:

- $\mathcal{X}_i$ is T_i-invariant for $i = 1, \ldots, \ell$,
- we have $A_N(f_1, \ldots, f_\ell) - A_N(\tilde{f}_1, \ldots, \tilde{f}_\ell) \to^{L^2(\mu)} 0$, where $\tilde{f}_i := \mathbb{E}(f_i|\mathcal{X}_i)$ for $i = 1, \ldots, \ell$, for all $f_1, \ldots, f_\ell \in L^\infty(\mu)$.

Equivalently, the second condition states that if $\mathbb{E}(f_i|\mathcal{X}_i) = 0$ for some $i \in \{1, \ldots, \ell\}$, then $A_N(f_1, \ldots, f_\ell) \to^{L^2(\mu)} 0$. If in addition one has $\mathcal{X}_1 = \cdots = \mathcal{X}_\ell$, then we call this common sub-σ-algebra a *characteristic factor* for the averages (5). When analyzing the limiting behavior of multiple ergodic averages, an important intermediate goal is to produce characteristic factors that are as simple as possible, and typically simple for our purposes means that the corresponding factor systems have very special (often algebraic) structure. For instance, it can be shown (see [151]) that for ergodic systems $(X, \mathcal{X}, \mu, T)$, the Kronecker factor $\mathcal{K}_T$, which is induced by the eigenfunctions of T, is a characteristic factor for the averages

$$\frac{1}{N} \sum_{n=1}^{N} T^{an} f_1 \cdot T^{bn} f_2,$$

whenever a, b are distinct integers. It is also well known that the system $(X, \mathcal{K}_T, \mu, T)$ is measure theoretically isomorphic to an ergodic rotation on a compact Abelian group with the Haar measure.

2.3. **Uniformity seminorms.** One way to find simple characteristic factors for the averages (5) is to show that their $L^2(\mu)$-norm is controlled by the so called *Gowers-Host-Kra uniformity seminorms*. Similar seminorms were first introduced in combinatorics by T. Gowers [165] and their ergodic counterpart (which is more relevant for our study) was introduced by B. Host and B. Kra [185]. For a given ergodic system $(X, \mathcal{X}, \mu, T)$ and function $f \in L^\infty(\mu)$, they are defined as follows:

$$\|\!|f|\!\|_1 := \Big| \int f \, d\mu \Big| \; ;$$

$$\|\!|f|\!\|_{k+1}^{2^{k+1}} := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \|\!|\bar{f} \cdot T^n f|\!\|_k^{2^k}.$$

It is shown in [185] that for every $k \in \mathbb{N}$ the above limit exists, and $\|\!|\cdot|\!\|_k$, thus defined, is a seminorm on $L^\infty(\mu)$. For non-ergodic systems the seminorms can be similarly defined, the only difference is that $\|\!|\cdot|\!\|_1$ is defined by $\|\!|f|\!\|_1 := \big\|\int f \, d\mu_x\big\|_{L^2(\mu)}$, where $\mu = \int \mu_x \, d\mu(x)$ is the ergodic decomposition of the measure μ with respect to T. If further clarification is needed, we write $\|\!|\cdot|\!\|_{k,\mu}$, or $\|\!|\cdot|\!\|_{k,T}$. We remark that if a measure preserving system is *weak mixing* (meaning, the product system $(X \times X, \mu \times \mu, T \times T)$ is ergodic), then $\|\!|f|\!\|_k = |\int f \, d\mu|$ for every $k \in \mathbb{N}$.

2.4. **Nilsystems and nilsequences.** The analytic part of the modern theory of characteristic factors requires familiarity with variants of the several classical Fourier analysis results that apply to functions defined on general nilmanifolds. In this non-Abelian setup the role of rotations on the circle play the nilsystems and the role of complex exponential sequences play the nilsequences.

2.4.1. *Nilsystems.* A *k-step nilmanifold* is a homogeneous space $X = G/\Gamma$, where G is a k-step nilpotent Lie group and Γ is a discrete cocompact subgroup of G. A *k-step nilsystem* is a system of the form $(X, \mathcal{G}/\Gamma, m_X, T_a)$, where $X = G/\Gamma$ is a k-step nilmanifold, $a \in G$, $T_a\colon X \to X$ is defined by $T_a(g\Gamma) := (ag)\Gamma$, $g \in G$, m_X is the normalized Haar measure on X, and $\mathcal{G}/\Gamma$ is the completion of the Borel σ-algebra of G/Γ.

Examples of nilsystems include all rotations on compact Abelian Lie groups, and more generally, every nilpotent affine transformation on a compact Abelian Lie group is measure theoretically isomorphic to a nilsystem (see Example 1). But these examples do not cover all the possible nilsystems (see Example 2).

Example 1. On the space $G = \mathbb{Z} \times \mathbb{R}^2$, define multiplication as follows:
if $g_1 = (m_1, x_1, y_1)$ and $g_2 = (m_2, x_2, y_2)$, let

$$g_1 \cdot g_2 = (m_1 + m_2, x_1 + x_2, y_1 + y_2 + m_1 x_2).$$

Then G is a 2-step nilpotent group and the discrete subgroup $\Gamma = \mathbb{Z}^3$ is cocompact. If $a = (m, \alpha, \beta)$. It turns out that the nilrotation $x \mapsto ax$ defined on $X = G/\Gamma$ is measure theoretically isomorphic to the unipotent affine transformation $S\colon \mathbb{T}^2 \to \mathbb{T}^2$ (with the Haar measure $m_{\mathbb{T}^2}$) defined by

$$S(x, y) = (x + \alpha, y + mx + \beta), \quad x, y \in \mathbb{T}.$$

Example 2. On the space $G = \mathbb{R}^3$, define multiplication as follows:
if $g_1 = (x_1, y_1, z_1)$ and $g_2 = (x_2, y_2, z_2)$, let

$$g_1 \cdot g_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2).$$

Then G is a 2-step nilpotent group and the discrete subgroup $\Gamma = \mathbb{Z}^3$ is cocompact. Let $a = (\alpha, \beta, 0)$, where $\alpha, \beta \in [0,1)$ are linearly independent. It turns out that the nilrotation $x \mapsto ax$ defined on $X = G/\Gamma$ is measure theoretically isomorphic to the skew product transformation $S\colon \mathbb{T}^3 \to \mathbb{T}^3$ that has the form

$$S(x,y) = (x+\alpha, y+\beta, z+f(x,y)), \quad x,y \in \mathbb{T},$$

where $f\colon \mathbb{T}^2 \to \mathbb{T}$ is defined by

$$f(x,y) = (x+\alpha)[y+\beta] - x[y] - \alpha y, \quad x,y \in \mathbb{T}.$$

It can be shown that the system induced by S is *not* measure theoretically isomorphic to a unipotent affine transformation on some finite dimensional torus.

2.4.2. *Nilsequences.* Following [47] we define a *basic k-step nilsequence* to be a sequence of the form $(F(b^n x))$ where $X = G/\Gamma$ is a k-step nilmanifold, $b \in G$, $x \in X$, and $F \in C(X)$. If in the previous definition we allow F to be Riemann-integrable we call $(F(b^n x))$ a *basic generalized k-step nilsequence.* A *k-step nilsequence* is a uniform limit of *basic k-step nilsequences* and similarly we define *generalized k-step nilsequences.* We remark that in [47] and subsequent work, the notion of a generalized nilsequence is not used, but for the purpose of formulating some problems later in this article it seems safer to extend the class of eligible sequences.

More generally, if $b_1, \ldots, b_\ell \in G$ commute, $x \in X$, and $F \in C(X)$ (or F is Riemann integrable on X), we call the sequence $(F(b_1^{n_1} \cdots b_\ell^{n_\ell} x))$ a *basic (generalized) k-step nilsequence in ℓ-variables.* A *k-step (generalized) nilsequence in ℓ-variables*, is a uniform limit of basic k-step (generalized) nilsequences in ℓ-variables.

As is easily verified, the collection of k-step (generalized) nilsequences in ℓ-variables, with the topology of uniform convergence, forms a closed algebra.

One can verify that for $\alpha, \beta \in \mathbb{R}$, the sequence $(e^{i(n^2\alpha + n\beta)})$ is a basic 2-step nilsequence defined on the nilmanifold given in Example 1 above. Also, for $\alpha, \beta \in \mathbb{R}$, the sequence $(e^{i[n\alpha]n\beta})$ is a basic generalized 2-step nilsequence defined on the nilmanifold given by the direct product of the nilmanifolds in Examples 1 and 2 above. It can be shown (see [55]) that if α and β are rationally independent, then the sequence $(e^{i[n\alpha]n\beta})$ is asymptotically orthogonal to all sequences of the form $(e^{i(n^2\gamma + n\delta)})$, meaning, it satisfies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{i[n\alpha]n\beta} \cdot e^{i(n^2\gamma + n\delta)} = 0$$

for every $\gamma, \delta \in \mathbb{R}$.

2.5. **The Host-Kra factors and their structure.** It is shown in [185] that the seminorms $\|\!|\cdot|\!\|_k$ induce T-invariant sub-σ-algebras $\mathcal{Z}_{k-1}$ that satisfy

$$\text{(6)} \qquad \textit{for } \ f \in L^\infty(\mu), \ \ \mathbb{E}(f|\mathcal{Z}_{k-1}) = 0 \quad \textit{if and only if} \quad \|\!|f|\!\|_k = 0.$$

As a consequence, if for some $k_1, \ldots, k_\ell \in \mathbb{N}$ we have an estimate of the form

$$\text{(7)} \qquad \limsup_{N\to\infty} \|A_N(f_1, \ldots, f_\ell)\|_{L^2(\mu)} \le C \min_{i=1,\ldots,\ell} \|\!|f_i|\!\|_{k_i, T_i}$$

for some constant $C = C_{f_1,\ldots,f_\ell}$, then the factors $\mathcal{Z}_{k_1-1,T_1}, \ldots, \mathcal{Z}_{k_\ell-1,T_\ell}$ are characteristic for mean convergence of the averages (5). Under such circumstances, one gets characteristic factors with the sought-after algebraic structure. This is a consequence of an important result of B. Host and B. Kra [185] which states that for ergodic systems the factor system $(X, \mathcal{Z}_k, \mu, T)$

is an inverse limit of k-step nilsystems. Note that for ergodic systems the factor $\mathcal{Z}_1$ coincides with the Kronecker factor.

Depending on the problem, it may be more useful to reinterpret the structure theorem of Host and Kra as a decomposition result: For every ergodic system $(X, \mathcal{X}, \mu, T)$ and function $f \in L^\infty(\mu)$, for every $k \in \mathbb{N}$ and $\varepsilon > 0$, there exist measurable functions f_s (the structured component), f_u (the uniform component), f_e (the error term), with $L^\infty(\mu)$ norm at most $2\,\|f\|_{L^\infty(\mu)}$, such that

- $f = f_s + f_u + f_e$;
- $|||f_u|||_{k+1} = 0$; $\|f_e\|_{L^1(\mu)} \leq \varepsilon$; and
- $(f_s(T^n x))_{n\in\mathbb{N}}$ is a basic k-step nilsequence for μ-almost every $x \in X$.

Such a decomposition also holds for non-ergodic systems (see Proposition 3.1 in [97]).

Combining the hypothetical seminorm estimates (7) with the aforementioned structure theorem (or the decomposition result), the problem of analyzing the limiting behavior of the averages (5) is reduced to a new problem which typically amounts to proving certain equidistribution properties of sequences on nilmanifolds. A lot of tools for handling such equidistribution problems have been developed in recent years, thus making such a reduction very much worthwhile. Some examples of useful equidistribution results on nilmanifolds can be found in [17, 130, 131, 141, 169, 170, 171, 185, 227, 228, 230, 231, 233, 241, 242, 299].

2.6. **A general strategy.** When one is aiming to prove a multiple recurrence or mean convergence property in ergodic theory by analyzing the limiting behavior of the multiple ergodic averages (5), very often one goes through the following three steps:

- Produce seminorm estimates like those in (7).
- Use a structure theorem or a decomposition result to reduce matters to the analysis of the averages (5) for nilsystems.
- Use qualitative or quantitative equidistribution results on nilmanifolds to complete the analysis.

This approach is also implicit in the foundational paper of Furstenberg [151] and subsequent works [52, 158, 161]. An important difference is that in these works, in place of nilsystems, one uses the much larger class of distal systems, but for several recent applications (for example for the precise evaluation of limits) the class of distal systems is too complicated to be able to deal with directly.

The reader can find several examples demonstrating how this general strategy is used in order to prove multiple recurrence and convergence results, as well as related applications in combinatorics, in the following articles: [9, 47, 59, 60, 92, 96, 97, 100, 129, 130, 132, 133, 140, 141, 142, 143, 145, 149, 163, 173, 184, 185, 186, 189, 190, 200, 229, 243, 261, 268, 296, 300, 301]. Depending on the problem, the difficulty of each step varies; typically the first step is elementary and is carried out by successive uses of the Cauchy-Schwarz inequality and an estimate of van der Corput (or Hilbert space variants of it), the second step involves the use of (a modification of) the structure theorem of B. Host and B. Kra, and the third step is a combination of algebraic and analytic techniques. In the next subsections we explain some techniques that help us execute the first and the third steps.

2.7. **The polynomial exhaustion technique.** We briefly explain a technique that is often used in order to produce seminorm estimates of the type (7). It is based on an induction scheme (often called PET induction) introduced by V. Bergelson in [35]. Let $\mathcal{F} := \{a_1, \dots, a_\ell\}$ be a family of real valued sequences, and suppose that one wishes to establish seminorm estimates of the form

$$\limsup_{N\to\infty} \|A_N(f_1,\ldots,f_\ell)\|_{L^2(\mu)} \leq C \min_{i=1,\ldots,\ell} \|\!|f_i|\!\|_{k_i}, \tag{8}$$

for some constant $C = C_{f_1,\ldots,f_\ell}$, where

$$A_N(f_1,\ldots,f_\ell) := \frac{1}{N}\sum_{n=1}^{N} T^{[a_1(n)]}f_1 \cdots T^{[a_\ell(n)]}f_\ell.$$

Variants of this technique could also be used to get similar estimates for some multiple ergodic averages involving commuting transformations, but this is usually a much more difficult task.

The main idea is to use the following Hilbert space variant of van der Corput's fundamental estimate

Theorem. *Let $v_1,\ldots,v_N$ be elements of an inner product space of norm less than 1. Then*

$$\Big\|\frac{1}{N}\sum_{n=1}^{N} v_n\Big\|^2 \leq 4\Big(\frac{1}{R}\sum_{r=1}^{R}(1-rR^{-1})\Re\big(\frac{1}{N}\sum_{n=1}^{N}\langle v_{n+r}, v_n\rangle\big) + R^{-1} + RN^{-1}\Big)$$

for every integer R between 1 and N where $\Re(z)$ denotes the real part of z.

Using this estimate, in various cases we can bound the left hand side in (8) by an expression that involves families of sequences of smaller "complexity". The goal is after a finite number of iterations to get families of sequences that are simple enough to handle directly. The details depend on the family of sequences at hand, but typically, after the first iteration, we get an upper bound by an average over $r \in \mathbb{N}$ of the $L^2(\mu)$-norm of multiple ergodic averages with iterates taken from the family of sequences (upon taking integer parts)

$$\mathcal{F}_{a,r} := \{\text{sequences of the form } a_i(n) - a(n),\ a_i(n+r) - a(n), i = 1,\ldots,\ell\} \tag{9}$$

where $a \in \mathcal{F}$ is fixed (so in particular independent of $r \in \mathbb{N}$) and chosen appropriately. If a choice of $a \in \mathcal{F}$ can be made so that the family $\mathcal{F}_{a,r}$ has smaller "complexity" than $\mathcal{F}$ except possibly for a finite number of $r \in \mathbb{N}$, then, practically, this means that after applying van der Corput's estimate a finite number of times, we will be able to bound the left hand side in (8) by a much simpler expression for which we can prove the desired seminorm estimates directly.

This strategy has been employed successfully in several instances and produced seminorm estimates of the form (8) for linear sequences [185], polynomial sequences [186, 212, 229], variable polynomials of fixed degree [149], and some sequences arising from smooth functions of polynomial growth [45, 132, 133]. A common desirable feature that these sequences share is that after taking successive differences (meaning iterating the operation $a(n) \mapsto a(n+r) - a(n)$) a finite number of times, we arrive to sequences that are either constant or asymptotically constant. This feature is not shared by several other sequences worth studying, for example, random sequences of integers, the sequence of primes, and the sequences $([n^{\log n}])$ or $([n\sin n])$. In these cases different approaches are needed in order to produce seminorm estimates for ergodic averages with iterates given by such sequences.

2.8. **Equidistribution of polynomial sequences on nilmanifolds.** Let $X = G/\Gamma$ be a nilmanifold. We say that a sequence $g\colon \mathbb{N} \to X$ is equidistributed in X if for every $F \in C(X)$

one has

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F(g(n))=\int F\ dm_X$$

where m_X denotes the Haar measure on X. A similar definition applies if X is a union of finitely many nilmanifolds.

Let $b_1,\dots,b_\ell\in G$, $x\in X$, and $a_1,\dots,a_\ell\colon\mathbb{N}\to\mathbb{Z}$ be sequences. In several of the applications we have in mind, one is at some point called to deal with equidistribution properties of sequences of the form $(g(n)x)$, defined by $g(n):=b_1^{a_1(n)}\cdots b_\ell^{a_\ell(n)}$, $n\in\mathbb{N}$. Such examples cover as special cases sequences of the form $\big((c_1^{a_1(n)}x_1,\dots,c_\ell^{a_\ell(n)}x_\ell)\big)$, defined on the product of the nilmanifolds $X_1,\dots,X_\ell$. To see this, let $X:=X_1\times\cdots\times X_\ell$, $x:=(x_1,\dots,x_\ell)$, $b_1:=(c_1,e_2,\dots,e_l)$, $\dots$, $b_\ell:=(e_1,\dots,e_{\ell-1},c_\ell)$ where e_i denotes the identity element of the group G_i for $i=1,\dots,\ell$.

Such problems are typically much easier to handle when $X=\mathbb{T}^d$, since in this case one can utilize the following equidistribution result of Weyl: If Y is a sub-torus of $\mathbb{T}^d$, then a sequence $g\colon\mathbb{N}\to Y$ is equidistributed on Y if and only if for every non-trivial character $\chi\colon Y\to\mathbb{C}$ one has

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\chi(g(n))=0.$$

This criterion allows us to verify equidistribution properties of arbitrary sequences on Y by estimating certain exponential sums. Unfortunately, such a convenient reduction is not available for non-Abelian nilmanifolds, and checking equidistribution in this broader setup can be very challenging even for simple sequences. The situation is much better understood when all the sequences $a_1,\dots,a_\ell\colon\mathbb{N}\to\mathbb{Z}$ are given by integer polynomials; in this case we call $(g(n)x)$ *a polynomial sequence on* X. Next we state some key results used to prove equidistribution properties of such polynomial sequences on nilmanifolds.

Let $X=G/\Gamma$ be a connected nilmanifold. By G_0 we denote the connected component of the identity element in G, and for technical reasons we assume that G_0 is simply connected and that $G=G_0\Gamma$. When X is connected, for the problems we are interested in, we can always arrange matters so that G_0 has these additional properties. We let $Z:=G/([G_0,G_0]\Gamma)$ and $\pi\colon X\to Z$ be the natural projection. It is very desirable to work with the nilmanifold Z instead of X because Z has much simpler structure. Indeed, if G is connected, then Z is a connected compact Abelian Lie group, hence, a torus (meaning $\mathbb{T}^d$ for some $d\in\mathbb{N}$), and as a consequence every nilrotation in Z is (isomorphic to) a rotation on some torus. In general, the nilmanifold Z may be more complicated, but it is the case that every nilrotation in Z is (isomorphic to) a *unipotent affine transformation* on some torus (see Proposition 3.1 in [141]), meaning, it has the form $T(x)=S(x)+b$, where $S\colon\mathbb{T}^d\to\mathbb{T}^d$ is a homomorphism which is unipotent (that is, there exists $k\in\mathbb{N}$ so that $(S-\mathrm{Id})^k=0$) and $b\in\mathbb{T}^d$. Iterates of such transformations can be computed explicitly and so one is much more comfortable to be dealing with equidistribution problems that involve unipotent affine transformations on some torus than with general nilsystems.

2.8.1. *Qualitative equidistribution.* The following qualitative equidistribution results were established by A. Leibman in [227]:

- A polynomial sequence $(g(n)x)$ is always equidistributed in a finite union of sub-nilmanifolds of X.
- A polynomial sequence $(g(n)x)$ is equidistributed in X if and only if the sequence $(g(n)\pi(x))$ is equidistributed in Z.

The second statement gives an efficient way for checking equidistribution of polynomial sequences on nilmanifolds. We illustrate this with a simple example. Suppose that $b \in G$ is an ergodic nilrotation (meaning the transformation $x \mapsto bx$ is ergodic) and we want to show that the polynomial sequence $(b^{n^2}x)$ is equidistributed in X for every $x \in X$. In the case where G is connected, the nilmanifold Z is a torus, therefore, according to the previous criterion, it suffices to show that if β is an ergodic element of $\mathbb{T}^d$ (this is the case if the coordinates of β are rationally independent), then for every $x \in X$ the sequence $(x + n^2\beta)$ is equidistributed in $\mathbb{T}^d$. This is a well known fact, and can be easily verified using Weyl's equidistribution theorem and van der Corput's estimate. If G is not necessarily connected, one needs to show that if $S\colon \mathbb{T}^d \to \mathbb{T}^d$ is an ergodic unipotent affine transformation, then the sequence $S^{n^2}x$ is equidistributed for every $x \in \mathbb{T}^d$. Although this is somewhat harder to establish, it follows again by Weyl's equidistribution theorem modulo some straightforward computations.

2.8.2. *Quantitative equidistribution.* Suppose that one seeks to study equidistribution properties of the sequence $(b^{[n^{3/2}]}x)$, or tries to prove uniform convergence of the sequence $\frac{1}{N}\sum_{n=1}^{N} F(b^{n^2}x)$ to the integral of F, where $b \in G$ is an ergodic element and $F \in C(X)$. In such cases (and several others) one needs to use quantitative variants of the previous qualitative equidistribution results. Such results were proved by B. Green and T. Tao [169, 170]. In order to state them we have to introduce some notation.

For simplicity we assume that we work on a nilmanifold $X = G/\Gamma$ with G connected. As before, we let $Z = G/([G,G]\Gamma) \cong \mathbb{T}^l$ and $\pi\colon X \to Z$ be the natural projection map. A *horizontal character* is a character $\chi\colon Z \to \mathbb{C}$. We have that $\chi(t) = e^{2\pi i\kappa\cdot t}$, $t \in \mathbb{T}$, for some $\kappa \in \mathbb{Z}^l$, where $\cdot$ denotes the inner product operation. We let $\|\chi\| = |\kappa|$. Suppose that $p\colon \mathbb{Z} \to \mathbb{R}$ has the form $p(t) = \sum_{j=0}^{d} n^j\alpha_j$ where $\alpha_j \in \mathbb{R}$ for $j = 0, \ldots, d$. We define

$$\left\|e^{2\pi i p(n)}\right\|_{C^\infty[N]} = \max_{1\leq j\leq d}(N^j \|\alpha_j\|) \tag{10}$$

where $\|x\| = d(x, \mathbb{Z})$.

Given $N \in \mathbb{N}$, a finite sequence $(g(n)\Gamma)_{1\leq n\leq N}$ is said to be δ-*equidistributed on* X, if

$$\Big|\frac{1}{N}\sum_{n=1}^{N} F(g(n)\Gamma) - \int\limits_X F\ dm_X\Big| \leq \delta\, \|F\|_{\text{Lip}(X)}$$

for every Lipschitz function $F\colon X \to \mathbb{C}$, where

$$\|F\|_{\text{Lip}(X)} = \|F\|_\infty + \sup_{x,y\in X, x\neq y} \frac{|F(x) - F(y)|}{d_X(x,y)}$$

and d_X is the Riemannian metric on X.

The quantitative equidistribution result in [169, Theorem 2.9] states that for every nilmanifold X, $d \in \mathbb{N}$, and $\delta > 0$, there exist a real number $M = M_{X,d,\delta}$ with the following property: For every $N \in \mathbb{N}$, if $g\colon \mathbb{Z} \to G$ is a polynomial sequence of degree at most d such that the finite sequence $(g(n)\Gamma)_{1\leq n\leq N}$ is not δ-equidistributed, then for some non-trivial horizontal character χ with $\|\chi\| \leq M$ we have

$$\|\chi(g(n))\|_{C^\infty[N]} \leq M, \tag{11}$$

where χ is thought of as a character of the horizontal torus $Z = \mathbb{T}^l$ and $(g(n))$ as a polynomial sequence on $\mathbb{T}^l$.

In the special case where $X = \mathbb{T}$ (with the standard metric) and the polynomial sequence on $\mathbb{T}$ is given by $p(n) = n^d\alpha + q(n) \pmod 1$ where $d \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and $q \in \mathbb{Z}[x]$ with $\deg(q) \leq d-1$, the previous result implies that: There exists $M = M(d,\delta) > 0$ such that for every $N \in \mathbb{N}$ and δ small enough, if the finite sequence $\big(n^d\alpha + q(n)\big)_{1\leq n\leq N}$ is not δ-equidistributed in $\mathbb{T}$, then $\|k\alpha\| \leq M/N^d$ for some non-zero $k \in \mathbb{Z}$ with $|k| \leq M$.

2.9. **Exploiting randomness.** In some cases we understand the limiting behavior of certain multiple ergodic averages and we would like to exploit weighted variants of those averages, or averages along subsequences of the iterates involved. If the weight (or the subsequence) enjoys some randomness features, it is often preferable to study the weighted averages indirectly, by comparing them with the unweighted ones and showing that the difference converges to 0. To give an example, suppose that R is the set of positive integers that have an even number of prime factors counted with multiplicity; this set is known to have density 1/2 and is expected to be distributed randomly. We are interested in proving mean convergence for the averages

$$\frac{1}{N}\sum_{n=1}^{N} w_n\, T^n f \cdot S^n g$$

where $w_n := \mathbf{1}_R(n)$, $n \in \mathbb{N}$. Since mean convergence of the unweighted averages is known, it suffices to show that the difference

$$\frac{1}{N}\sum_{n=1}^{N}(w_n - 1/2)\, T^n f \cdot S^n g \tag{12}$$

converges to 0 in $L^2(\mu)$. This can be done using structural decomposition results for multiple correlation sequences (see [137]), but the most economical way (given existing knowledge) is to use the van der Corput Lemma twice in order to bound the $L^2(\mu)$ norm of the averages (12) by the $U^3(\mathbb{Z}_N)$ Gowers uniformity norm of the sequence $(w_n - 1/2)$ (note that $w_n - 1/2 = \lambda(n)/2$ where λ is the Liouville function). Since these norms are known to converge to 0 as $N \to \infty$ (see [166, 171]), the averages (12) converge to 0 in $L^2(\mu)$, and we are done. In a similar way, we can treat weights of the form $(\mathbf{1}_R(n))$ where $R = \{n \in \mathbb{N}\colon \big\|n^k\alpha\big\| \leq 1/2\}$ for $k \geq 3$ and α irrational (see [144]), or $R = \{n \in \mathbb{N}\colon \phi(n) = 1\}$ where ϕ is an aperiodic multiplicative function with values in $\{\pm 1\}$ (see [137]). Moreover, we can cover weights of the form $(\Lambda(n))$ where Λ is the von Mangoldt function (see [138]), or (unbounded) weights supported on random subsets of the integers of zero-density (see [146]).

2.10. **The approaches of Tao and Walsh.** Mean convergence for averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_\ell^{p_\ell(n)} f_\ell,$$

was established for linear polynomials $p_1, \dots, p_\ell$ by T. Tao [280] and for general polynomials by M. Walsh [290]. Although the technical details in these two arguments are different, the general strategy is rather similar. Both approaches proceed by reformulating the mean convergence result as a finitary quantitative convergence problem for averages of products of general bounded sequences. After decomposing one of these sequences into a sum of a structured component and a component that contributes negligibly in the averaging, one is left with analyzing the contribution of the structured component. This component is defined using an averaging operation and its precise structure is not further analyzed; this contrasts previous ergodic convergence arguments where the bulk of the proof involves a detailed analysis of the structured component.

What is important, is that when the original sequence is replaced by its structured component, the corresponding averages reduce to "lower complexity" ones. The proof then concludes with an induction on the complexity of the averages involved (which is similar but not identical to the PET induction of Section 2.7), and eventually reduces matters to the trivial case where one deals with constant polynomials.

To give an example, suppose that one seeks to prove convergence for averages of the form $\frac{1}{N}\sum_{n=1}^{N} T^n f\, S^{n^2} g$, where the transformation T and S commute. After a few applications of the previous method, one is reduced to studying averages of the form $\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 \cdots T_r^n f_r$, for some $r \in \mathbb{N}$, where the transformations $T_1, \ldots, T_r$ commute. After each subsequent application of the previous method, the number of transformations is reduced by one, hence, after r additional steps one is reduced to a trivial convergence result. Let us remark though, that the induction hypothesis demands certain uniformity conditions on the various parameters involved, and these parameters unfortunately live strictly on the finitary universe. This makes it impossible to carry out this argument entirely on the infinite world of ergodic theory. On the other hand, see [25] for an ergodic proof of mean convergence inspired by the argument of Walsh.

A key advantage of the methods of Tao and Walsh is that they use "weak" decomposition results that do not require detailed information on the structured component of the sequences involved. This is the main reason for the effectiveness and brevity of their arguments and lead to significantly simpler proofs even for previously known mean convergence results that relied on complicated structure theorems. The price to pay is that these methods do not give explicit information for the limit function and it is also not clear if they can be adjusted in order to prove mean convergence results even in the simplest cases where the iterates involved are not given by polynomial sequences. For such problems, it seems that in several cases one still has to rely on the more elaborate theory of characteristic factors described in Sections 2.2-2.8.

2.11. **Pleasant and magic extensions.** Motivated by work of T. Tao [280], several people, including H. Towsner [288], T. Austin [18], and B. Host [181], introduced new tools that help us handle multiple ergodic averages with commuting transformations. In particular, a key conceptual breakthrough that first appeared in [18], is that in some instances by working with suitable extensions of a family of systems (called "pleasant extensions" in [18] and "magic extensions" in [181]), characteristic factors of the corresponding multiple ergodic averages may be chosen to have particularly simple structure, a structure that is not visible when one works with the original system (the idea of passing to an extension in order to simplify some convergence problems already appears implicitly in [163]). This is a rather counterintuitive statement since characteristic factors of extensions are extensions of characteristic factors of the original systems. So in order to clarify matters, we explain a simple instance where such an approach works.

Suppose that one wants to prove mean convergence for the averages

$$A_N(T,S,f_i) := \frac{1}{N^2}\sum_{1\leq m,n\leq N} T^m f_1 \cdot S^n f_2 \cdot T^m S^n f_3$$

where $(X, \mathcal{X}, \mu, T, S)$ is a system and $f_1, f_2, f_3 \in L^\infty(\mu)$. Although an estimate that relates the $L^2(\mu)$-norm of these averages with the Gowers-Host-Kra seminorms of the individual functions with respect to either T or S is not feasible, the following estimate is valid

$$\limsup_{N\to\infty} \|A_N(T,S,f_i)\|_{L^2(\mu)} \leq C \min_{i=1,2,3} \|\!|f_i|\!\|_{T,S,\mu},$$

for some constant $C = C_{f_1,f_2,f_3}$, where

$$|||f|||^4_{T,S,\mu} := \lim_{M\to\infty} \frac{1}{M}\sum_{m=1}^{M} \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int f\cdot T^m\bar{f}\cdot S^n\bar{f}\cdot T^mS^nf\ d\mu$$

(it is shown in [181] that $|||f|||_{T,S,\mu} = |||f|||_{S,T,\mu}$). Now, although factors of the original systems that control the seminorms $|||\cdot|||_{T,S,\mu}$ may not admit particularly neat structure, it is shown in [181] that there exists a new system (X^*, μ^*, T^*, S^*) that extends the system (X, μ, T, S) and in addition enjoys the following key property (the term "magic extension" from [181] alludes to this property):

$$|||f^*|||_{T^*,S^*,\mu^*} = 0 \Leftrightarrow f^* \perp \mathcal{I}_{T^*} \vee \mathcal{I}_{S^*}$$

where $f^* \in L^\infty(\mu^*)$ and $\mathcal{I}_{T^*}$ denotes the σ-algebra of T^*-invariant sets and similarly for $\mathcal{I}_{S^*}$. In fact, one can take $X^* := X^4$, $T^* := (\mathrm{id}, T, \mathrm{id}, T)$, $S^* := (\mathrm{id}, \mathrm{id}, S, S)$, and define the measure μ^* by

$$\int f_1\otimes f_2\otimes f_3\otimes f_4\ d\mu^* := \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int f_1\cdot T^mf_2\cdot S^nf_3\cdot T^mS^nf_4\ d\mu.$$

Note that mean convergence for the averages $A_N(T, S, f_i)$ follows if we prove mean convergence for the averages $A_N(T^*, S^*, f_i^*)$. Combining all these observations, we can easily reduce matters to proving mean convergence for the averages $A_N(T^*, S^*, f_i^*)$ when all functions f_i^* are $\mathcal{I}_{T^*}\vee\mathcal{I}_{S^*}$-measurable. This is a significant simplification of our original problem, and in fact it is now straightforward to deduce the required convergence property from the mean ergodic theorem. Indeed, this is immediate if f_3^* is a product of a T^*-invariant and a S^*-invariant function, and the general case follows by linearity and approximation.

This approach has proved particularly useful for handling mean convergence problems of multiple ergodic averages of commuting transformations with linear iterates (and in some cases for handling multiple recurrence problems) that previously seemed intractable [18, 19, 93, 95, 181] (see also [21, 26] for more general group actions). A drawback to this approach is that it does not give information about the precise form of the limiting function, and also, up to now, it has not proved to be very useful when some of the iterates are non-linear (for polynomial iterates though there is some progress in this direction [23, 24]).

2.12. **Using structural results for multiple correlation sequences.** Suppose we want to give sufficient conditions on a bounded sequence of complex numbers (w_n) so that the weighted averages

$$\frac{1}{N}\sum_{n=1}^{N} w_n\, T_1^nf_1\cdots T_\ell^nf_\ell \qquad (13)$$

converge for all choices of systems and functions. In the absence of a useful description for the characteristic factors of these averages, an alternate way to proceed is to use structural results for the corresponding multiple correlation sequences. Using the fact that the multiple correlation sequence

$$\int f_0\cdot T_1^nf_1\cdot\ldots\cdot T_\ell^nf_\ell\, d\mu,\quad n\in\mathbb{N},$$

is equal to a basic ℓ-step nilsequence modulo a sequence that is small in uniform density (see the theorem in Section 5.2.6), we deduce that the averages (13) converge weakly for all choices of systems and functions if and only if the averages

$$\frac{1}{N}\sum_{n=1}^{N} w_n\,\psi(n)$$

converge for all basic ℓ-step nilsequences $(\psi(n))$ (this condition is also necessary). Similar criterions also apply for weighted averages of multiple correlation sequences with polynomial iterates (see [47, 95] for powers of the same transformation and [134, 137] for general commuting transformations).

Somewhat stronger conditions imply mean convergence; it suffices to assume that either the averages

$$\frac{1}{N-M}\sum_{n=M}^{N-1} w_n\,\psi(n)$$

converge as $N-M\to\infty$ for every basic ℓ-step nilsequence ψ (see [137]), or that the averages

$$\frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} w_n\,\overline{w_m}\,\psi(m,n)$$

converge as $M,N\to\infty$ for every basic $(2\ell-1)$-step nilsequence $(\psi(m,n))$ (see [137]).

Although this approach is rather effective for bounded weights (see [134, 137, 189] for applications involving weights of dynamical and arithmetic origin), it is inadequate for unbounded weights (w_n), or if one is interested in ergodic theorems along zero density subsequences of the iterates involved. For such purposes, and in order to fully exploit the potential of this approach, it seems that one needs to prove stronger structural decomposition results like those conjectured in Section 4.1.

2.13. **Equivalent problems for sequences.** It is sometimes useful to be aware of the fact that mean convergence and multiple recurrence problems in ergodic theory are intimately related to similar problems involving bounded sequences of complex numbers. We give some explicit examples below.

Given a collection of sequences of integers $\{a_1,\ldots,a_\ell\}$, it turns out that the following two properties are equivalent:

- For every system $(X,\mathcal{X},\mu,T)$ and set $A\in\mathcal{X}$ with $\mu(A)>0$, there exists $n\in\mathbb{N}$, such that
$$\mu(A\cap T^{a_1(n)}A\cap\cdots\cap T^{a_\ell(n)}A)>0.$$
- For every bounded sequence $(z(n))$ of non-negative real numbers that satisfies
$$\limsup_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} z(m)>0,$$
there exists $n\in\mathbb{N}$, such that
$$\limsup_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} z(m)\cdot z(m+a_1(n))\cdots z(m+a_\ell(n))>0.$$

Using the correspondence principle of Furstenberg it is not hard to see that the first statement implies the second. To see that the second statement implies the first it suffices to set $z(m):=\mathbf{1}_A(T^m x)$, $m\in\mathbb{N}$, for a suitable point $x\in X$ (μ-almost every $x\in X$ works) and use the pointwise ergodic theorem.

We say that a sequence of complex numbers $(z(n))$ *admits correlations along the sequence of intervals* $([1,M_k])$, where $M_k\to\infty$, if for every $\ell\in\mathbb{N}$ and $n_1,\ldots,n_\ell\in\mathbb{Z}$, the averages

$$\frac{1}{M_k}\sum_{m=1}^{M_k} z_1(m+n_1)\cdots z_\ell(m+n_\ell)$$

converge as $k\to\infty$ for all sequences $z_1,\ldots,z_\ell\in\{z,\overline{z}\}$. Using a diagonal argument it is easy to show that if $M_k\to\infty$, then any bounded sequence of complex numbers $(z(n))$ admits correlations along some subsequence of $([1,M_k])$. Given a collection of sequences of integers $\{a_1,\ldots,a_\ell\}$, it turns out that the following two properties are equivalent:

- For every system $(X,\mathcal{X},\mu,T)$ and $f\in L^\infty(\mu)$, the averages
$$\frac{1}{N}\sum_{n=1}^{N}\int f\cdot T^{a_1(n)}f\cdots T^{a_\ell(n)}f\ d\mu$$
converge as $N\to\infty$.
- For every bounded sequence $(z(n))$ of complex numbers, which admits correlations along the sequence of intervals $([1,M_k])$, the averages
$$\frac{1}{N}\sum_{n=1}^{N}\Big(\lim_{k\to\infty}\frac{1}{M_k}\sum_{m=1}^{M_k} z(m)\cdot z(m+a_1(n))\cdots z(m+a_\ell(n))\Big)$$
converge as $N\to\infty$.

The proof of the previous equivalence is based on a variation of the correspondence principle of Furstenberg that applies to bounded sequences and can be found, for example, in [137, Proposition 6.4].

One can get similar statements for mean convergence, as well as for convergence and recurrence properties involving several commuting transformations (in this case one has to use sequences in several variables).

3. Some useful notions

To ease our exposition we collect some notions that are frequently used in subsequent sections.

3.1. Recurrence.

The next notions are used to describe multiple recurrence properties of a collection of possibly different sequences:

Definition 3.1. The collection of sequences of integers $\{(a_1(n)),\ldots,(a_\ell(n))\}$ is

- *good for ℓ-recurrence of a single transformation* if for every system $(X,\mathcal{X},\mu,T)$ and set $A\in\mathcal{X}$ with $\mu(A)>0$, we have
$$\mu(A\cap T^{a_1(n)}A\cap\cdots\cap T^{a_\ell(n)}A)>0$$
for some $n\in\mathbb{N}$ with $a_i(n)\neq 0$ for $i=1,\ldots,\ell$.
- *good for ℓ-recurrence of commuting transformations* if for every system
$$(X,\mathcal{X},\mu,T_1,\ldots,T_\ell)$$
and set $A\in\mathcal{X}$ with $\mu(A)>0$, we have
$$\mu(A\cap T_1^{a_1(n)}A\cap\cdots\cap T_\ell^{a_\ell(n)}A)>0$$
for some $n\in\mathbb{N}$ with $a_i(n)\neq 0$ for $i=1,\ldots,\ell$.

We remark that in the previous statements the existence of a single $n \in \mathbb{N}$ for which the multiple intersection has positive measure, forces the existence of infinitely many $n \in \mathbb{N}$ with the same property.

Examples of collections of sequences that are known to be good for ℓ-recurrence of commuting transformations include collections of: integer polynomials with zero constant term [52] and integer polynomials with zero constant term evaluated at the shifted primes [139]. For arbitrary collections of integer polynomials necessary and sufficient conditions for ℓ-recurrence of a single transformation are given in [59]. For every $\ell \in \mathbb{N}$, the collection $\{([n^{c_1}]), \ldots, ([n^{c_\ell}])\}$, where $c_1, \ldots, c_\ell \in \mathbb{R} \setminus \mathbb{Z}$, is known to be good for ℓ-recurrence of a single transformation [132], and for ℓ-recurrence of commuting transformations if the non-integers $c_1, \ldots, c_\ell$ are distinct [133].

Next we define notions related to multiple recurrence properties of collections defined by a single sequence:

Definition 3.2. The sequence of integers $(a(n))$ is

- *good for ℓ-recurrence of powers* if for all non-zero $k_1, \ldots, k_\ell \in \mathbb{Z}$ the collection of sequences $\{(k_1 a(n)), \ldots, (k_\ell a(n))\}$ is good for ℓ-recurrence of a single transformation.
- *good for multiple recurrence of powers* if it is good for ℓ-recurrence of powers for every $\ell \in \mathbb{N}$.
- *good for ℓ-recurrence of commuting transformations* if the collection of sequences
$$\{(a(n)), \ldots, (a(n))\}$$
is good for ℓ-recurrence of commuting transformations.
- *good for multiple recurrence of commuting transformations* if it is good for ℓ-recurrence of commuting transformations for every $\ell \in \mathbb{N}$.

The fact that the sequence (n) is good for multiple recurrence of powers corresponds to the multiple recurrence result of H. Furstenberg [151], and the fact that it is good for multiple recurrence of commuting transformations corresponds to the multidimensional extension of this result of H. Furstenberg and Y. Katznelson [158]. Further examples of sequences that are good for multiple recurrence of commuting transformations include: integer polynomials with zero constant term [52], the shifted primes (see [296] for powers and [62] in general), integer polynomials with zero constant term evaluated at the shifted primes (see [296] for powers and [139] in general), arbitrary shifts of integers with an even (or odd) number of prime factors [136], several generalized polynomial sequences [65], and some random sequences of integers of zero density [146]. The sequence $([n^c])$, where $c \in \mathbb{R}$ is positive, is known to be good for multiple recurrence of powers [149] (see also [132]), but if $c \in \mathbb{R} \setminus \mathbb{Q}$ is greater than 1, it is not known whether it is good for multiple recurrence of commuting transformations. Examples of sequences that are good for ℓ-recurrence of powers but are not good for $(\ell+1)$-recurrence of powers can be found in [144, 147, 194].

It is also useful to know some obstructions to recurrence. One can check that if the sequence $(a(n))$ is good for 1-recurrence, then the equation $a(n) \equiv 0 \mod r$ has a solution for every $r \in \mathbb{N}$; as a consequence, the sequences (p_n), where p_n is the n-th prime, (n^2+1), (2^n), are not good for 1-recurrence. More generally, if the sequence $(a(n))$ is good for 1-recurrence, then for every $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ the inequality $\|a(n)\alpha\| \leq \varepsilon$ has a solution (letting $\alpha = 1/r$ gives the previous congruence condition); as a consequence, the sequence $([\sqrt{5}n + 2])$ is not good for 1-recurrence (take $\alpha = 1/\sqrt{5}$ and $\varepsilon = 1/10$). This obstruction also implies that if a sequence $(a(n))$ is lacunary, meaning, $\liminf_{n\to\infty} a(n+1)/a(n) > 1$, then it is not good for 1-recurrence, since it is well known that in this case there exist an irrational α and a positive ε_0 such that $\|a(n)\alpha\| \geq \varepsilon_0$ for every $n \in \mathbb{N}$.

Furthermore, if a sequence $(a(n))$ is good for 2-recurrence of powers, then for every $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ the inequality $\|a(n)^2\alpha\| \leq \varepsilon$ has a solution (this is implicit in [152] Section 9.1, and is proved in detail in [144, 194]). It follows that if one forms a sequence by putting the elements of the set $\{n \in \mathbb{N}\colon \|n^2\sqrt{2}\| \in [1/4, 1/2]\}$ in increasing order, then this sequence is not going to be good for 2-recurrence of powers (one can show that it is going to be good for 1-recurrence [144, 194]). And there are also additional more restrictive obstructions, if the sequence $(a(n))$ is good for 2-recurrence of powers, then for every $\alpha, \beta \in \mathbb{R}$ and $\varepsilon > 0$ the inequality $\|[a(n)\alpha]a(n)\beta\| \leq \varepsilon$ has a solution [145]. More generally, for ℓ-recurrence of powers, one can state further obstructions by using higher degree polynomials with zero constant term, or other generalized polynomials [196].

The correspondence principle of Furstenberg enables to deduce statements in density Ramsey theory from multiple recurrence statements in ergodic theory. Using this and some elementary arguments, we can reformulate the previous ergodic notions in purely combinatorial terms:

Theorem. *The collection of sequences of integers* $\{a_1(n), \dots, a_\ell(n)\}$ *is*

- *good for ℓ-recurrence of a single transformation if and only if every set $E \subset \mathbb{Z}$ with $\bar{d}(E) > 0$ contains patterns of the form*
$$\{m, m + a_1(n), \dots, m + a_\ell(n)\}$$
for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $a_i(n) \neq 0$ for $i = 1, \dots, \ell$.
- *good for ℓ-recurrence of commuting transformations if and only if for every $\mathbf{v}_1, \dots, \mathbf{v}_\ell \in \mathbb{Z}^d$, every set $E \subset \mathbb{Z}^d$ with $\bar{d}(E) > 0$ contains patterns of the form*
$$\{\mathbf{m}, \mathbf{m} + a_1(n)\mathbf{v}_1, \dots, \mathbf{m} + a_\ell(n)\mathbf{v}_\ell\}$$
for some $\mathbf{m} \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ with $a_i(n) \neq 0$ for $i = 1, \dots, \ell$.

For instance, the fact that the sequence (n) is good for multiple recurrence of powers corresponds to the theorem of Szemerédi on arithmetic progressions, and the fact that for polynomials $p_1, \dots, p_\ell \in \mathbb{Z}[t]$ with zero constant term the collection $\{p_1, \dots, p_\ell\}$ is good for multiple recurrence of powers corresponds to the polynomial Szemerédi theorem.

Let us also remark that the previous notions admit equivalent uniform versions that are often useful for applications. For instance, one can prove the following (see [48] for an argument that works for polynomials and [145] for an argument that works for general sequences):

Theorem. *Let $(a_1(n)), \dots, (a_\ell(n))$ be sequences of integers. Then the following statements are equivalent:*

- *The collection $\{a_1(n), \dots, a_\ell(n)\}$ is good for ℓ-recurrence of a single transformation.*
- *For every $\varepsilon > 0$ there exist $\delta > 0$ and N_0, such that for every $N \geq N_0$ and integer set $E \subset [-N, N]$ with $|E| \geq \varepsilon N$, we have*
$$|E \cap (E - a_1(n)) \cap \dots \cap (E - a_\ell(n))| \geq \delta N$$
for some $n \in [1, N_0]$.
- *For every $\varepsilon > 0$ there exist $\gamma > 0$ and N_1, such that for every system $(X, \mathcal{X}, \mu, T)$ and $A \in \mathcal{X}$ with $\mu(A) \geq \varepsilon$, we have that*
$$\mu(A \cap T^{a_1(n)}A \cap \dots \cap T^{a_\ell(n)}A) \geq \gamma$$
for some $n \in [1, N_1]$.

We remark that the constants γ, δ, and N_0 in the previous theorem depend only on ε and the choice of the sequences $a_1, \ldots, a_\ell$.

3.2. **Convergence.** The next notion is used to describe multiple convergence properties of a collection of sequences:

Definition 3.3. The collection of sequences of integers $\{(a_1(n)), \ldots, (a_\ell(n))\}$ is

- *good for ℓ-convergence of a single transformation* if for every system $(X, \mathcal{X}, \mu, T)$ and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages
$$\frac{1}{N}\sum_{n=1}^{N} T^{a_1(n)} f_1 \cdots T^{a_\ell(n)} f_\ell$$
converge in the mean.
- *good for ℓ-convergence of commuting transformations* if for every system
$$(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$$
and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages
$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a_1(n)} f_1 \cdots T_\ell^{a_\ell(n)} f_\ell$$
converge in the mean.

Examples of collections of sequences (not coming from multiples of the same sequence) that are known to be good for ℓ-convergence of commuting transformations include those defined by: integer polynomials [290], the integer part of polynomials with real coefficients [212], any such sequence evaluated at the prime numbers [139, 213], and pairs of sequences of the form $\{(n), (a_n(\omega))\}$ where $(a_n(\omega))$ is a certain random sequence of integers of zero density [146]. Additional examples of collections of sequences that are good for ℓ-convergence of a single transformation include the collection $\{([n^{c_1}]), \ldots, ([n^{c_\ell}])\}$, where $c_1, \ldots, c_\ell \in \mathbb{R} \setminus \mathbb{Z}$ are positive and several other Hardy field sequences of polynomial growth [132], these sequences are also known to be good for ℓ-convergence of commuting transformations if the non-integers $c_1, \ldots, c_\ell$ are distinct [133].

Next we define notions related to multiple convergence properties of collections defined by a single sequence:

Definition 3.4. The sequence of integers $(a(n))$ is

- *good for ℓ-convergence of powers* if for every $k_1, \ldots, k_\ell \in \mathbb{Z}$ the collection of sequences $\{(k_1 a(n)), \ldots, (k_\ell a(n))\}$ is good for ℓ-convergence of a single transformation.
- *good for multiple convergence of powers* if it is good for ℓ-convergence of powers for every $\ell \in \mathbb{N}$.
- *good for ℓ-convergence of commuting transformations* if the collection of sequences $\{(a(n)), \ldots, (a(n))\}$ is good for ℓ-convergence of commuting transformations.
- *good for multiple convergence of commuting transformations* if it is good for ℓ-convergence of commuting transformations for every $\ell \in \mathbb{N}$.

Examples of sequences that are good for multiple convergence of commuting transformations include sequences given by integer polynomials [290], the integer part of polynomials with real coefficients [212], any such sequence evaluated at the prime numbers [139, 213], and some random sequences of integers of zero density [146, 147]. Additional examples of sequences that are known to be good for multiple convergence of powers include sequences of the form $([n^a])$ where $a > 0$ and several other Hardy field sequences of polynomial growth [132]. Examples of sequences that are good for ℓ-convergence of powers but are not good for $(\ell+1)$-convergence of powers can be found in [144].

4. Problems related to general sequences

In this section we give a list of problems related to the study of multiple ergodic averages and multiple recurrence problems involving iterates given by general sequences of integers.

4.1. **The structure of multiple correlation sequences.** Let $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ be a system and $f_0, f_1, \dots, f_\ell \in L^\infty(\mu)$ be functions. We are interested in determining the structure of the multiple correlation sequences $\mathcal{C} \colon \mathbb{Z}^\ell \to \mathbb{C}$ defined by the formula

$$\mathcal{C}(n_1, \dots, n_\ell) := \int f_0 \cdot T_1^{n_1} f_1 \cdot \ldots \cdot T_\ell^{n_\ell} f_\ell \, d\mu, \quad n_1, \dots, n_\ell \in \mathbb{Z}. \tag{14}$$

The next result gives a very satisfactory solution to this problem for $\ell = 1$ and serves as our model for possible generalizations. It can be deduced from Herglotz's theorem on positive definite sequences (the sequence $n \mapsto \int \bar{f} \cdot T^n f \, d\mu$ is positive definite) and a polarization identity.

Theorem. *Let $(X, \mathcal{X}, \mu, T)$ be a system and $f, g \in L^\infty(\mu)$. Then there exists a complex Borel measure σ on $[0, 1)$, with bounded variation, such that for every $n \in \mathbb{Z}$ we have*

$$\int f \cdot T^n g \, d\mu = \int_0^1 e^{2\pi i n t} \, d\sigma(t). \tag{15}$$

Finding a formula analogous to (15), with the multiple correlation sequences (14) in place of the single correlation sequences, is a problem of fundamental importance which has been in the mind of experts for several years. A satisfactory solution is going to give us new insights and significantly improve our ability to deal with difficult questions involving multiple ergodic averages which at the moment seem out of reach. There are indications that sequences of polynomial nature should replace the "linear" sequences $(e^{2\pi i n t})$. The most reasonable candidates at this point seem to be some collection of generalized multivariable nilsequences (defined in Section 2.4). For instance, examples of generalized 2-step nilsequences in 1-variable are the sequences $\big(e^{i([n\alpha]n\beta + n\gamma)}\big)$ ($[x]$ denotes the integer part of x), where $\alpha, \beta, \gamma \in \mathbb{R}$, and examples of generalized 2-step nilsequences in 2 variables are the sequences $\big(e^{i([m\alpha]n\beta + m\gamma + n\delta)}\big)$, where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Let $\mathcal{S}$ be a subset of $\ell^\infty(\mathbb{N}^\ell)$. We say that a bounded sequence $a \colon \mathbb{Z}^\ell \to \mathbb{C}$ is

- *an integral combination of elements of $\mathcal{S}$*, if there exist a complex Borel measure σ of bounded variation on a compact metric space X and sequences $a_x \in \mathcal{S}$, $x \in X$, such that for each $\mathbf{n} \in \mathbb{Z}^\ell$ the map $x \mapsto a_x(\mathbf{n})$ is in $L^\infty(\sigma)$ and for every $\mathbf{n} \in \mathbb{Z}^\ell$ we have
$$a(\mathbf{n}) = \int_X a_x(\mathbf{n}) \, d\sigma(x).$$
- *an approximate integral combination of elements of $\mathcal{S}$*, if for every $\varepsilon > 0$ we have $a = b + c$ where b is an integral combination of elements of $\mathcal{S}$ and $\|c\|_\infty \le \varepsilon$.

Problem 1. *Determine the structure of the multiple correlation sequences* $(\mathcal{C}(n_1,\ldots,n_\ell))$ *defined by* (14). *Is it true that any such sequence is an (approximate) integral combination of generalized* ℓ*-step nilsequences in* ℓ*-variables?*

A similar result may hold even if the transformations $T_1,\ldots,T_\ell$ generate a nilpotent group. But some commutativity assumption on the transformations is needed, otherwise simple examples show that generalized nilsequences cannot be the only building blocks. For instance, let $T,S\colon \mathbb{T}\to\mathbb{T}$ be given by $Tx=2x$, $Sx=2x+\alpha$, and $f(x)=e^{-ix}, g(x)=e^{ix}$. Then $\int T^nf\cdot S^ng\;dx=e^{i\cdot(2^n-1)\alpha}$ and one can show that the sequence $(e^{i\cdot 2^n\alpha})$ is not a generalized nilsequence for $\alpha\in\mathbb{R}\setminus\mathbb{Q}$.

Even resolving special cases of this problem would be extremely interesting. One particular instance is the following:

Special Case of Problem 1. *Let* $(X,\mathcal{X},\mu,T)$ *be a system and* $f,g,h\in L^\infty(\mu)$. *Is it true that the sequence* $(\mathcal{C}(n))$ *defined by*

$$\mathcal{C}(n):=\int f\cdot T^ng\cdot T^{2n}h\;d\mu,\quad n\in\mathbb{N},$$

is an (approximate) integral combination of generalized 2*-step nilsequences?*

In [47] it is shown that for ergodic systems one has the decomposition

$$\mathcal{C}(n)=\psi(n)+e(n)$$

where $\psi(n)$ is a (single variable) 2-step nilsequence and $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N|e(n)|=0$. A variant of this result regarding correlation sequences defined by commuting transformations was established in [133] and an extension regarding correlation sequences in several variables in [137]. Unfortunately, these results do not provide information on the error term $e(n)$ for zero density subsets of the integers, and as a consequence they are of little use when one studies sparse subsequences of the sequence $\mathcal{C}(n)$. Lastly, we remark that an essentially equivalent problem is to characterize the structure of sequences $(\mathcal{C}(n))$ defined by

$$\mathcal{C}(n):=\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}a(m)\cdot a(m+n)\cdot a(m+2n),\quad n\in\mathbb{N},$$

where $a\in\ell^\infty(\mathbb{Z})$ is any bounded sequence that admits correlations along the sequence of intervals $([1,M])$.

It is natural to inquire whether correlation sequences defined using commuting transformations provide new examples of sequences that cannot be constructed using correlation sequences that use only powers of the same transformation. To formulate precise statements let us define as $\mathcal{C}_{T,S}$ to be the set of all sequences of the form

$$\Big(\int f\cdot T^ng\cdot S^nh\,d\mu\Big)$$

where $(X,\mathcal{X},\mu,T,S)$ ranges over all systems and f,g,h over all functions in $L^\infty(\mu)$. Let also $\mathcal{C}_T$ be the set of sequences of the previous form where we impose the restriction that T and S are powers of the same transformation. It was shown in [133], that modulo sequences that are small in uniform density, the set $\mathcal{C}_{T,S}$ coincides with the set of basic 2-step nilsequences, and also that modulo terms that are small in $\|\cdot\|_\infty$, every basic 2-step nilsequence belongs to the set $\mathcal{C}_T$. Hence, letting $\|a\|_2:=\limsup_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M}^{N-1}|a(n)|^2$ for $a\in\ell^\infty(\mathbb{N})$, one deduces the following:

Theorem. *Let $a \in \mathcal{C}_{T,S}$. Then for every $\varepsilon > 0$ there exists $b \in \mathcal{C}_T$ such that $\|a - b\|_2 \leq \varepsilon$.*

In fact, we believe that the sets $\mathcal{C}_{T,S}$ and $\mathcal{C}_T$ coincide:

Problem 2. *Show that $\mathcal{C}_{T,S} = \mathcal{C}_T$.*

It can be shown that a positive answer to Problem 1 will give a positive answer to Problem 2.

4.2. **Necessary and sufficient conditions for ℓ-convergence.** The next result can be deduced from formula (15) and serves as our model for giving usable necessary and sufficient conditions for ℓ-convergence:

Theorem. *If $(a(n))$ is a sequence of integers, then the following statements are equivalent:*

- *The sequence $(a(n))$ is good for 1-convergence.*
- *The sequence $(a(n))$ is good for 1-convergence for rotations on the circle.*
- *The sequence $\big(\frac{1}{N}\sum_{n=1}^{N} e^{ia(n)t}\big)$ converges for every $t \in \mathbb{R}$.*

Since a formula that generalizes (15) to multiple correlation sequences is not available, we are unable to prove analogous necessary and sufficient conditions for ℓ-convergence. Nevertheless, inspired by Problem 1 we make the following natural guess:

Problem 3. *If $(a_1(n)), \ldots, (a_\ell(n))$ are sequences of integers, then show that the following statements are equivalent:*

- *The sequences $(a_1(n)), \ldots, (a_\ell(n))$ are good for ℓ-convergence of commuting transformations.*
- *The sequences $(a_1(n)), \ldots, (a_\ell(n))$ are good for ℓ-convergence of ℓ-step nilsystems.*
- *The sequence $\big(\frac{1}{N}\sum_{n=1}^{N} \psi(a_1(n), \ldots, a_\ell(n))\big)$ converges for every basic generalized ℓ-step nilsequence ψ in ℓ-variables.*

A similar problem was formulated in [140]. We are mainly interested in knowing if the third (or the second) condition implies the first. Even the following very special case is open:

Special Case of Problem 3. *Let $(a(n))$ be a sequence of integers such that the averages $\frac{1}{N}\sum_{n=1}^{N} \psi(a(n))$ converge for every basic generalized 2-step nilsequence ψ. Show that for every system $(X, \mathcal{X}, \mu, T, S)$ and functions $f, g \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^{a(n)} f \cdot S^{a(n)} g$$

converge weakly in $L^2(\mu)$.

The problem has been solved when $(a(n))$ is strictly increasing and its range has positive density; this follows from [47, Theorem 1.9] for $S = T^2$ and in the general case from [134]. For general sequences the problem is open even when $S = T^2$.

A similar problem that is probably of equivalent difficulty is the following one:

Variant of Special Case of Problem 3. *Let (w_n) be a sequence of complex numbers such that the averages $\frac{1}{N}\sum_{n=1}^{N} w_n\, \psi(n)$ converge for every generalized 2-step nilsequence ψ. Show that for every system $(X, \mathcal{X}, \mu, T, S)$ and functions $f, g \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} w_n\, T^n f \cdot S^n g$$

converge weakly in $L^2(\mu)$.

This problem is solved for bounded sequences (w_n) (see [133]) and is open for general sequences even when $S = T^2$.

4.3. **Sufficient conditions for ℓ-recurrence.** Our model result is the next theorem of T. Kamae and M. Mendès-France [204] that gives usable conditions for checking that a sequence is good for 1-recurrence:

Theorem. *Let $(a(n))$ be sequence of integers that satisfies:*

- *the sequence $(a(n)\alpha)_{n\in\mathbb{N}}$ is equidistributed in $\mathbb{T}$ for every irrational α, and*
- *the set $\{n \in \mathbb{N} \colon r|a(n)\}$ has positive upper density for every $r \in \mathbb{N}$.*

Then the sequence $(a(n))$ is good for 1-recurrence.

Once again, the proof of this result relies upon the identity (15). Since appropriate generalizations of this identity are not known for multiple correlation sequences, we are unable to give a similar criterion for ℓ-recurrence when $\ell \geq 2$. To state a conjectural criterion, we extend the notion of an irrational rotation on the circle to general connected nilmanifolds: Given a connected nilmanifold $X = G/\Gamma$, an *irrational nilrotation in X* is an element $b \in G$ such that the sequence $(b^n\Gamma)_{n\in\mathbb{N}}$ is equidistributed on X.

Problem 4. *Let $(a(n))$ be a sequence that satisfies:*

- *for every connected ℓ-step nilmanifold X and every irrational nilrotation b in X the sequence $(b^{a(n)}\Gamma)_{n\in\mathbb{N}}$ is equidistributed in X, and*
- *the set $\{n \in \mathbb{N} \colon r|a(n)\}$ has positive upper density for every $r \in \mathbb{N}$.*

Show that the sequence $(a(n))$ is good for ℓ-recurrence of commuting transformations.

This problem was first formulated in [145] and in the same article it was solved for ℓ-recurrence of powers for sequences that have range a set of integers with positive density. The stated conditions are satisfied by any non-constant integer polynomial sequence with zero constant term (follows from [227]), the sequence $([n^c])$ for every $c > 0$ (follows from [131]), the sequence of shifted primes $(p_n - 1)$ (follows from [171]), and random non-lacunary sequences of integers (follows from results in [131]).

4.4. **Powers of sequences and recurrence.** It is known that if a sequence is good for ℓ-convergence of powers, then its first ℓ powers are good for 1-convergence. More precisely, the following holds (this is implicit in [152] Section 9.1, and is proved in detail in [144]):

Theorem. *If $(a(n))$ is good for ℓ-convergence of powers, then $(a(n)^k)$ is good for 1-convergence for $k = 1, \dots, \ell$.*

It is unclear whether a similar property holds for recurrence.

Problem 5. *If $(a(n))$ is good for ℓ-recurrence of powers, is then $(a(n)^k)$ good for 1-recurrence for $k = 1, \dots, \ell$?*

This problem was first stated in [144] and it is open even when $\ell = 2$. It is known that if $(a(n))$ is good for 2-recurrence of powers, then the sequence $(a(n)^2)$ is good for Bohr recurrence, meaning it is good for 1-recurrence for all rotations on tori (see [152] Section 9.1, or [144]). A well known question of Y. Katznelson asks whether a set of Bohr recurrence is necessarily a set of topological recurrence (for background on this question see [77, 206, 293, 194]). Although there exist examples of sets of topological recurrence that are not sets of 1-recurrence [219], all known examples are rather complicated. As a consequence, an example showing that the answer to Problem 5 is negative is probably going to be complicated.

4.5. **Commuting vs powers of a single transformation.** If a sequence is good for 2-convergence of commuting transformations, then, of course, it is also good for 2-convergence of powers. Interestingly, no example that distinguishes the two notions is known and we believe that there is none:

Problem 6. *If a sequence is good for* 2*-convergence of powers, then show that it is good for* 2*-convergence of commuting transformations.*

The corresponding question for recurrence is also open:

Problem 7. *Is there a sequence that is good for* 2*-recurrence of powers but is not good for* 2*-recurrence of commuting transformations?*

This question was first stated in [37] (Question 8) where V. Bergelson states that the answer is very likely yes.

4.6. **Fast growing sequences.** Despite the successes in dealing with multiple recurrence and convergence problems of sequences that do not grow faster than polynomials, when it comes down to fast growing sequences our knowledge is very limited. Say that a sequence $(a(n))$ of positive integers is *fast growing* if $\lim_{n\to\infty} \log(a(n))/\log n = \infty$ (equivalently, if it is of the form $(n^{b(n)})$ with $b(n) \to \infty$).

Problem 8. *Give an explicit example of a fast growing sequence that is good for multiple recurrence and convergence of powers and commuting transformations.*

Even for 2-recurrence and convergence of powers, no such example is known. It is known that if a sequence grows exponentially fast (for example (2^n) or $(n!)$), then it is not good for 1-recurrence and 1-convergence (even for irrational circle rotations). On the other hand, several natural examples of fast growing sequences that do not grow exponentially fast should be good for multiple recurrence and convergence, for instance, the sequences $([n^{(\log n)^a}])$, $(n^{[(\log n)^b]})$, $([e^{n^c}])$ for $a, b > 0$ and $c \in (0, 1)$. Unfortunately, it is very hard to work with these sequences, even for issues related to 1-recurrence and 1-convergence. Probably a sequence like $(n^{[\log \log n]})$ is easier to work with. One could also try to construct (not so explicit) examples using random sequences of integers (more on this in Section 7).

5. Problems related to polynomial sequences

In this section we give a list of problems related to the study of multiple ergodic averages involving iterates given by polynomial sequences, and related applications to multiple recurrence.

5.1. **Powers of a single transformation.** Let $\mathcal{P} := \{p_1, \dots, p_\ell\}$ be a family of integer polynomials that are *essentially distinct*, meaning, all polynomials and their differences are nonconstant. First, we consider averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{p_1(n)} f_1 \cdots T^{p_\ell(n)} f_\ell, \tag{16}$$

where $(X, \mathcal{X}, \mu, T)$ is a system and $f_1, \dots, f_\ell \in L^\infty(\mu)$.

We remark that all the mean convergence results stated in this section work equally well for uniform Cesaro averages, that is, averages of the form $\frac{1}{|\Phi_N|}\sum_{n\in\Phi_N}$ in place of the averages $\frac{1}{N}\sum_{n=1}^{N}$ where $(\Phi_N)_{N\in\mathbb{N}}$ is a Følner sequence of subsets of $\mathbb{N}$, meaning, for every $h \in \mathbb{N}$ it satisfies $|(\Phi_N + h)\triangle\Phi_N|/|\Phi_N| \to 0$ as $N \to \infty$. This property is no longer true if the iterates are given by Hardy field sequences or random sequences of integers (see Sections 6 and 7), or if we deal with pointwise convergence (not even when $\ell = 1$ and $p_1(n) = n$).

5.1.1. *Optimal characteristic factors.* Before discussing some problems related to the characteristic factors of the averages (16), we state a result of B. Host and B. Kra [186] and A. Leibman [229] that gives useful information about their structure.

Theorem. *There exists $d = d(\mathcal{P})$ such that the factor $\mathcal{Z}_{d,T}$ (defined in 2.5) is characteristic for mean convergence of the averages* (16).

We note that in the previous statement the value of $d(\mathcal{P})$ can be chosen to depend only on the number and the maximum degree of the polynomials in $\mathcal{P}$. Given a family of polynomials $\mathcal{P}$, we denote by $d_{min}(\mathcal{P})$ the minimal value of $d(\mathcal{P})$ that works in the previous theorem. This value is in general hard to pin down and depends on the algebraic relations that the polynomials satisfy. For instance, we know that $d_{min}(\{n, 2n, \ldots, \ell n\}) = \ell - 1$ ([185, 301]), and $d_{min}(\mathcal{P}) = 1$ when $\mathcal{P}$ consists of (at least two) rationally independent polynomials ([141, 143]). But it is not only linear relations between the polynomials that matter, for instance, we know that $d_{min}(\{n, 2n, n^2\}) = 2$ while $d_{min}(\{n, 2n, n^3\}) = 1$ ([130, 233]). More examples of families $\mathcal{P}$ where $d_{min}(\mathcal{P})$ has been computed can be found in [130, 233]. Furthermore, in [233] a (rather complicated) algorithm is given for computing this value. Despite such progress, the following is still open (the problem is implicit in [58] and was stated explicitly in [233]):

Problem 9. *If $|\mathcal{P}| \geq 2$, show that $d_{min}(\mathcal{P}) \leq |\mathcal{P}| - 1$.*

The estimate is known when $|\mathcal{P}| = 2, 3$ ([130]) and it is open when $|\mathcal{P}| = 4$. The problem is open even when one is restricted to the class of Weyl systems, meaning, systems of the form $(\mathbb{T}^d, \mathcal{X}_{\mathbb{T}^d}, m_{\mathbb{T}^d}, T)$ where $T\colon \mathbb{T}^d \to \mathbb{T}^d$ is a unipotent affine transformation. We denote with $d^{\mathrm{W}}_{min}(\mathcal{P})$ the minimum value of d such that the factor $\mathcal{Z}_{d,T}$ is characteristic for mean convergence of the averages (16) for all Weyl systems (properties of $d^{\mathrm{W}}_{min}(\mathcal{P})$ were studied in [58]).

Special Case of Problem 9. *If $|\mathcal{P}| \geq 2$, show that $d^{W}_{min}(\mathcal{P}) \leq |\mathcal{P}| - 1$.*

This problem was first stated in [58] (set $W(P) := d^{W}_{min}(\mathcal{P}) + 1$ in the remark after [58, Proposition 5.3]). The estimate is known when $|\mathcal{P}| = 2, 3, 4$ ([130, 247]) and it is open when $|\mathcal{P}| = 5$. We also remark that no example is known where $d_{min}(\mathcal{P}) \neq d^{\mathrm{W}}_{min}(\mathcal{P})$, so it is natural to suspect that these two values coincide. This is known to be the case when $|\mathcal{P}| = 3$ ([130]) and it is open when $|\mathcal{P}| = 4$. Obviously one has $d^{\mathrm{W}}_{min}(\mathcal{P}) \leq d_{min}(\mathcal{P})$. Some bounds in the other direction are given in [233].

5.1.2. *Variable polynomials.* Mean convergence of the averages (16) was established after a long series of intermediate results; the papers [99, 100, 101, 151, 163, 182, 185, 268, 301] dealt with the important case of linear polynomials, and using the machinery introduced in [185], convergence for arbitrary polynomials was finally obtained by B. Host and B. Kra in [186] except for a few cases that were treated by A. Leibman in [229].

Theorem. *Let $(X, \mathcal{X}, \mu, T)$ be a system, $f_1, \ldots, f_\ell \in L^\infty(\mu)$ be functions, and $p_1, \ldots, p_\ell$ be integer polynomials. Then the averages* (16) *converge in the mean as $N \to \infty$.*

Furthermore, explicit formulas for the limit can be given for special families of polynomials [130, 141, 143, 233, 299], but no explicit formula is known for general families of polynomials.

We record here a related open problem involving variable polynomials. We say that

- the sequence of polynomials (p_N) where $p_N \in \mathbb{R}[t]$, $N \in \mathbb{N}$, is *good* if the polynomials have bounded degree and for every non-zero $\alpha \in \mathbb{R}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{ip_N(n)\alpha} = 0.$$

- the sequence of ℓ-tuples of polynomials $(p_{1,N}, \dots, p_{\ell,N})$ where $p_{i,N} \in \mathbb{R}[t]$, $N \in \mathbb{N}$, $i \in \{1, \dots, \ell\}$ is *good* if every non-trivial linear combination of the sequences of polynomials $(p_{1,N}), \dots, (p_{\ell,N})$ is good.

It can be shown (see for example [169, Lemma 4.4]) that if $p_N(t) = \sum_{k=1}^{d} c_{k,N} t^k$, $c_{k,N} \in \mathbb{R}$, $N \in \mathbb{N}$, then the sequence (p_N) is good if and only if for every non-zero $\alpha \in \mathbb{R}$ we have $\lim_{N\to\infty} N^j \|c_{j,N}\alpha\| = \infty$ for at least one $j \in \{1, \dots, d\}$ where $\|\cdot\|$ denotes the distance from the closest integer. Hence, for $\ell = 2$, letting $p_{1,N} := n/N^a$ and $p_{2,N} := n/N^b$, $n, N \in \mathbb{N}$, where $0 < a < b < 1$, produces an example of a sequence of good pairs of polynomials of degree 1. Another example is defined by the polynomials $p_{1,N} := n/N^a$, $p_{2,N} := n^2/N^a$, ..., $p_{\ell,N} := n^\ell/N^a$, $n, N \in \mathbb{N}$, where $a \in (0, 1)$. The next problem is of interest even for these collections of variable polynomials.

Problem 10. *Suppose that the sequence of ℓ-tuples of polynomials $(p_{1,N}, \dots, p_{\ell,N})$ is good. Show that for every ergodic system $(X, \mathcal{X}, \mu, T)$ and functions $f_1, \dots, f_\ell \in L^\infty(\mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{[p_{1,N}(n)]} f_1 \cdots T^{[p_{\ell,N}(n)]} f_\ell = \int f_1 \, d\mu \cdots \int f_\ell \, d\mu$$

where convergence takes place in $L^2(\mu)$.

For $\ell = 1$ the problem can be solved using the spectral theorem, that is, the identity in (15).

5.1.3. *Pointwise convergence.* In most cases, it is still unknown whether mean convergence of multiple ergodic averages can be strengthened to pointwise convergence. We mention two particular cases that are open:

Problem 11. *Let $(X, \mathcal{X}, \mu, T)$ be a system and $f, g, h \in L^\infty(\mu)$ be functions. Show that the averages*

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \cdot g(T^{2n} x) \cdot h(T^{3n} x) \quad \textit{and the averages} \quad \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \cdot g(T^{n^2} x)$$

converge pointwise almost everywhere.

Pointwise convergence of the averages (16) is known when $\ell = 1$ [80] and is also known when $\ell = 2$ and both polynomials are linear [84] (see also [105] for an alternative proof). In all other cases the problem is open even for weak mixing systems. Partial results that deal with special classes of transformations can be found in [5, 7, 28, 29, 108, 197, 227, 243, 244].

We record also some problems of arithmetic nature regarding pointwise convergence of double averages. We say that a multiplicative function $\phi\colon \mathbb{N} \to \mathbb{C}$ (meaning, $\phi(mn) = \phi(m)\phi(n)$ for all $m, n \in \mathbb{N}$ with $(m, n) = 1$) has *convergent means* if the averages $\frac{1}{N}\sum_{n=1}^{N} \phi(an + b)$ converge as $N \to \infty$ for every $a, b \in \mathbb{N}$. It can be shown that every multiplicative function that takes values in $[-1, 1]$ has convergent means, but there are multiplicative functions with values on the complex unit disc that do not have a mean value (for example take $\phi(n) = n^{it}, n \in \mathbb{N}$, for some non-zero $t \in \mathbb{R}$).

Problem 12. *Let $(X, \mathcal{X}, \mu, T)$ be a system and $f, g \in L^\infty(\mu)$ be functions. If Λ is the von Mangoldt function and ϕ is a multiplicative function that takes values on the complex unit disc and has convergent means, then show that the averages*

$$\frac{1}{N}\sum_{n=1}^{N} \Lambda(n)\, f(T^n x)\cdot g(T^{2n}x) \quad \text{and the averages} \quad \frac{1}{N}\sum_{n=1}^{N} \phi(n)\, f(T^n x)\cdot g(T^{2n}x)$$

converge pointwise almost everywhere.

Pointwise convergence of the first averages would imply pointwise convergence of the averages $\frac{1}{N}\sum_{n=1}^{N} f(T^{p_n}x)\cdot g(T^{2p_n}x)$ where p_n denotes the n-th prime. Pointwise convergence of the second averages when ϕ is the Liouville function would imply pointwise convergence of the averages $\frac{1}{N}\sum_{n=1}^{N} f(T^{s_n}x)\cdot g(T^{2s_n}x)$, where s_n is the n-th natural number that has an even number of prime factors counted with multiplicity. Of course, one also expects similar properties to hold for higher order variants of the previous statements. Mean convergence for both averages is known ([138] for the first, [136] for the second). Also pointwise convergence is known when $g = 1$ ((([294] for the first, [136] for the second). The second problem is open even when ϕ is the Möbius or the Liouville function (in which case the L^2-limit is known to be 0 [136]).

5.1.4. *Subsequences of multiple correlation sequences.* Lastly, we record a problem about correlation sequences involving powers of a single transformations. We start with the following result of V. Bergelson, B. Host, and B. Kra [47]:

Theorem. *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $f_0, \dots, f_\ell \in L^\infty(\mu)$. Then one has a decomposition*

$$\int f_0 \cdot T^n f_1 \cdot \ldots \cdot T^{\ell n} f_\ell \; d\mu = \psi(n) + e(n)$$

where $(\psi(n))$ is an ℓ-step nilsequence and $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |e(n)| = 0$.

This result was extended to non-ergodic systems in [236, 237] and to polynomial iterates in [234]; in the latter case the level of nilpotency of ψ depends on ℓ and the degree of the polynomials. A key ingredient in the proofs is the fact that the factor $\mathcal{Z}_{\ell,T}$ is characteristic for convergence of the averages $\frac{1}{N}\sum_{n=1}^{N} |\int f_0 \cdot T^n f_1 \cdot \ldots \cdot T^{\ell n} f_\ell \; d\mu|$. The next problem seeks to explore the extend to which the previous theorem continues to hold for subsequences of multiple correlation sequences.

Problem 13. *Let $(a(n))$ be the sequence of integers (p_n), where p_n is the n-th prime, or $([n^c])$ where $c > 0$, or (2^n). Is it true that for every ergodic system $(X, \mathcal{X}, \mu, T)$ and all functions $f_0, \dots, f_\ell \in L^\infty(\mu)$, one has a decomposition*

$$\int f_0 \cdot T^{a(n)} f_1 \cdot \ldots \cdot T^{\ell a(n)} f_\ell \; d\mu = \psi(a(n)) + e(n),$$

where $(\psi(n))$ is an (ℓ-step) nilsequence and $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |e(n)| = 0$*?*

We believe that the answer is yes for the first two sequences and no for the third sequence. One could also ask similar questions for correlation sequences of the form

$$\mathcal{C}(n) = \int f \cdot T^{a(n)} g \cdot T^{b(n)} h \; d\mu, \quad n \in \mathbb{N},$$

where $(a(n)), (b(n))$ are particular sequences of integers. In this case, the expected decomposition has the form

$$\mathcal{C}(n) = \psi(a(n), b(n)) + e(n), \quad n \in \mathbb{N},$$

where $(\psi(m,n))$ is a (2-step) nilsequence in two variables and $(e(n))$ is as before.

5.2. **Commuting transformations.** Throughout this section, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ is a system and $f_1, \ldots, f_\ell$ are functions in $L^\infty(\mu)$.

We start with problems related to mean convergence of multiple ergodic averages. After a long series of partial results that dealt with commuting transformations [18, 23, 24, 97, 99, 141, 158, 181, 200, 243, 280, 288, 297], M. Walsh in [290] proved the following result:

Theorem. *Let $p_1, \ldots, p_\ell$ be integer valued polynomials. Then the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_\ell^{p_\ell(n)} f_\ell \tag{17}$$

converge in the mean as $N \to \infty$.

A similar result also holds when we work with averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} \Big(\prod_{i=1}^{\ell} T_i^{p_{i,1}(n)}\Big) f_1 \cdots \Big(\prod_{i=1}^{\ell} T_i^{p_{i,\ell}(n)}\Big) f_\ell$$

and the transformations $T_1, \ldots, T_\ell$ generate a nilpotent group [290] and when one averages over Følner sequences in $\mathbb{Z}$ and uses polynomials in several variables [304]. Moreover, we have mean convergence when the iterates are given by integer parts of real valued polynomials [212].

5.2.1. *Generalized polynomials.* A *generalized polynomial* is a real valued function that is obtained from the identity function and real constants by using the operations of addition, multiplication, and taking integer parts. An example is the sequence $([[n\alpha]n\beta + n^2\gamma + n\delta])$ where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. It remains a challenge to extend known mean convergence results with polynomial iterates to the case where the iterates are given by generalized polynomials:

Problem 14. *Let $p_1, \ldots, p_\ell$ be integer valued generalized polynomials. Show that the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_\ell^{p_\ell(n)} f_\ell$$

converge in the mean as $N \to \infty$.

This problem was first stated in [41]. For $\ell = 1$ convergence was proved in [55] and follows from the spectral theorem and the fact (proved in [55]) that any sequence of the form $(e^{ip(n)})$, where p is a generalized polynomial, can be represented as a generalized nilsequence. For $\ell = 2$ the problem is open even when the transformations are equal and weak mixing.

5.2.2. *Characteristic factors.* As mentioned previously, if all transformations are equal, and the polynomials are essentially distinct, then characteristic factors of the averages (17) can be chosen to have very special algebraic structure. For general commuting transformations this is no longer the case; if one chooses $p_1 = p_2 = n$, $T_1 = T_2$, and $f_2 = \bar{f}_1$, then the averages (17) do not converge to 0 unless $f_1 = f_2 = 0$. The same problem persists when two of the polynomials are *dependent*, meaning, their quotient is some non-trivial linear combination of the polynomials is constant. But in all other cases, there is no obvious obstruction to having "simple" characteristic factors with algebraic structure:

Problem 15. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are pairwise independent. Show that there exists $d \in \mathbb{N}$ such that the factors $\mathcal{Z}_{d,T_1}, \ldots, \mathcal{Z}_{d,T_\ell}$ are characteristic factors for the averages* (17).

This is known to be the case when the polynomials have distinct degrees [97]. But it is not known for some simple families of integer polynomials, for instance, for the family $\{n^3, n^3+n\}$ and the family $\{n, n^2, n^2+n\}$. Even for weak mixing transformations the following problem is open:

Special Case of Problem 15. *Suppose that all the transformations $T_1, \ldots, T_\ell \colon X \to X$ are weak mixing and the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are pairwise independent. Show that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T_1^{p_1(n)} f_1 \cdots T_\ell^{p_\ell(n)} f_\ell = \int f_1 \, d\mu \cdots \int f_\ell \, d\mu.$$

The problem is open even when $\ell = 2$ and the family of polynomials is $\{n^3, n^3+n\}$.

When all transformations are equal and the polynomials are in general position, characteristic factors for the averages (17) turn out to be extremely simple; they are given by the *rational Kronecker factor* of the system which is defined as follows: Given a system $(X, \mathcal{X}, \mu, T)$ we let $\mathcal{K}_{rat}(T) = \bigvee_{d\in\mathbb{N}} \mathcal{I}_{T^d}$ where $\mathcal{I}(T^d) := \{f \in L^2(\mu) \colon T^d f = f\}$. The next result is proved in [141, 143]:

Theorem. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are rationally independent (meaning, non-trivial linear combinations of them are non-constant). Then the rational Kronecker factor $\mathcal{K}_{rat}(T)$ is a characteristic factor for the averages* (16).

We believe that this result generalizes to the case of several commuting transformations:

Problem 16. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are rationally independent. Show that $\mathcal{K}_{rat}(T_1), \ldots, \mathcal{K}_{rat}(T_\ell)$ are characteristic factors for the averages* (17).

This was proved in [97] when $\ell = 2$ for families of the form $\{n, p(n)\}$ where $\deg(p) \geq 2$. In the same article a somewhat weaker property was proved for all monomials with distinct degrees. Using techniques from [97] and [254] it is very likely that the problem can be solved for all polynomial families with distinct degrees.

5.2.3. *Optimal lower bounds for multiple recurrence.* We mention also a closely related multiple recurrence problem:

Problem 17. *Suppose that the polynomials $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ are rationally independent and have zero constant term. Show that for every $A \in \mathcal{X}$ and every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that*

$$\mu(A \cap T_1^{p_1(n)} A \cap \cdots \cap T_\ell^{p_\ell(n)} A) \geq \mu(A)^{\ell+1} - \varepsilon. \tag{18}$$

In fact, the set of integers n for which (18) holds is expected to have bounded gaps. The lower bounds are known when all transformations are equal [142] and they are also known for general commuting transformations when the polynomials are monomials with distinct degrees [97]. Combining techniques from [97] and [254] it is very likely that the problem can be solved for arbitrary polynomials with distinct degrees. The result fails if two of the polynomials are distinct and dependent; in this case no fixed power of $\mu(A)$ works as a lower bound in (18) for arbitrary systems and sets [47]. On the other hand, the assumption that the polynomials are rationally independent is not necessary, for instance, the result is expected to work for the family of polynomials $\{n, n^2, n^2+n\}$ (this is known to be the case when all transformations are equal [130]). We remark that if Problem 16 is solved, then the conjectured lower bounds of Problem 17 will follow rather easily. On the other hand, Problem 17 is open even when $\ell = 2$.

5.2.4. *Multiple recurrence for intersective polynomials.* A family of integer valued polynomials $\{p_1, \ldots, p_\ell\}$ is called *intersective* if for every $r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $p_i(n) \equiv 0 \pmod r$ for $i = 1, \ldots, \ell$. When $\ell = 1$ an example of an intersective polynomial with no linear factors is $p(n) = (n^2-13)(n^2-17)(n^2-221)$. Examples of periodic systems show that if a family of polynomials is good for multiple recurrence of commuting transformations, then this family has to be intersective. The next problem was first stated in [59] and seeks to show that the condition of intersectivity is also sufficient for multiple recurrence of commuting transformations.

Problem 18. *Let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system and $\{p_1, \ldots, p_\ell\}$ be a family of intersective integer polynomials. Show that for every set $A \in \mathcal{X}$ with $\mu(A) > 0$ one has*

$$\mu(A \cap T_1^{p_1(n)} A \cap \cdots \cap T_\ell^{p_\ell(n)} A) > 0$$

for some $n \in \mathbb{N}$.

The problem is open even when $\ell = 2$ and $p_1 = p_2$. The case where all the polynomials have zero constant term is covered by the Polynomial Szemerédi Theorem [52]. The case where all the transformations are equal was dealt in [59]. The argument used in [52] depends crucially on the fact that the polynomials have zero constant term. The argument used in [59] depends crucially on the fact that characteristic factors for the corresponding multiple ergodic averages are (inverse limits) of nilsystems; a substitute for this result that is useful for the problem at hand is not currently available.

Note that a solution to Problem 18 would imply that for every collection of intersective polynomials $\{p_1, \ldots, p_\ell\}$, for every $d, \ell \in \mathbb{N}$, set $E \subset \mathbb{N}^d$ with $\bar{d}(E) > 0$, and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_\ell \in \mathbb{N}^d$, there exist $\mathbf{m} \in \mathbb{N}^d$ and $n \in \mathbb{N}$ such that

$$\mathbf{m}, \mathbf{m} + p_1(n)\mathbf{v}_1, \ldots, \mathbf{m} + p_\ell(n)\mathbf{v}_\ell \in E.$$

5.2.5. *Pointwise convergence for commuting transformations.* Regarding pointwise convergence of multiple ergodic averages of commuting transformations, progress has been scarce, even in some seemingly simple cases. The following is a well known open problem:

Problem 19. *Let $(X, \mathcal{X}, \mu, T, S)$ be a system and $f, g \in L^\infty(\mu)$ be functions. Show that the averages*

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) \cdot g(S^n x)$$

converge pointwise almost everywhere.

For a list of partial results that apply to special classes of transformations see the list after Problem 11. Moreover, pointwise convergence in the case where the system $(X, \mathcal{X}, \mu, T, S)$ is distal was recently established in [113], and prior to this, in the case where $S = T^k$ for some $k \in \mathbb{N}$ in [197]. See also [114] for more general statements involving an arbitrary number of commuting transformations.

5.2.6. *Optimal error term for decomposition results.* We end this subsection with a problem related to the structure of multiple correlation sequences defined by commuting transformations. The following result was proved in [134]:

Theorem. *Let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system. Then for every $\varepsilon > 0$ one has a decomposition*

$$\int f_0 \cdot T_1^n f_1 \cdot \ldots \cdot T_\ell^n f_\ell \, d\mu = \psi(n) + e(n)$$

where $(\psi(n))$ is a basic ℓ-step nilsequence and $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^N |e(n)| \leq \varepsilon$.

A similar result also holds when the iterates are given by integer polynomials [134], or integer parts of real valued polynomials [212], and when one uses correlation sequences in several variables [137]. When $T_1, \ldots, T_\ell$ are powers of the same transformation, the main results in [47, 236, 237] give an analogous decomposition with $\varepsilon = 0$ and ψ an ℓ-step nilsequence. It is thus natural to ask whether a similar result holds for commuting transformations.

Problem 20. *Is it true that one always has a decomposition*

$$\int f_0 \cdot T_1^n f_1 \cdot \ldots \cdot T_\ell^n f_\ell \, d\mu = \psi(n) + e(n)$$

where $(\psi(n))$ is an ℓ-step nilsequence and $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N |e(n)| = 0$?

5.3. **Transformations that do not commute.** All problems in the previous sections were stated for families of transformations that commute. When one works with arbitrary families of invertible measure preserving transformations the next result shows that one cannot expect to have similar convergence and recurrence results:

Theorem. *Let $a, b\colon \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ be $1-1$ sequences. Then there exist invertible Bernoulli measure preserving transformations T and S acting on the same probability space $(X, \mathcal{X}, \mu)$ such that*

- *for some $f, g \in L^\infty(\mu)$ the averages $\frac{1}{N}\sum_{n=1}^N \int T^{a(n)} f \cdot S^{b(n)} g \, d\mu$ diverge;*
- *for some $A \in \mathcal{X}$ with $\mu(A) > 0$ we have $T^{a(n)}A \cap S^{b(n)}A = \emptyset$ for every $n \in \mathbb{N}$.*

To construct such examples it suffices to modify examples of D. Berend (see Ex 7.1 in [28]) and H. Furstenberg (page 40 in [152]) that cover the case $a(n) = b(n) = n$ (the details appear in [146]). When $a(n) = b(n)$, it is also known that given any finitely generated solvable group G of exponential growth, there exist invertible measure preserving transformations T, S, with $\langle T, S\rangle \subset G$, and such that the conclusion of the previous theorem holds for those T and S. It is interesting that despite such negative news, once one introduces an extra variable, several convergence (and very likely recurrence) results can be extended to arbitrary families of measure preserving transformations. We mention a positive result from [96]:

Theorem. *Let $(X, \mathcal{X}, \mu)$ be a probability space, $T_1, \ldots, T_\ell\colon X \to X$ be invertible measure preserving transformations, $f_1, \ldots, f_\ell \in L^\infty(\mu)$ be functions, $p_1, \ldots, p_\ell$ be essentially distinct polynomials of degree at most $k \in \mathbb{N}$, and $a \in (0, 1/k)$. Then the averages*

$$\frac{1}{N^{1+a}} \sum_{1\leq m\leq N, 1\leq n\leq N^a} f_1(T_1^{m+p_1(n)}x) \cdots f_\ell(T_\ell^{m+p_\ell(n)}x) \tag{19}$$

converge pointwise almost everywhere as $N \to \infty$.[2]

[2]For $k = 1$ one can take $a = 1$, this was established in [197] but also follows from the method of proof in [96].

The assumption that the polynomials are essentially distinct is necessary. It was also shown in [96] that there exists $d \in \mathbb{N}$ such that the factors $\mathcal{Z}_{d,T_1}, \ldots, \mathcal{Z}_{d,T_\ell}$ are characteristic for pointwise convergence of the averages (19). Despite these facts, the corresponding multiple recurrence result (that would generalize the polynomial Szemerédi theorem) remains open:

Problem 21. *Let $(X, \mathcal{X}, \mu)$ be a probability space, $T_1, \ldots, T_\ell \colon X \to X$ be invertible measure preserving transformations, and $p_1, \ldots, p_\ell$ be distinct polynomials with zero constant term. Show that for every $A \in \mathcal{X}$ with $\mu(A) > 0$ we have*

$$\mu(A \cap T_1^{m+p_1(n)} A \cap \cdots \cap T_\ell^{m+p_\ell(n)} A) > 0$$

for some $m, n \in \mathbb{N}$.

A solution to this problem would imply that given a countable amenable group G and arbitrary elements $g_1, \ldots, g_\ell \in G$, for every $E \subset G$ that has positive upper density with respect to some Følner sequence in G, there exist $g \in G$ and $m, n \in \mathbb{N}$ such that

$$g,\ g_1^{m+p_1(n)} g, \ldots, g_\ell^{m+p_\ell(n)} g \in E.$$

The assumption in Problem 21 that the polynomials are distinct is necessary since, as mentioned before, there exist (non-commuting) transformations T, S, acting on the same probability space $(X, \mathcal{X}, \mu)$, and a set $A \in \mathcal{X}$ with $\mu(A) > 0$ such that $\mu(T^n A \cap S^n A) = 0$ for every $n \in \mathbb{N}$. The multiple recurrence property is known when the polynomials are rationally independent [150] and when all the transformations are weak mixing (because in this case the characteristic factors are trivial [96]). For general measure preserving transformations and linear polynomials some of the simplest cases are open:

Special Case of Problem 21. *Let $(X, \mathcal{X}, \mu)$ be a probability space and $T, S, R \colon X \to X$ be invertible measure preserving transformations. Show that for every $A \in \mathcal{X}$ with $\mu(A) > 0$ there exist $m, n \in \mathbb{N}$ such that*

$$\mu(A \cap T^m A \cap S^{m+n} A \cap R^{m+2n} A) > 0.$$

6. Problems related to sequences arising from smooth functions

In this section we give a list of problems related to the study of multiple ergodic averages involving iterates given by sequences arising from smooth functions, and related applications to multiple recurrence.

We are going to restrict ourselves, almost entirely, to a class of non-oscillatory functions that is rich enough to contain several interesting examples. Its formal definition is the following: Let B be the collection of equivalence classes of real valued functions defined on some half-line (c, ∞), where we identify two functions if they agree eventually. A *Hardy field* is a subfield of the ring $(B, +, \cdot)$ that is closed under differentiation. With $\mathcal{H}$ we denote the *union of all Hardy fields*. It is easy to check that if a function belongs in $\mathcal{H}$, then it is eventually monotonic and the same holds for its derivatives, so if $a \in \mathcal{H}$, all limits $\lim_{t\to\infty} a^{(k)}(t)$ exist (they may be infinite). We call a *Hardy sequence* any sequence of the form $([a(n)])$ where $a \in \mathcal{H}$.

An explicit example of a Hardy field to keep in mind is the set $\mathcal{LE}$ that consists of all *logarithmic-exponential functions* (introduced by Hardy in [179]), meaning all functions defined on some half-line (c, ∞) using a finite combination of the symbols $+, -, \times, :$, log, exp, operating on the real variable t and on real constants. For example, all rational functions and the functions $t^{\sqrt{2}}$, $t \log t$, $t^{\sqrt{\log\log t}}/\log(t^2+1)$ belong to $\mathcal{LE}$. Also all *polynomials with fractional powers*, meaning, functions of the form $\sum_{k=1}^d \alpha_k\, t^{c_k}$, where $\alpha_k, c_k, \in \mathbb{R}$, $k = 1, \ldots, d$, belong to $\mathcal{LE}$.

The main advantage we get by working with elements of $\mathcal{H}$ is that it is possible to relate their growth rates with the growth rates of their derivatives. As a consequence, a single growth condition encodes a lot of useful information and this enables us to give more transparent and aesthetically pleasing statements. To give an example note that if $a \in \mathcal{H}$ and $b \in \mathcal{LE}$, then there exists a Hardy field that contains both a and b. As a consequence, the limit $\lim_{t\to\infty} a'(t)/b'(t)$ exists (it may be infinite), and so assuming that $a(t), b(t) \to \infty$, we get (using L'Hospital's rule) that the quotients $a(t)/b(t)$ and $a'(t)/b'(t)$ have the same limit as $t \to \infty$. We deduce, for instance, that if $a \in \mathcal{H}$ satisfies $a(t)/t^2 \to \infty$, then $a'(t)/t \to \infty$ and $a''(t) \to \infty$.

Background material on Hardy fields can be found in [73, 74, 75, 179, 180, 266].

6.1. **Powers of a single transformation.** To avoid repetition, we remark that in this subsection we always work with a family $\mathcal{F} := \{a_1(t), \dots, a_\ell(t)\}$ of functions of *polynomial growth* (meaning, for some $k \in \mathbb{N}$ we have $a_i(t)/t^k \to 0$ for $i = 1, \dots, \ell$) that belong to the same Hardy field. With $\text{span}^*(\mathcal{F})$ we denote the set of all *non-trivial* linear combinations of elements of $\mathcal{F}$.

6.1.1. *Necessary and sufficient conditions for ℓ-convergence.* We first state two problems from [132] related to the mean convergence of multiple ergodic averages involving iterates given by Hardy sequences. The next result was proved in [132] (the case $\ell = 1$ was first handled in [78]):

Theorem. *Let $a \in \mathcal{H}$ have polynomial growth. Then the sequence $([a(n)])$ is good for multiple convergence of powers if and only if one of the following conditions is satisfied:*

- $|a(t) - cp(t)|/\log t \to \infty$ *for every* $c \in \mathbb{R}$ *and* $p \in \mathbb{Z}[t]$*; or*
- $a(t) - cp(t) \to d$ *for some* $c, d \in \mathbb{R}$*; or*
- $|a(t) - t/m| \leq C \log t$ *for some* $m \in \mathbb{Z}$ *and* $C > 0$.

For instance, the sequences (n^2), $([n^{3/2}])$, $([n \log n])$, $([n^2 + (\log n)^2])$, $([n^2 + n\sqrt{2} + \log\log n])$ are all good for multiple convergence of powers, but the sequences $([n^2 + \log n])$, $([n^2\sqrt{2} + \log\log n])$ are not good for 1-convergence. Unlike the case of polynomial sequences, if $a \in \mathcal{H}$ satisfies $a(t)/t^{k-1} \to \infty$ and $a(t)/t^k \to 0$ for some $k \in \mathbb{N}$, then it can be shown that the sequence $([a(n)])$ takes odd (respectively even) values in arbitrarily large intervals. As a consequence, when T is the rotation by $1/2$ on the circle and $f = \mathbf{1}_{[0,1/2]}$, the $L^2(\mu)$-limit $\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} T^{[a(n)]} f$ does not exist for some appropriately chosen Følner sequence (Φ_N) of subsets of $\mathbb{N}$.

The next problem seeks to give similar necessary and sufficient conditions for ℓ-convergence of arbitrary collections of sequences arising from functions of polynomial growth that belong to the same Hardy field. We remind the reader that in such circumstances one is seeking to prove mean convergence for averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{[a_1(n)]} f_1 \cdots T^{[a_\ell(n)]} f_\ell \tag{20}$$

for all systems $(X, \mathcal{X}, \mu, T)$ and functions $f_1, \dots, f_\ell \in L^\infty(\mu)$.

Problem 22. *Let $\mathcal{F}$ be as above. Show that the family of sequences $\{([a_1(n)]), \dots, ([a_\ell(n)])\}$ is good for ℓ-convergence of a single transformation if and only if every function $a \in span^*(\mathcal{F})$ satisfies one of the following conditions:*

- $|a(t) - cp(t)|/\log t \to \infty$ *for every* $c \in \mathbb{R}$ *and* $p \in \mathbb{Z}[t]$*; or*
- $a(t) - cp(t) \to d$ *for some* $c, d \in \mathbb{R}$*; or*
- $|a(t) - t/m| \leq C \log t$ *for some non-zero* $m \in \mathbb{Z}$ *and* $C > 0$.

Mean convergence of the averages (20) was proved in [132] but under much more restrictive conditions than those advertised here. The problem is open even when one assumes that the functions $a_1, \dots, a_\ell$ are polynomials with fractional powers. Also the collection of sequences $\{([n \log n]), ([n^2 \log n]), \dots, ([n^\ell \log n])\}$ is another explicit example that is expected to be good for ℓ-convergence of a single transformation but this is not known yet (not even for all weak mixing systems, or all nilsystems).

6.1.2. *Convergence to the product of the integrals.* When the multiple ergodic averages of a collection of Hardy sequences of polynomial growth converge in the mean, one would like to have an explicit formula for their limit. In general, such a limit formula can be extremely complicated, but when the sequences are in "general position" the limit is expected to be very simple:

Problem 23. *Let $\mathcal{F}$ be as above and suppose that for every function $a \in span^*(\mathcal{F})$ we have $|a(t) - cp(t)|/\log t \to \infty$ for every $c \in \mathbb{R}$ and $p \in \mathbb{Z}[t]$. Show that for every ergodic system $(X, \mathcal{B}, \mu, T)$ and functions $f_1, \dots, f_\ell \in L^\infty(\mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{[a_1(n)]} f_1 \cdots T^{[a_\ell(n)]} f_\ell = \int f_1 \, d\mu \cdots \int f_\ell \, d\mu \tag{21}$$

where the convergence takes place in $L^2(\mu)$.

The limit formula is known when $a_i(t) = t^{c_i}$, $i = 1, \dots, \ell$, where $c_1, \dots, c_\ell \in \mathbb{R} \setminus \mathbb{Z}$ are distinct and positive [132] (this was established first in [45] when $c_i \in (0, 1)$, or when the system is weak mixing). More generally, identity (21) is established in [132] when the functions $a_1, \dots, a_\ell$ and their pairwise differences belong to the set $\mathcal{LE} \cap \{a\colon a(t)/t^{k+\varepsilon} \to \infty, a(t)/t^{k+1} \to 0, \text{ for some } k \geq 0 \text{ and } \varepsilon > 0\}$ and $a_i(t)/a_j(t) \to 0$ or ∞ as $t \to \infty$ for all $i \neq j$. Problem 23 is open even when one assumes that the functions $a_1, \dots, a_\ell$ are polynomials with fractional powers. If some function $a \in \text{span}(\mathcal{F})$ satisfies the estimate $|a(t) - cp(t)| \leq C \log t$ for some $c \in \mathbb{R}$, $C > 0$, and $p \in \mathbb{Z}[t]$ with $\deg(p) \geq 2$, then one easily sees that (21) fails for T given by an appropriate rotation on $\mathbb{T}^\ell$.

An intermediate step that would help solve the previous two problems is to find suitable characteristic factors for the relevant multiple ergodic averages: If $\mathcal{F}$ is as above, $a_i(t)/\log t \to \infty$, and $(a_i(t) - a_j(t))/\log t \to \infty$ whenever $i \neq j$, then for large enough $d \in \mathbb{N}$ the factor $\mathcal{Z}_{d,T}$ is expected to be characteristic for mean convergence of the averages (20). This is known when for some $\varepsilon > 0$ we have $a_i(t)/t^\varepsilon \to \infty$ and $(a_i(t) - a_j(t))/t^\varepsilon \to \infty$ whenever $i \neq j$ [132], and the methods of [132] (see the proof of Theorem 2.4 there) can be used to show that it also holds when $a_i(t) = ia(t)$ for $i = 1, \dots, \ell$ and $a(t)/\log t \to \infty$.

6.1.3. *Pointwise convergence.* Regarding variants of the identity (21) that deal with pointwise convergence, progress has been very scarce. The case $\ell = 1$ was treated in [78], but other than this, even the simplest cases remain open.

Problem 24. *Let a, b be distinct positive non-integers. Show that for every ergodic system $(X, \mathcal{X}, \mu, T)$ and functions $f, g \in L^\infty(\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{[n^a]}x) \cdot g(T^{[n^b]}x) = \int f \; d\mu \cdot \int g \; d\mu$$

for almost every $x \in X$.

As mentioned before, mean convergence is known [132]. For pointwise convergence all cases where both a and b are greater than 1 are open.

6.1.4. *Necessary conditions for ℓ-recurrence.* Next, we state some problems related to multiple recurrence. The following result was proved in [132] (see also [149] for a special case):

Theorem. *Let $a \in \mathcal{H}$ have polynomial growth and suppose that $|a(t) - cp(t)| \to \infty$ for every $c \in \mathbb{R}$ and $p \in \mathbb{Z}[t]$. Then the sequence $([a(n)])$ is good for multiple recurrence of powers.*

It follows that the sequences $([n^{\sqrt{2}}])$, $([n \log n])$, $([n^2 + (\log n)^2])$, $([n^2 + \log n])$, $([n^2\sqrt{2} + \log\log n])$ $([n^2 + n\sqrt{2}])$ are all good for multiple recurrence of powers.

Next, we seek to give necessary conditions for ℓ-recurrence of collections of sequences arising from functions of polynomial growth that belong to the same Hardy field. We remind the reader that in such circumstances we seek to prove that for every system $(X, \mathcal{X}, \mu, T)$ and $A \in X$ with $\mu(A) > 0$, we have $\mu(A \cap T^{-[a_1(n)]}A \cap \cdots \cap T^{-[a_\ell(n)]}A) > 0$ for some $n \in \mathbb{N}$.

Problem 25. *Let $\mathcal{F}$ be as above and suppose that for every function $a \in span^*(\mathcal{F})$ we have $|a(t) - cp(t)| \to \infty$ for every $c \in \mathbb{R}$ and $p \in \mathbb{Z}[t]$. Show that the collection of sequences $\{([a_1(n)]), \ldots, ([a_\ell(n)])\}$ is good for ℓ-recurrence of a single transformation.*

This is known for $\ell = 1$ [149]. For $\ell = 2$ the problem is open even when one assumes that the sequences are polynomials with fractional powers.

6.1.5. *Hardy sequences of super-polynomial growth.* Despite the fact that multiple recurrence and convergence properties of Hardy sequences of polynomial growth are relatively well understood, when it comes down to sequences that grow faster than polynomials, even the most basic problems are open.

Problem 26. *Find an example of a function $a \in \mathcal{H}$ that grows faster than polynomials, meaning, $a(t)/t^k \to \infty$ for every $k \in \mathbb{N}$, such that the sequence $[a(n)]$ is good for multiple recurrence and convergence of powers.*

The sequences $([n^{(\log n)^a}])$, $([e^{n^b}])$, where $a > 0$ and $b \in (0, 1)$, seem to be natural candidates; unfortunately they are extremely hard to work with. Even when $\ell = 1$, the relevant exponential sum estimates needed to prove convergence appear to be out of reach in most cases; for the first sequence such estimates are available only when $a \in (0, 1/2)$ [205], and no estimates are available for the second sequence. On the other hand, a slower growing sequence, like the sequence $([n^{\log\log n}])$ may be easier to handle. But even for this sequence, 2-recurrence and 2-convergence is not known for all weak mixing systems or all nilsystems.

6.1.6. *Hardy sequences evaluated at the primes.* With p_n we denote the n-th prime.

Problem 27. *Let c be a positive non-integer. Show that the sequence $([p_n^c])$ is good for multiple recurrence and convergence of powers.*

Proving multiple recurrence is known when $c < 1$ since in this case the range of the sequence $([p_n^c])$ misses at most finitely many positive integer values. It is known that if c is a positive non-integer, then the sequence of fractional parts $(\{p_n^c\})$ is equidistributed in $[0, 1]$ (see [274] or [295] for $c < 1$ and [238] for $c > 1$). Probably the techniques used to prove these equidistribution results also gives 1-recurrence and 1-convergence (it suffices to show that the sequence $(\{p_n^c \alpha\})$ is equidistributed in the unit interval for every non-zero $\alpha \in \mathbb{R}$), but the problem is open for ℓ-recurrence and ℓ-convergence when $\ell \geq 2$.

6.1.7. *Oscillatory sequences.* All the previous problems deal with sequences that do not oscillate. Multiple recurrence and convergence properties of oscillatory sequences are not well studied and even analyzing some simple looking sequences leads to very challenging problems:

Problem 28. *Show that the sequence* $([n \sin n])$ *is good for multiple recurrence and convergence of powers.*

Quite likely one can say more; the averages $\frac{1}{N}\sum_{n=1}^{N} T^{a(n)}f_1 \cdot T^{2a(n)}f_2 \cdot \ldots \cdot T^{\ell a(n)}f_\ell$ have the same limit in $L^2(\mu)$ when $a(n) = [n \sin n]$ and when $a(n) = n$. This is known for $\ell = 1$; and follows from equidistribution results in [34] (see also related results in [32, 33]). The problem has not been studied for $\ell \geq 2$, even for particular classes of measure preserving systems, like nilsystems or weakly mixing systems.

6.2. **Commuting transformations.** As mentioned before, if a Hardy sequence has polynomial growth and stays away from constant multiples of integer polynomials, then it is going to be good for multiple recurrence and convergence of powers. The next problem seeks to extend these results to the case of commuting transformations.

Problem 29. *Show that if* $c > 1$ *is not an integer, then the sequence* $([n^c])$ *is good for multiple recurrence and convergence of commuting transformations. Moreover, show that if* $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ *is a system and* $f_1, \ldots, f_\ell \in L^\infty(\mu)$*, then the* $L^2(\mu)$*-limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T_1^{[n^c]}f_1 \cdots T_\ell^{[n^c]}f_\ell \tag{22}$$

is equal to the $L^2(\mu)$*-limit* $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 \cdots T_\ell^n f_\ell$.

Both the recurrence and the convergence problem is open even when $\ell = 2$ and all transformations are assumed to be weak mixing. The result is known when the $T_1, \ldots, T_\ell$ are powers of the same transformation [132], but the proof relies crucially on the precise algebraic structure of suitable characteristic factors for the corresponding multiple ergodic averages; an advantage that we do not have when we work with general commuting transformations. Moreover, the methods used in [290] to prove mean convergence when the iterates are polynomial, seem hard to adjust in order to deal with fractional powers; in any case, they give no information about the limit and so do not seem suitable for proving multiple recurrence results or obtaining limit formulas.

6.3. **Configurations in the primes.** The theorem of Szemerédi on arithmetic progressions [276], and its polynomial extension [52], have been instrumental in proving that the primes contain arbitrarily long arithmetic progressions [167] and polynomial progressions [284]. Thus, it is natural to expect that the various known Hardy field extensions of the theorem of Szemerédi [132, 149] can be used to prove that the primes contain the corresponding Hardy field patterns. We mention a relevant problem:

Problem 30. *Let* $\ell \in \mathbb{N}$ *and* $c, c_1, \ldots, c_\ell$ *be positive real numbers. Show that the prime numbers contain patterns of the form*

$$\{m, m+[n^c], m+2[n^c], \ldots, m+\ell[n^c]\} \quad \text{and} \quad \{m, m+[n^{c_1}], \ldots, m+[n^{c_\ell}]\}$$

for infinitely many $n \in \mathbb{N}$.

When all exponents are rational the existence of such patterns follows immediately from [284].

7. Problems related to random sequences

In this section we give a list of problems related to the study of multiple ergodic averages involving iterates given by random sequences of integers.

The random sequences that we work with are constructed by selecting a positive integer n to be a member of our sequence with probability $\sigma_n \in [0,1]$. More precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent $0-1$ valued random variables with

$$\mathbb{P}(X_n = 1) := \sigma_n \ \text{ and } \ \mathbb{P}(X_n = 0) := 1 - \sigma_n$$

where σ_n is a decreasing sequence of positive real numbers that satisfies $\sum_{n=1}^{\infty} \sigma_n = \infty$ (in which case $\sum_{n=1}^{\infty} X_n(\omega) = +\infty$ almost surely). The random sequence $(a_n(\omega))_{n\in\mathbb{N}}$ is constructed by taking the positive integers n for which $X_n(\omega) = 1$ in increasing order. Equivalently, $a_n(\omega)$ is the smallest $k \in \mathbb{N}$ such that $X_1(\omega) + \cdots + X_k(\omega) = n$. If $\sigma_n = n^{-a}$ for some $a \in (0,1)$, then one can show that almost surely $a_n(\omega)/n^{1/(1-a)}$ converges to a non-zero constant. On the other hand, if $\sigma_n = 1/n$, then almost surely there exists a subsequence (n_k) of the integers, of density arbitrarily close to one, such that the sequence $(a_{n_k}(\omega))$ is lacunary [201] (this is no longer the case if $n\sigma_n \to \infty$). So it makes sense to call *random non-lacunary sequences* the random sequences of integers one gets when σ_n satisfies $n\sigma_n \to \infty$.

We say that a property holds almost surely for the sequences $(a_n(\omega))$, if there exists a universal set $\Omega_0 \in \mathcal{F}$, such that $\mathbb{P}(\Omega_0) = 1$, and for every $\omega \in \Omega_0$ the sequence $(a_n(\omega))$ satisfies the given property.

The next result was proved by M. Boshernitzan [74] for mean convergence and by J. Bourgain [80] for pointwise convergence (see also [265] for a nice exposition of these results).

Theorem. *If $n\sigma_n \to \infty$, then almost surely the following holds: For every system $(X, \mathcal{X}, \mu, T)$ and function $f \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T^{a_n(\omega)} f \tag{23}$$

converge in the mean and their limit equals the $L^2(\mu)$-limit of the averages $\frac{1}{N}\sum_{n=1}^{N} T^n f$. Furthermore, if $n\sigma_n/(\log\log n)^{1+\delta} \to \infty$ for some $\delta > 0$, then the conclusion also holds pointwise.

It is known that the mean convergence result fails if $\sigma_n = 1/n$ and the pointwise convergence result fails if $\sigma_n = (\log\log n)^{1/3}/n$ (for both results see [201]). It is not known whether the pointwise convergence result fails when say $\sigma_n = \log\log n/n$.

One would naturally like to extend the previous convergence result to multiple ergodic averages:

Problem 31. *Suppose that $n\sigma_n \to \infty$. Show that almost surely the sequence $(a_n(\omega))$ is good for multiple recurrence and convergence of commuting transformations. Moreover, show that almost surely the following holds: For every system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and functions $f_1, \ldots, f_\ell \in L^\infty(\mu)$, the averages*

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{a_n(\omega)} f_1 \cdots T_\ell^{a_n(\omega)} f_\ell$$

converge in $L^2(\mu)$ and their limit equals the limit of the averages $\frac{1}{N}\sum_{n=1}^{N} T_1^n f_1 \cdots T_\ell^n f_\ell$.

This problem was first mentioned in 2004 by M. Wierdl in [207]. For $\ell \geq 2$ the result is known when $\sigma_n = n^{-a}$ where $a \in (0, 1/2^{\ell-1})$ [146] (the argument in [91] also gives this). When

$T_1, \ldots, T_\ell$ are powers of the same transformation, a different argument was given in [147] that works for all $a \in (0, 1/(\ell+1))$ (this result is superior to the previous one for $\ell > 3$). Under the assumption that $n\sigma_n \to \infty$, pointwise convergence is known when the transformations are nilrotations and the functions are continuous [131]. On the other hand, when $\sigma_n = n^{-a}$ for some $a > 1/2$, $\ell = 2$, and $T_2 = T_1^2$, mean convergence is not known even when we restrict ourselves to the class of weak mixing systems.

A solution to Problem 31 would imply that almost surely the following holds: For every $d, \ell \in \mathbb{N}$, every $\mathbf{v_1}, \ldots, \mathbf{v}_\ell \in \mathbb{Z}^d$, every set $\Lambda \subset \mathbb{Z}^d$ with $\bar{d}(\Lambda) > 0$ contains patterns of the form

$$\{\mathbf{m}, \mathbf{m} + a_n(\omega)\mathbf{v_1}, \ldots, \mathbf{m} + a_n(\omega)\mathbf{v}_\ell\}$$

for some $\mathbf{m} \in \mathbb{Z}^d$ and $n \in \mathbb{N}$.

Next we mention a problem that combines deterministic and random iterates.

Problem 32. *Suppose that $n\sigma_n \to \infty$. Show that almost surely the following holds: For every system $(X, \mathcal{X}, \mu, T, S)$ and functions $f, g \in L^\infty(\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f \cdot S^{a_n(\omega)} g = \mathbb{E}(f|\mathcal{I}_T) \cdot \mathbb{E}(g|\mathcal{I}_S)$$

where the limit is taken in $L^2(\mu)$. Furthermore, if $\sigma_n = n^{-a}$ for some $a \in (0,1)$, show that the convergence also holds pointwise almost everywhere.

The limit formula (in $L^2(\mu)$ and pointwise) is only known when $a \in (0, 1/14)$ [146]. When $T = S$, using a different method, the eligible range was improved to $a \in (0, 1/2)$ in [147].

In the next problem we are going to work with two random sequences with different growth rates that are chosen independently of each other. More precisely, let $X_1, Y_1, X_2, Y_2, \ldots$ be a sequence of independent 0-1 valued random variables with $\mathbb{P}(X_n = 1) := n^{-a}$ and $\mathbb{P}(Y_n = 1) := n^{-b}$ for some $a, b \in (0,1)$. We construct the random sequence $(a_n(\omega))$ by taking the positive integers n for which $X_n(\omega) = 1$ in increasing order, and the random sequence $(b_n(\omega))$ by taking the positive integers n for which $Y_n(\omega) = 1$ in increasing order.

Problem 33. *Suppose that $a, b \in (0,1)$ and $a \neq b$. Show that almost surely the following holds: For every system $(X, \mathcal{X}, \mu, T, S)$ and functions $f, g \in L^\infty(\mu)$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{a_n(\omega)} f \cdot S^{b_n(\omega)} g = \mathbb{E}(f|\mathcal{I}_T) \cdot \mathbb{E}(g|\mathcal{I}_S)$$

where the limit is taken in $L^2(\mu)$ or pointwise.

The problem seems non-trivial even when $T = S$ is a weak mixing transformation. Moreover, no values of $a, b \in (0,1)$ are known for which the conclusion holds.

8. Problems related to systems with multiplicative structure

Let $T_n, n \in \mathbb{Z}$, be invertible measure preserving transformations acting on a probability space $(X, \mathcal{X}, \mu)$ that satisfy $T_0 = T_1 = \mathrm{id}$ and $T_m \circ T_n = T_{mn}$ for every $m, n \in \mathbb{N}$. We say that the quadruple $(X, \mathcal{X}, \mu, T_n)$ is a *measure preserving system with multiplicative structure.* Motivated by partition regularity problems of quadratic equations in three variables, in [135] multiple recurrence properties of systems with multiplicative structure were studied. A sample result is the following:

Theorem. *Let $(X, \mathcal{X}, \mu, T_n)$ be a measure preserving system with multiplicative structure and $A \in \mathcal{X}$ with $\mu(A) > 0$. Then there exist $m, n \in \mathbb{N}$ such that*

$$\mu\big(T_{m(m+n)}A \cap T_{(m+2n)(m+3n)}A\big) > 0. \tag{24}$$

Using variants of the previous result one can prove that for every partition of $\mathbb{N}$ into finitely many cells, there exist distinct x and y belonging to the same cell such that $9x^2 + 16y^2 = \lambda^2$ for some $\lambda \in \mathbb{N}$. In fact, our result proves density regularity, namely, that such solutions can be found on every subset of $\mathbb{N}$ that has positive density with respect to any dilation invariant density on the integers. In order to prove a similar statement for the equation $x^2 + y^2 = \lambda^2$ (i.e., prove partition regularity of Pythagorean pairs) it suffices to get a positive answer for the following variant of the previous result:

Problem 34. *Let $(X, \mathcal{X}, \mu, T_n)$ be a measure preserving system with multiplicative structure and $A \in \mathcal{X}$ with $\mu(A) > 0$. Is it true that there exist $m, n \in \mathbb{N}$, $m > n$, such that*

$$\mu\big(T_{2mn}A \cap T_{(m-n)(m+n)}A\big) > 0\,?$$

In order to prove (24) it turns out to be useful to analyse the limiting behavior of averages of the form

$$\frac{1}{N^2} \sum_{1 \le m,n \le N} \phi\big(m(m+n)\big)\, \overline{\phi}\big((m+2n)(m+3n)\big)$$

where $\phi\colon \mathbb{N} \to \mathbb{C}$ is an arbitrary completely multiplicative function (i.e. satisfies $\phi(mn) = \phi(m)\phi(n)$ for all $m, n \in \mathbb{N}$) with modulus 1. A key technical point in the proof is that the expressions $\phi\big(m(m+n)\big)\, \overline{\phi}\big((m+2n)(m+3n)\big)$ are real and non-negative when $n = 0$; this is an advantage that we do not have when we deal with similar expressions related to Problem 34.

It would also be interesting to prove higher order multiple recurrence results for systems of multiplicative structure. The following problem is a typical one:

Problem 35. *Let $(X, \mathcal{X}, \mu, T_n)$ be a measure preserving system with multiplicative structure and $A \in \mathcal{X}$ with $\mu(A) > 0$. Is it true that there exist $m, n \in \mathbb{N}$ such that*

$$\mu\big(T_{m(m+n)}A \cap T_{(m+2n)(m+3n)} \cap T_{(m+4n)(m+5n)}A\big) > 0\,?$$

If the answer is positive, then this will imply partition regularity (in fact, density regularity with respect to any dilation invariant density on the integers) for some non-trivial quadratic equation in three variables with all three variables belonging to the same partition cell. A fundamental new difficulty is that although the single correlation sequences $\mu(T_rA \cap T_sA)$ can be expressed as integral combinations of sequences of the form $\phi(r)\,\overline{\phi}(s)$ where ϕ is a completely multiplicative function, this is no longer the case for the higher order correlation sequences $\mu(T_rA \cap T_sA \cap T_tA)$. So one has to take an alternate approach.

9. Extended bibliography organized by topic

We give a rather extensive bibliography with material that is directly related to the problems discussed before, organized by topic. We caution the reader that this is *not* a comprehensive list of articles in ergodic Ramsey theory, and in fact articles in several important topics in this area are missing from this list. For instance, the reader will find very few articles related to actions of measure preserving transformations for groups other than $\mathbb{Z}^d$, the richness of return times in various multiple recurrence results, topological dynamics, and applications in partition Ramsey theory. There are several excellent places to look for such topics, for instance, the survey articles of V. Bergelson [37, 40, 41] cover a substantial part of related material and contain an extensive bibliography up to 2006.

Linear sequences: [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 22, 25, 27, 28, 29, 30, 31, 38, 43, 47, 49, 53, 54, 70, 76, 77, 86, 93, 95, 98, 99, 100, 101, 104, 105, 106, 107, 112, 113, 118, 119, 121, 125, 127, 128, 129, 142, 151, 158, 159, 161, 162, 172, 174, 181, 182, 183, 184, 185, 189, 191, 194, 197, 198, 199, 209, 226, 240, 243, 244, 248, 252, 253, 264, 268, 269, 270, 277, 280, 282, 283, 286, 288, 297, 298, 300, 301, 302, 304, 305, 306].

Polynomial sequences: [21, 23, 24, 35, 42, 44, 48, 52, 56, 58, 59, 60, 63, 64, 65, 68, 69, 71, 80, 81, 82, 83, 84, 85, 87, 92, 96, 97, 108, 124, 126, 130, 132, 134, 135, 136, 137, 140, 141, 143, 145, 150, 163, 173, 178, 186, 193, 200, 208, 211, 212, 224, 229, 236, 237, 239, 245, 247, 254, 261, 287, 290].

Other sequences (rational, smooth, random, prime numbers, generalized polynomials): [44, 45, 50, 51, 55, 61, 62, 66, 67, 74, 78, 79, 91, 116, 120, 123, 132, 133, 138, 139, 146, 147, 149, 167, 168, 201, 213, 221, 222, 235, 250, 251, 255, 275, 284, 294, 296, 307].

Equidistribution on nilmanifolds and other nil-stuff: [17, 57, 88, 89, 90, 110, 109, 122, 131, 169, 170, 171, 175, 176, 177, 187, 188, 190, 192, 195, 196, 210, 225, 227, 228, 230, 231, 232, 233, 234, 235, 241, 242, 246, 256, 257, 258, 259, 271, 272, 273, 285, 289, 299].

Books, survey articles, and thesis on related topics: [1, 6, 17, 20, 36, 37, 39, 40, 41, 94, 102, 103, 111, 115, 117, 148, 152, 153, 154, 155, 156, 157, 164, 179, 202, 203, 207, 214, 215, 216, 217, 218, 220, 223, 246, 249, 260, 262, 265, 267, 278, 279, 281, 291, 292, 293, 303].

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