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GUINAND'S MEASURES ARE ALMOST PERIODIC DISTRIBUTIONS

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Abstract. A.P. Guinand discovered new Poisson's summation formulae which are not given by lattice Dirac combs. These formulae are provided by atomic measures μ whose support and spectrum are locally finite sets. Guinand's measures are almost periodic distributions in a sense which is discussed in this note.

1. Definitions and notations

A weighted Dirac comb is a locally finite sum $\sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ where $c(\lambda)$ are real or complex coefficients δ_λ is the Dirac measure located at λ. A lattice Dirac comb is a sum $\mu = \sum_{\gamma\in\Gamma}\delta_\gamma$ of Dirac measures δ_γ on a lattice $\Gamma \subset \mathbb{R}^n$. The Fourier transform of the lattice Dirac comb on a lattice Γ is, up to a constant factor, the lattice Dirac comb on the dual lattice Γ^*. This is the *standard Poisson summation formula* which plays a seminal role in X-ray crystallography and molecular biology. Nir Lev and Alexander Olevskii proved the existence of *exotic Poisson summation formulae*. Soon after this discovery some surprisingly simple examples were found [7]. These examples are rooted in an old and almost forgotten paper by A.P. Guinand [2]. This explains why the corresponding measures will be named Guinand's measures. Guinand measures are almost periodic distributions as it will be proved below. They are not almost periodic measures. Some weak limits of translated Guinand's measures are computed in this note and the surprising results show how much Guinand's differ from almost periodic measures.

The Fourier transform $\mathcal{F}(f) = \widehat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by $\widehat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi\, ix\cdot y)f(x)\,dx$. This extends to tempered distributions. The space of tempered distributions is denoted $\mathcal{S}'(\mathbb{R}^n)$ and the corresponding space of test functions is $\mathcal{S}(\mathbb{R}^n)$.

Definition 1.1. *A set of points $\Lambda \subset \mathbb{R}^n$ is locally finite if $\Lambda \cap B$ is finite for every compact set B.*

Definition 1.2. *A generalized lattice Dirac comb is a sum $\mu = \mu_1 + \cdots + \mu_N$ where*
(a) $\mu_j = g_j\,\sigma_j,\ 1 \le j \le N,$
(b) *σ_j is a lattice Dirac comb supported by a coset $x_j + \Gamma_j$ of a lattice $\Gamma_j \subset \mathbb{R}^n$*
(c) *$g_j(x) = \sum_{k\in F_j} c(j,k)\exp(2\pi\, i\,\omega_{j,k}\, x)$ is a finite trigonometric sum.*

The Fourier transform of a generalized lattice Dirac comb is a generalized lattice Dirac comb.

Definition 1.3. *A purely atomic signed measure μ on $\mathbb{R}^n$ is a crystalline measure if*
(a) *the support Λ of μ is a locally finite set*
(b) *μ is a tempered distribution*

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(c) *the distributional Fourier transform $\widehat{\mu}$ of μ is also a purely atomic measure which is supported by a locally finite set S (the spectrum of μ).*

This definition was proposed by the author in [7]. Crystalline measures generalize lattice Dirac combs (true crystals) which explains the word "crystalline". We are interested in exotic crystalline measures (crystalline measures which are not generalized lattice Dirac combs).

Let μ be a crystalline measure. We then have

$$\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda, \quad \widehat{\mu} = \sum_{y\in S} b(y)\delta_y \tag{1}$$

It implies the following generalized Poisson summation formula

$$\sum_{\lambda\in\Lambda} a(\lambda)\widehat{f}(\lambda) = \sum_{y\in S} b(y)f(y) \tag{2}$$

which holds for every $f \in \mathcal{S}(\mathbb{R}^n)$.

The measure $\widehat{\mu}$ is not the diffraction pattern of the measure μ. The diffraction pattern of μ is $\sum_{y\in S} |b(y)|^2\delta_y$. The vector space consisting of all crystalline measures is strictly contained in the set of weighted Dirac combs whose diffraction pattern is also a weighted Dirac comb (pure point diffraction spectrum).

If Λ is a "model set" in the sense of [8] and if Λ is not a lattice, then the Fourier transform of $\mu = \sum_{\lambda\in\Lambda}\delta_\lambda$ is a tempered distribution which is not a measure. In full contrast the diffraction pattern of μ is an atomic measure with a dense support [9]. Using the cut and projection construction of model sets one can find some weights $c(\lambda) \in [0, 1]$ such that the Fourier transform $\widehat{\mu}$ of the measure $\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ is also an atomic measure. However in this construction the support of the atomic measure $\widehat{\mu}$ is necessarily dense in $\mathbb{R}^n$. Therefore μ cannot be a crystalline measure. This was explained by N. Lev and A. Olevskii. They proved that the support of an exotic crystalline measure cannot be contained in a model set [6] (see also [10]). That explains why N. Lev and A. Olevskii did not use a single model set but a ladder of model sets to construct exotic crystalline measures in [3].

On the other hand N. Lev and A. Olevskii conjectured that a crystalline measure μ with uniformly discrete support and spectrum is a generalized lattice Dirac comb. They proved this property in one dimension. They also proved it in any dimension if μ is non negative. The general case is still open [5] since P.D. Palamodov withdrew his paper entitled "Uniformly discrete quasi-crystals are crystals" from the arXiv.

A set of points $\Lambda \subset \mathbb{R}^n$ is uniformly discrete if

(3). $$\inf_{\{\lambda,\,\lambda'\in\Lambda,\,\lambda'\neq\lambda\}} |\lambda' - \lambda| = \beta > 0$$

2. Almost periodic distributions

A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is almost periodic if for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ the convolution product $f * \phi$ is an almost periodic function in the sense of H. Bohr. The definition of an almost periodic measure is more demanding.

Definition 2.1. *A Borel measure μ on $\mathbb{R}^n$ is an almost periodic measure if for every compactly supported continuous function ϕ the convolution product $\mu \star \phi$ is an almost periodic function in the sense of Bohr.*

Lemma 2.1. *An almost periodic measure μ is translation bounded: if $B \subset \mathbb{R}^n$ denotes the unit ball there exists a finite constant C such that*

$$\sup_{x\in\mathbb{R}^n} |\mu|(B+x) \le C \tag{4}$$

Conversely a translation bounded measure which is an almost periodic distribution is an almost periodic measure.

Let $\mathcal{C}_0([-T,T]^n)$ be the Banach space consisting of continuous functions supported by the cube $[-T,T]^n$ and $\mathcal{A}P$ be the Banach space of almost periodic functions in the sense of Bohr. If μ is an almost periodic measure for every $T \ge 1$ the map $G_T : f \mapsto \mu * f$ is continuous from $\mathcal{C}_0([-T,T]^n)$ to $\mathcal{A}P$. This obviously implies (4). The converse implication is easy. It suffices to observe that the space of test functions $\mathcal{C}_0^\infty([-T,T]^n)$ is dense in $\mathcal{C}_0([-T,T]^n)$.□

Guinand's measure is not translation bounded as will be proved below. But Guinand's measure is an almost periodic distribution. We conjecture that this is a general fact.

Conjecture 2.1. *Every crystalline measure is an almost periodic distribution.*

This was stated as a theorem in [7]. However the proof which was given there is incorrect.

Definition 2.2. *If S is a locally finite set of real numbers we denote by $\mathcal{S}'_S$ the translation invariant space consisting of all tempered distributions S whose Fourier transform is a sum of Dirac measures on S. The orbit of $f \in \mathcal{S}'_S$ is the collection of translated $f(\cdot - y)$, $y \in \mathbb{R}$, of f.*

Theorem 2.1. *Let $S = \{\pm\sqrt{n},\, n \in \mathbb{N}\}$. Then every $f \in \mathcal{S}'_S$ is an almost periodic distribution.*

Let $s_k = \pm\sqrt{|k|}$, $k \in \mathbb{Z}$, where $\pm$ is the sign of k. We have

$$\widehat{f} = \sum_{k\in\mathbb{Z}} a_k \delta_{s_k} \tag{5}$$

Then

$$|a_k| \le C(1+|k|)^N \tag{6}$$

for some exponent N and a constant C. For proving this estimate one computes $\langle \widehat{f}, \phi_k\rangle$ where ϕ_k is a smooth bump function supported by the interval $[s_{k-1}, s_{k+1}]$ and such that $\phi_k(s_k) = 1$. The $m-$th derivative of ϕ_k satisfies

$$\|\phi_k^{(m)}\|_\infty = O(|k|^{m/2}) \tag{7}$$

Then (6) follows from (7) and from the definition of a distribution. Finally for every test function $\phi \in \mathcal{S}(\mathbb{R})$ the Fourier transform of the convolution product $f * \phi$ is the product $\widehat{f}\widehat{\phi} = \sum_{k\in\mathbb{Z}} a_k \widehat{\phi}(s_k)\delta_{s_k}$. This an atomic measure with a finite total mass. Therefore $f * \phi$ is almost periodic in the sense of Bohr. However Theorem 2.1 is no longer valid if S is replaced by an arbitrary locally finite set (see Theorem 2.2 below). This is the mistake we made in [7]. As was stressed by the Referee Theorem 2.1 does not depend on the arithmetic structure of S but only on the two following properties: $s_{k+1} - s_k \ge C(|k|+1)^{-N}$ for some constant C and some exponent N and $|a_k| \le C(|k|+1)^N$.

Definition 2.3. *A distribution $f \in \mathcal{S}'_S$ is translation bounded if its orbit is a bounded subset $\mathcal{B} \subset \mathcal{S}'(\mathbb{R})$.*

Here is another result.

Theorem 2.2. *Let $\alpha > 0$ be an irrational number and $S = \mathbb{Z} \cup (\alpha\mathbb{Z})$. Then the four following properties are equivalent:*

(*a*) *There exists an exponent $q \geq 1$ and a constant $C > 0$ such that $|\exp(2\pi i m\alpha) - 1| \geq Cm^{-q}$, $m \in \mathbb{N}$.*
(*b*) *Every tempered distribution $f \in \mathcal{S}'_S$ is an almost periodic distribution.*
(*c*) *Every tempered distribution $f \in \mathcal{S}'_S$ is a sum $f = u + v$ where u is a periodic distribution of period 1 and v is periodic of period $1/\alpha$.*
(*d*) *The orbit of every tempered distribution $f \in \mathcal{S}'_S$ is bounded in $\mathcal{S}'(\mathbb{R})$.*

Before beginning the proof of Theorem 2.2 let us treat the case of almost periodic functions in the sense of Bohr.

Lemma 2.2. *If F is an almost periodic function in the sense of Bohr and if the spectrum of F is contained in $S = \mathbb{Z} \cup (\alpha\mathbb{Z})$ then $F = U + V$ where U is a continuous function which is periodic of period 1 and V is a continuous function which is periodic of period $1/\alpha$. Moreover $\|U\|_\infty \leq \|F\|_\infty$.*

This is trivial if the spectrum of F is a finite subset of $\mathbb{Z} \cup (\alpha\mathbb{Z})$. The general case follows from the following observation. Let ϕ be a non negative function in the Schwartz class such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and whose Fourier transform is compactly supported. Let $\phi_N(x) = N\phi(Nx)$. Since F is uniformly continuous the convolution product $F * \phi_N$ uniformly converges to F as N tends to infinity. We are reduced to the first case. Moreover $U(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} F(x+k)$ and the limit is uniform on the line.

The implication $(a) \Rightarrow (b)$ can be proved by the argument which was used in Theorem 2.1. We give a simpler proof of the stronger assertion $(a) \Rightarrow (c)$. If $\widehat{f} = \sum_{s\in S} b_s\delta_s$ then $g(x) = f(x + 1/\alpha) - f(x)$ is a tempered distribution which is periodic of period 1. Indeed the Fourier transform of g is $\widehat{g}(\xi) = (\exp(-2\pi i\xi/\alpha) - 1)\widehat{f}(\xi)$ which is supported by $\mathbb{Z}$. Moreover $\widehat{g}(0) = 0$. Since α is not a Liouville number there exists a 1−periodic tempered distribution u such that $g(x) = u(x + 1/\alpha) - u(x)$. Finally $f - u = v$ is a tempered distribution which is periodic of period $1/\alpha$ by construction.

The implication $(c) \Rightarrow (b)$ is obvious and we now prove $(b) \Rightarrow (a)$ by contradiction. If x is an irrational number we denote by $[x]$ the nearest integer. Let us assume that α is a Liouville number and let us construct a tempered distribution ω which is not an almost periodic distribution but whose Fourier transform is a sum of Dirac measures on S. Here is the construction. There exists a sequence of integers $n_j \to \infty$ such that $n_{j+1} \geq 4n_j$ and $|n_j\alpha - [n_j\alpha]| \leq n_j^{-j}$. Let

$$\omega(x) = \sum_{j\in\mathbb{N}} j^{-2} n_j^j \Big(\exp(2\pi i n_j\alpha x) - \exp(2\pi i [n_j\alpha] x) \Big) \tag{8}$$

Since $|\exp(2\pi i n_j\alpha x) - \exp(2\pi i[n_j\alpha]x)| \leq 2\pi n_j^{-j}|x|$ the series (8) converges to a continuous function which is $O(|x|)$ at infinity. We then argue by contradiction and assume that ω is an almost periodic distribution. Then for every $\phi \in \mathcal{S}(\mathbb{R})$ the convolution product $\omega * \phi$ is an almost periodic function. But $\omega * \phi(x) = \sum_{j\in\mathbb{N}} j^{-2} n_j^j \Big(\widehat{\phi}(n_j\alpha) \exp(2\pi i n_j\alpha x) - \widehat{\phi}([n_j\alpha]) \exp(2\pi i[n_j\alpha]x) \Big)$.

Lemma 2.2 implies that

$$\omega_1(x) = \sum_{j\in\mathbb{N}} j^{-2} n_j^j \widehat{\phi}(n_j\alpha) \exp(2\pi i n_j \alpha x) \tag{9}$$

satisfies $\|\omega_1\|_\infty \le \|f * \phi\|_\infty$. But

$$\omega_1(0) = \sum_{j\in\mathbb{N}} j^{-2} n_j^j \widehat{\phi}(n_j\alpha) \tag{10}$$

Therefore $\phi \mapsto \sum_{j\in\mathbb{N}} j^{-2} n_j^j \widehat{\phi}(n_j\alpha)$ is a continuous linear form on $\mathcal{S}(\mathbb{R})$. It follows that

$$\sum_{j\in\mathbb{N}} j^{-2} n_j^j \exp(2\pi i n_j \alpha x)$$

is a tempered distribution. We reach the expected contradiction since the growth of the Fourier coefficients is too fast.

Property (d) is trivial for an periodic distribution f. Indeed for every tempered distribution f the mapping from $\mathbb{R}$ to $\mathcal{S}'$ defined by $y \mapsto f(\cdot - y)$ is continuous. In the periodic case y belongs to a compact interval. Therefore the collection of translated $f(\cdot - y)$, $y \in \mathbb{R}$, of f is compact in $\mathcal{S}'$.

Property (d) seems to be the weakest one. For proving $(d) \Rightarrow (a)$ it suffices to return to the function ω defined by the series (8) and to prove the following lemma:

Lemma 2.3. *If α is a Liouville number the collection $\{\omega(\cdot - y),\ y \in \mathbb{R}\}$ is not bounded in $\mathcal{S}'$.*

We argue by contradiction and assume that the collection $\{\omega(\cdot - y),\ y \in \mathbb{R}\}$ is a bounded set $\mathcal{B} \subset \mathcal{S}'$. We first replace $\mathcal{B}$ by its closed convex hull $\mathcal{B}'$ which is still a bounded subset of $\mathcal{S}'$. For every $\theta \in [0, 1)$ let k_l, $l \in \mathbb{N}$, be a sequence of integers such that $\exp(2\pi i k_l \alpha) = \exp(2\pi i \theta_l) \to \exp(2\pi i\theta)$ as $l \to \infty$. We want to prove the following

Lemma 2.4. *If α is a Liouville number the $\omega(\cdot + k_l)$ is not bounded in $\mathcal{S}'$.*

In the first version of this manuscript it was claimed that the following lemma was obviously true.

Claim 2.1. *The limit of $\omega(x + k_l)$ as $l \to \infty$ is given by*

$$\omega(x, \theta) = \sum_{j\in\mathbb{N}} j^{-2} n_j^j \Big(\exp(2\pi i n_j(\alpha x + \theta) - \exp(2\pi i [n_j\alpha] x) \Big). \tag{11}$$

The Referee observed that this claim was nonsense. Let us proceed however and reach the expected contradiction. The correct proof will easily follow from the wrong one. These functions $\omega(\cdot, \theta)$ are assumed to belong to a bounded set $\mathcal{B} \subset \mathcal{S}'(\mathbb{R})$. It implies that

$$\int_0^1 \omega(x, \theta)\, d\theta = -\sum_{j\in\mathbb{N}} j^{-2} n_j^j \exp(2\pi i [n_j\alpha] x))$$

belongs to $\mathcal{B}'$ which is absurd since this periodic object is not a distribution.

The correct proof is quite similar. We observe that $\omega * \phi_N$ belongs to $\mathcal{B}'$. Indeed this convolution product is an average of translates of ω. Then Claim 2.1 becomes obvious if ω is replaced by $\omega_N = \omega * \phi_N$. We then proceed as above and conclude that $\sum_{j\in\mathbb{N}} j^{-2} n_j^j \exp(2\pi i [n_j\alpha] x) \hat{\phi}([n_j\alpha]/N)$ belongs to $\mathcal{B}'$. It suffices to let N tend to infinity to obtain the required contradiction.

Guinand's measures are constructed in the next sections. Guinand's measures are crystalline measures. They are almost periodic distributions (Theorem 2.1) but cannot be almost periodic measures (Theorem 4.3).

Conjecture 2.2. *A crystalline measure which is an almost periodic measure is a generalized lattice Dirac comb.*

3. Guinand's distribution

In the late fifties A. P. Guinand constructed a remarkable tempered distribution σ enjoying the following two properties: (a) the support of σ is locally finite and (b) $\widehat{\sigma} = -i\,\sigma$ [2]. As it is proved in the next section an exotic crystalline measure can be easily built on Guinand's distribution. We return to Guinand's distribution. Following Guinand let $r_3(n)$ be the number of decompositions of n as a sum of three squares (with $r_3(n) = 0$ if n is not a sum of three squares). More precisely $r_3(n)$ is the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. By Legendre's theorem, an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k+7)$. For instance 0, 1, 2, 3, 4, 5, 6 are sums of three squares but 7 is not. We have $r_3(4n) = r_3(n)$, $\forall n \in \mathbb{N}$, $r_3(0) = 1$, $r_3(1) = 6, r_3(2) = 12, \ldots$. Then $r_3(2^j) = 6$ if j is even and 12 if j is odd. The behavior of $r_3(n)$ as $n \to \infty$ is erratic. The mean behavior is more regular since [3]

$$\sum_{0\leq n\leq x} r_3(n) = \frac{4}{3}\pi x^{3/2} + O(x^{3/4}). \tag{12}$$

It implies

$$\sum_{0\leq n\leq x} r_3(n)/\sqrt{n} = 2\pi x + O(\sqrt{x}). \tag{13}$$

Guinand's distribution is the one dimensional tempered distribution defined by

$$\sigma = \sum_1^\infty r_3(n) n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) - 2\frac{d}{dx}\delta_0 \tag{14}$$

Guinand proved the following result in [2]

Lemma 3.1. *The distributional Fourier transform of σ is $-i\sigma$.*

4. Guinand's measure

Guinand's distribution is not a measure, but a simple modification of σ yields a crystalline measure. If $\alpha \in (0,1)$ we consider

$$\tau_\alpha = (\alpha^2 + 1/\alpha)\sigma - \alpha\sigma(\alpha x) - \sigma(x/\alpha) \tag{15}$$

and Lemma 3.1 implies $\widehat{\tau}_\alpha = -i\tau_\alpha$. On the other hand $\frac{d}{dx}\delta_0$ has been wiped out from the RHS of (14). Therefore τ_α is a crystalline measure. Let us fix $\alpha = 1/2$ and write τ instead of τ_α. Let $\chi : \mathbb{Z} \mapsto \{-1/2, 0, 4\}$ be defined by $\chi = 0$ on $16\,\mathbb{Z}$, $\chi = 4$ on $4\,\mathbb{Z} \setminus 16\,\mathbb{Z}$ and $\chi = -1/2$ on $\mathbb{Z} \setminus 4\,\mathbb{Z}$. Then we have [7]

Theorem 4.1. *The Fourier transform of the one dimensional measure*

$$\tau = \sum_1^\infty \chi(n) r_3(n) n^{-1/2}(\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2}) \tag{16}$$

is $-i\tau$.

The measure τ will be named Guinand's measure.

Guinand's measure appears naturally in another context as the following theorem is showing [1]

Theorem 4.2. *Let* ν *be a real, finitely supported measure on* $\mathbb{T}^3$ *such that*
 (a) 0 *does not belong to the support of* ν
 (b) $\int_{\mathbb{T}^3} d\,\nu = 0.$
Let $u : \mathbb{T}^3 \times \mathbb{R} \mapsto \mathbb{R}$ *be the solution of the Cauchy problem*
 (i) $\frac{\partial^2}{\partial t^2} u(x,t) = \Delta\, u(x,t)$
 (ii) $u(x,0) = 0,\ \frac{\partial}{\partial t} u(x,0) = \nu.$
Then $t \mapsto u(0,t)$ *is a crystalline measure.*

Guinand's measure can be built using this theorem. It suffices to define ν by the following four conditions: ν is supported by $\mathbb{Z}^3/4$, ν does not charge $\mathbb{Z}^3$, the mass of ν on each element of $(\mathbb{Z}^3/2) \setminus \mathbb{Z}^3$ is $1/2$, and the charge of ν on each element of $(\mathbb{Z}^3/4) \setminus (\mathbb{Z}^3/2)$ is $-1/16$.

Observe that τ is odd. More generally if μ is an odd distribution and if $\widehat{\mu} = \lambda\mu$, $\lambda \in \mathbb{C}$, then $\lambda = \pm i$.

Theorem 4.3. *Guinand's measure* τ *is not an almost periodic measure. It is an almost periodic distribution.*

It suffices to observe that τ is not translation bounded. Let us define ξ on $\mathbb{Z}^3$ by $\xi(k) = 0$ on $4\mathbb{Z}^3$, $\xi(k) = 4$ on $2\mathbb{Z}^3 \setminus 4\mathbb{Z}^3$ and $\xi(k) = -1/2$ on $\mathbb{Z}^3 \setminus 2\mathbb{Z}^3$. Then $\chi(n) = \xi(k)$ if $n = |k|^2$. Let us estimate $\gamma(x) = \int_x^{x+10} d\,|\tau|$ when $x \to \infty$. We have $\gamma(x) = \sum_{x \le \sqrt{n} \le x+C} |\chi(n)| r_3(n) n^{-1/2}$. This sum is calculated on $\mathbb{Z}^3$ and we obtain $\sum_{x \le |k| \le x+C} |\xi(k)||k|^{-1}$. Comparing this sum with the corresponding integral gives $\gamma(x) \simeq x$. Theorem 2.1 implies that Guinand's measure is an almost periodic distribution.

Theorem 4.4. *Let* τ *be Guinand's measure as above. There exists a sequence* u_j *of real numbers tending to infinity such that the translated measures* τ_{u_j} *converge in the distributional sense to* $-4\frac{d}{dx}\delta_0 + \rho$ *where* ρ *is an atomic measure carried by a locally finite set.*

The proof of Theorem 4.4 is given in paragraph 6 below. It is amusing to observe that the derivative of the Dirac mass at 0 which was erased in (16) shows up again at infinity. This would be impossible if τ was an almost periodic measure. Then every weak limit of a sequence τ_{u_j} of translated of μ would be a measure. The measure τ is an almost periodic distribution which is not an almost periodic measure. Theorem 4.4 should be completed with a full description of the closed orbit of τ. The proof of Theorem 4.4 implies that there exists another sequence u'_j of real numbers tending to infinity such that $\tau_{u'_j}$ tends to τ as j tends to ∞. No derivatives of Dirac measures appear. Therefore the closed orbit of τ in $\mathcal{S}'(\mathbb{R})$ has a complicated structure.

5. Diophantine approximations

Theorem 4.4 relies on the following remark.

Theorem 5.1. *There exists a sequence* u_j *of real numbers tending to infinity such that*
 (a) $\lim_{j\to\infty} \exp(\pi i u_j \sqrt{n}) = 1,\ \forall n \in 2\mathbb{N}$

(*b*) $\lim_{j\to\infty} \exp(\pi i u_j \sqrt{n}) = -1,\ \forall n \in 2\mathbb{N}+1.$

The proof of Theorem 5.1 is based on the following lemma:

Lemma 5.1. *Let $SF \subset \mathbb{N}$ be the set of square free integers. Then the set $\{\sqrt{n},\, n \in SF\}$ is linearly independent over $\mathbb{Q}$.*

This is well known [11] and we return to the proof of Theorem 5.1. Dirichlet's theorem implies the existence of a sequence u_j tending to infinity such that

(*i*) $\exp(\pi i u_j \sqrt{n}) \to 1,\ n \in 2\mathbb{N} \cap SF$

(*ii*) $\exp(\pi i u_j \sqrt{n}) \to -1,\ n \in (2\mathbb{N}+1) \cap SF.$

If n is an odd integer, we have $n = m^2 q$ where both m and q are odd integers, and q is square free. Then $\exp(\pi i u_j \sqrt{n}) = (\exp(\pi i u_j \sqrt{q}))^m \to -1$. If n is an even integer, $n = m^2 q$ where either m or q is even and q is square free. In both cases $\exp(\pi i u_j \sqrt{n}) = (\exp(\pi i u_j \sqrt{q}))^m \to 1$. This ends the proof of Theorem 5.1.

6. Proof of Theorem 4.4

We begin with a simple lemma describing the measure τ of Theorem 4.1.

Lemma 6.1. *We have*

$$\tau = -\frac{1}{2}\tau_1 - \frac{1}{2}\tau_2 + 2\tau_3 + 2\tau_4 \tag{17}$$

where

(*i*) $\tau_1 = \sum_{m\in 2\mathbb{N}+1} r_3(m) m^{-1/2}(\delta_{\sqrt{m}/2} - \delta_{-\sqrt{m}/2})$

(*ii*) $\tau_2 = \sum_{m\in 2\mathbb{N}+1} r_3(2m)(2m)^{-1/2}(\delta_{\sqrt{2m}/2} - \delta_{-\sqrt{2m}/2})$

(*iii*) $\tau_3 = \sum_{m\in 2\mathbb{N}+1} r_3(m) n^{-1/2}(\delta_{\sqrt{m}} - \delta_{-\sqrt{m}})$

(*iv*) $\tau_4 = \sum_{n\in 2\mathbb{N}+1} r_3(2m)(2m)^{-1/2}(\delta_{\sqrt{2m}} - \delta_{-\sqrt{2m}})$

To prove this lemma it suffices to write $n = 4^j m,\ j, m \in \mathbb{N}$, in (16) and to observe that $\chi(n) = 0$ if $n = 4^j m,\ j \geq 2$. The four terms in (17) correspond to $j = 0$, m odd, $j = 0$, m even (but not a multiple of 4), $j = 1$, m odd, and $j = 1$, m even (but not a multiple of 4). The measure τ_1 is not crystalline (Lemma 6.3).

Lemma 6.2. *If the sequence u_j is defined by Theorem 5.1 we have*

(*a*) $\exp(-2\pi i u_j x)\tau_1(x) \to -\tau_1(x)$

(*b*) $\exp(-2\pi i u_j x)\tau_2(x) \to \tau_2(x)$

(*c*) $\exp(-2\pi i u_j x)\tau_3(x) \to \tau_3(x)$

(*d*) $\exp(-2\pi i u_j x)\tau_4(x) \to \tau_4(x)$

Theorem 5.1 implies Lemma 6.2. We now conclude the proof of Theorem 4.4. The sequence u_j is defined by Theorem 5.1 and we consider the sequence τ_{u_j}. Instead of computing τ_{u_j} we compute its Fourier transform. The Fourier transform of τ is $-i\tau$. Therefore we obtain

$$\widehat{\tau_{u_j}} = \frac{i}{2}[\exp(-2\pi i u_j x)\tau_1(x) + \exp(-2\pi i u_j x)\tau_2(x)]$$
$$-2i[\exp(-2\pi i u_j x)\tau_3(x) + \exp(-2\pi i u_j x)\tau_4(x)]$$

which tends to $-i[\frac{1}{2}\tau_1 - \frac{1}{2}\tau_2 + 2\tau_3 + 2\tau_4] = -i[\tau + \tau_1]$. Theorem 4.4 will be proved if the following lemma is accepted.

Lemma 6.3. *The measure* τ_1 *is the Fourier transform of* $-4i\frac{d}{dx}\delta_0 + \rho$ *where* ρ *is an atomic measure carried by a locally finite set.*

For proving Lemma 6.3 we return to the Guinand distribution

$$\sigma = \sum_1^\infty r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) - 2\frac{d}{dx}\delta_0 \tag{18}$$

and consider its companion defined by

$$\widetilde{\sigma} = \sum_1^\infty (-1)^n r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) - 2\frac{d}{dx}\delta_0 \tag{19}$$

Let us compute the Fourier transform of $\widetilde{\sigma}$. We mimic Guinand's approach in [2] and consider the family $\psi_t(x) = x\exp(-\pi tx^2)$ indexed by $t > 0$. The Fourier transform of $\psi_t(x)$ is $-it^{(-3/2)}\psi_{t^{-1}}(x)$. Then $I(t) = \langle\widetilde{\sigma}, \psi_t\rangle =$
$2 + 2\sum_1^\infty (-1)^n r_3(n)\exp(-\pi tn) = 2\sum_{k\in\mathbb{Z}^3}(-1)^{k_1+k_2+k_3}\exp(-\pi t|k|^2)$ since m and m^2 have the same parity. Therefore

$$I(t) = 2(\sum_{m\in\mathbb{Z}}(-1)^m\exp(-\pi tm^2))^3 \tag{20}$$

We now apply the one dimensional Poisson formula to

$$J(t) = \sum_{m\in\mathbb{Z}}(-1)^m\exp(-\pi tm^2) \tag{21}$$

and obtain $J(t) = t^{-1/2}\sum_{m\in\mathbb{Z}}\exp(-\pi t^{-1}(m+1/2)^2)$. Therefore $J(t)^3 = t^{-3/2}\sum_{k\in\mathbb{Z}^3}\exp(-\pi t^{-1}(k_1^2 + k_2^2 + k_3^2 + k_1 + k_2 + k_3 + 3/4)$. Let $s_3(n)$ denote the number of decompositions of n as a sum $n = k_1^2 + k_2^2 + k_3^2 + k_1 + k_2 + k_3$, $k_j \in \mathbb{Z}$. Finally this calculation yields $\langle\widetilde{\sigma}, \psi_t\rangle = i\langle\omega, \widehat{\psi_t}\rangle$ where

$$\omega = \sum_1^\infty s_3(n)n^{-1/2}(\delta_{\sqrt{n+3/4}} - \delta_{-\sqrt{n+3/4}}). \tag{22}$$

Moreover ω and $\widetilde{\sigma}$ are odd measures and the collection of functions ψ_t is total in the space of odd functions of the Schwartz class. We conclude

Lemma 6.4. *The Fourier transform of* $\widetilde{\sigma}$ *is* $-i\omega$.

We now consider $w = \frac{1}{2}\sigma - \frac{1}{2}\widetilde{\sigma}$. Its Fourier transform is $-\frac{i}{2}\sigma + \frac{i}{2}\omega$. Let us observe that $w = \sum_{n\in 2\mathbb{N}+1} r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$ and $\tau_1(x) = 2w(2x)$. Therefore the Fourier transform of τ_1 is $-\frac{i}{2}[\sigma(x/2) - \omega(x/2)] = 4i\frac{d}{dx} - \rho$ where ρ is an atomic measure carried by a locally finite set. Lemma 6.3 is proved now and it implies Theorem 4.4.

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References

[1] J. I. Díaz and Y. Meyer, Poisson summation formulae and the wave equation with a finitely supported measure as initial velocity. Afr. Diaspora J. Math. Vol 20, No 1 (2017) 1-13.
[2] A. P. Guinand, Concordance and the harmonic analysis of sequences. Acta Math. 101 (1959), 235-271.
[3] N. Lev, A. Olevskii, Quasicrystals with discrete support and spectrum. Revista Matemática Iberoamericana, Volume 32, Issue 4, (2016) 1341-1352.

[4] N. Lev, A. Olevskii, Measures with uniformly discrete support and spectrum. C. R. Math. Acad. Sci. Paris 351 (2013), 613-617.
[5] N. Lev, A. Olevskii, Quasicrystals and Poisson's summation formula. Invent. Math., Volume 200, Issue 2, 585-606, 2015.
[6] N. Lev, A. Olevskii, Fourier quasi-crystals and discreteness of the diffraction spectrum, arXiv 1512.08735.
[7] Y. Meyer, Measures with locally finite support and spectrum. PNAS, March 22, 2016, vol. 113, no. 12, 3152-3158.
[8] Y. Meyer, Algebraic Numbers and Harmonic Analysis. North-Holland, Amsterdam, (1972).
[9] Y. Meyer, Quasi-crystals, Almost periodic patterns, mean-periodic functions, and irregular sampling. Afr. Diaspora J. Math. Vol 13, No. 1 (2012) 1-45.
[10] N. Strungaru, On weighted Dirac combs supported inside model sets. J. Phys. A: Math. Theor. 47 (2014).
[11] K.S. Williams, Note on extensions of the rationals by square roots. Delta 3. Waukesha Mathematical Society. (1973) 11-15.

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