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ON THE EIGENVALUES OF THE INFINITESIMAL GENERATOR OF A SEMIGROUP OF COMPOSITION OPERATORS ON BERGMAN SPACES

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ABSTRACT. Suppose that (ϕ_t) is a one-parameter semigroup of holomorphic self-maps of the unit disk with associated planar domain Ω. Let (T_t) be the corresponding semigroup of composition operators on the Bergman space A^p_α. When the semigroup (ϕ_t) is elliptic or hyperbolic, we describe the point spectrum of the infinitesimal generator of (T_t) in terms of the geometry of Ω.

1. Introduction

A semigroup of holomorphic self-maps ϕ_t, $t \geq 0$, of the unit disk $\mathbb{D}$ induces a strongly continuous semigroup of composition operators T_t on the Hardy space $H^p, 1 \leq p < \infty$, or on the Bergman space A^p_α, $1 \leq p < \infty$, $-1 < \alpha < \infty$. The study of composition semigroups began by Berkson and Porta [4] on H^p and by Siskakis [19] on A^p_α. The main results of the theory are reviewed in [21]. The infinitesimal generator of the semigroup (ϕ_t) is a holomorphic function $G : \mathbb{D} \to \mathbb{C}$ such that (see [4], [1])

$$\frac{\partial \phi_t(z)}{\partial t} = G(\phi_t(z)), \quad t \geq 0, \ z \in \mathbb{D}. \tag{1.1}$$

The infinitesimal generator of the operator semigroup (T_t) on H^p (respectively, on A^p_α) is the operator Γ given by

$$\Gamma(f) = Gf' \tag{1.2}$$

and defined for all $f \in H^p$ (respectively, all $f \in A^p_\alpha$) for which $Gf' \in H^p$ (respectively, $Gf' \in A^p_\alpha$); see [19, 20].

In the article [6] we study the eigenvalues of the infinitesimal generator Γ on Hardy spaces. That paper contains an introduction including the definitions and main results of the theory of holomorphic semigroups and the associated composition operator theory. Now we will deal with the point spectrum Λ of the operator Γ on Bergman spaces. Suppose that h is the Koenigs function of the semigroup (ϕ_t) and $\Omega = h(\mathbb{D})$ is the associated planar domain. Let $\tau \in \mathbb{D} \cup \partial\mathbb{D}$ be the Denjoy-Wolff point of the (ϕ_t); (for the definitions, see [1], [21], [6] and references therein). Siskakis [19, 20] proved that
(i) If $\tau \in \mathbb{D}$, then $\Lambda \subset \{kG'(0) : k = 0, 1, 2, \dots\}$ and $kG'(0) \in \Lambda$ if and only if $h^k \in A^p_\alpha$.
(ii) If $\tau \in \partial\mathbb{D}$, then $\Lambda = \{\lambda G(0) : e^{\lambda h} \in A^p_\alpha\}$.

In the present work we will provide an exact description of the eigenvalues of Γ on A^p_α in the cases of elliptic and hyperbolic semigroups. Recall that a semigroup (ϕ_t) is elliptic if its Denjoy-Wolff point τ belongs to $\mathbb{D}$ and no ϕ_t is an elliptic automorphism of $\mathbb{D}$; in this case

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we assume (without essential loss of generality) that $\tau = 0$. Therefore [4], the infinitesimal generator G of (ϕ_t) has the form $G(z) = -zF(z)$, where F is a function with positive real part.

To describe the point spectrum Λ of Γ on A^p_α, we need to review some definitions and facts from geometric function theory. If $\sigma \in (-\pi/2, \pi/2)$, we say that a domain $\Omega \subsetneq \mathbb{C}$ is σ-spiral-like (with respect to 0) if for every $w \in \Omega$ the open spiral segment

$$\{we^{-te^{i\sigma}} : t > 0\} \tag{1.3}$$

joining w to 0 is contained in Ω. Note that the point 0 may or may not belong to Ω. In the former case we obtain the classical notion of spiral-like domains; see e.g. [12], [15] and references therein. In the latter case, $0 \in \partial\Omega$ and we obtain the class of spiral-like domains with respect to a boundary point; see e.g. [18], [10]. If Ω is a σ-spiral-like domain for some σ, we say that Ω is spiral-like. If f is a conformal mapping of $\mathbb{D}$ onto a σ-spiral-like domain, we say that f is a σ-spiral-like function. The set of all functions in $\mathbb{D}$ that are σ-spiral-like for some $\sigma \in (-\pi/2, \pi/2)$ is denoted by $\mathcal{S}$.

Given $\sigma \in (-\pi/2, \pi/2)$, $\theta_o \in [0, 2\pi)$, and $\eta \in [0, 2\pi]$, the *σ-spiral sector of opening* η with center angle θ_o is the set

$$S_\sigma(\theta_o, \eta) = \left\{e^{i\theta}e^{-te^{i\sigma}} : t \in \mathbb{R}, |\theta - \theta_o| < \frac{\eta}{2}\right\}. \tag{1.4}$$

The spiral curve $\{e^{i\theta_o}e^{-te^{i\sigma}} : t \in \mathbb{R}\}$ is the *spiral bisector* of $S_\sigma(\theta_o, \eta)$. It is easy to see that the spiral bisector is a hyperbolic geodesic for the domain $S_\sigma(\theta_o, \eta)$. If Ω is a σ-spiral-like domain and contains a σ-spiral sector, then it contains a maximal σ-spiral sector of the form $S_\sigma(\theta_o, \eta(\Omega))$, in the sense that there is no $\theta_1 \in [0, 2\pi)$ and no $\eta > \eta(\Omega)$ such that $S_\sigma(\theta_1, \eta) \subset \Omega$. The number $\eta(\Omega) \in [0, 2\pi]$ is called the *maximal angular opening* of Ω. When there is no danger of confusion we will use the simpler notation η for $\eta(\Omega)$.

If (ϕ_t) is a semigroup of holomorphic self-maps of the unit disk, then its Koenigs function h is spiral-like and the associated planar domain $\Omega = h(\mathbb{D})$ is σ-spiral-like, where

$$\sigma := \arg(-G'(0)) = \arg(F(0)) \in (-\pi/2, \pi/2); \tag{1.5}$$

see [19], [21].

By the result of Siskakis mentioned above, the problem of the determination of the point spectrum of Γ is reduced to a well known problem in geometric function theory. In its general form the problem can be stated as follows: Find conditions on a (univalent) holomorphic function f in $\mathbb{D}$ so that f belongs to some space of holomorphic functions (Hardy, Bergman, Dirichlet, etc.); see [3], [16] for recent contributions. The membership of a spiral-like function to the Hardy space has been studied in [12]; see also [15].

Theorem 1. *Let $-\pi/2 < \sigma < \pi/2$. Suppose that f is a σ-spiral-like function and let $\eta \in [0, 2\pi]$ be the maximal angular opening of $f(\mathbb{D})$.*
(i) *Assume $\eta = 0$. Then $f \in A^p_\alpha$ for every $p \in (0, +\infty)$ and every $\alpha \in (-1, +\infty)$.*
(ii) *Assume that $\eta > 0$. Then $f \in A^p_\alpha$ if and only if*

$$p < \frac{(\alpha + 2)\pi}{\eta \cos^2 \sigma}. \tag{1.6}$$

Baernstein, Girela, and Peláez [3] proved that for $p \geq 1/2$, there exists a univalent function in $A^{2p}_0 \setminus H^p$. Theorem 1 implies that such a function cannot be spiral-like:

Corollary 1. *For every $p \in (0, +\infty)$,*

$$\mathcal{S} \cap H^p = \mathcal{S} \cap A^{2p}_0.$$

With the aid of Theorem 1, we can easily obtain a transparent description of the point spectrum of the operator Γ on A^p_α in the case of interior Denjoy-Wolff point.

Theorem 2. *Let $(\phi_t)_{t\geq 0}$ be an elliptic semigroup of holomorphic self maps of $\mathbb{D}$. Suppose that the Denjoy-Wolff point of the semigroup is 0. Let h be the Koenigs function of (ϕ_t) and $\Omega = h(\mathbb{D})$ be the associated spiral-like domain. Let G be the infinitesimal generator of (ϕ_t) and Γ be the infinitesimal generator of the induced semigroup of composition operators on the Bergman space A^p_α, $1 \leq p < \infty$, $-1 < \alpha < \infty$. Let Λ be the point spectrum of Γ. Let $\sigma = \arg(-G'(0)) \in (-\pi/2, \pi/2)$ and let $\eta(\Omega)$ be the maximal angular opening of Ω.*
(a) *If Ω does not contain any σ-spiral sector then*

$$\Lambda = \{kG'(0) : k = 0, 1, 2, \dots\}. \tag{1.7}$$

(b) *If Ω contains a σ-spiral sector then*

$$\Lambda = \{kG'(0) : k = 0, 1, 2, \dots, k_o\}, \tag{1.8}$$

where

$$k_o = \max\left\{k \in \mathbb{N} : k < \frac{\pi(\alpha+2)}{p\,\eta(\Omega)\cos^2\sigma}\right\}. \tag{1.9}$$

The proofs of Theorems 1, 2 and Corollary 1 are in section 3.

We now turn to hyperbolic semigroups. Recall that a semigroup (ϕ_t) is hyperbolic if $\tau \in \partial\mathbb{D}$ and for the angular derivative of every function ϕ_t, we have $0 < \phi_t'(\tau) < 1$; see [19, 8]. For hyperbolic semigroups, we assume (without essential loss of generality) that $\tau = 1$, $h(0) = 0$ and $G(0) = 1$. The associated planar domain is *convex in the positive direction* (see [19]). A domain is called convex in the positive direction if it has the following property: Whenever $w \in \Omega$, the horizontal half-line $\{w + t : t \geq 0\}$ is contained in Ω. Suppose that such a domain is contained in a horizontal strip. Let $\nu = \nu(\Omega)$ be the width of the smallest horizontal strip containing Ω. If Ω contains a horizontal strip, then it contains one of maximal width. Let $\beta = \beta(\Omega)$ be this maximal width. We also set $\beta(\Omega) = 0$ if Ω does not contain any horizontal strip.

Contreras and Díaz-Madrigal [8] have proved that the associated planar domain Ω of a hyperbolic semigroup is contained in a horizontal strip. So we can consider the quantities $\nu(\Omega)$ and $\beta(\Omega)$. We will need one more geometric quantity that quantifies the proximity of the boundary of Ω to the boundary lines of the smallest strip S containing Ω. For $x > 0$, let $\ell(x)$ be the vertical line through x. Let $J(x)$ be the component of $\ell(x) \cap \Omega$ that intersects the midline of S and set $\theta(x) = |J(x)|$ (the length of $J(x)$). Note that $\theta(x)$ is an increasing function of x, that $0 < \theta(x) \leq \nu$ for $x \in (0, +\infty)$, and that $\lim_{x\to+\infty} \theta(x) = \nu$. Consider the quantity

$$W_\alpha(\Omega) = \int_1^\infty \exp\left[-\pi(\alpha+2)\int_0^t \left(\frac{1}{\theta(x)} - \frac{1}{\nu}\right)dx\right]dt. \tag{1.10}$$

Observe that $W_\alpha(\Omega) < +\infty$ if $\theta(x)$ converges to ν (as $x \to +\infty$) slowly enough.

Theorem 3. *Let (ϕ_t) be a hyperbolic semigroup of holomorphic self-maps of $\mathbb{D}$. Let Γ be the infinitesimal generator of the induced composition semigroup on A^p_α, $1 \leq p < +\infty$, $-1 < \alpha < +\infty$. Let Λ be the point spectrum of Γ.*
(a) *If $\beta(\Omega) = 0$ and $W_\alpha(\Omega) < +\infty$ then*

$$\Lambda = \left\{\lambda \in \mathbb{C} : -\infty < \mathtt{Re}\,\lambda \leq \frac{\pi(\alpha+2)}{p\nu(\Omega)}\right\}. \tag{1.11}$$

(b) *If* $\beta(\Omega) = 0$ *and* $W_\alpha(\Omega) = +\infty$ *then*

$$\Lambda = \left\{ \lambda \in \mathbb{C} : -\infty < \mathtt{Re}\,\lambda < \frac{\pi(\alpha+2)}{p\nu(\Omega)} \right\}. \tag{1.12}$$

(c) *If* $\beta(\Omega) > 0$ *and* $W_\alpha(\Omega) < +\infty$ *then*

$$\Lambda = \left\{ \lambda \in \mathbb{C} : -\frac{\pi(\alpha+2)}{p\beta(\Omega)} < \mathtt{Re}\,\lambda \leq \frac{\pi(\alpha+2)}{p\nu(\Omega)} \right\}. \tag{1.13}$$

(d) *If* $\beta(\Omega) > 0$ *and* $W_\alpha(\Omega) = +\infty$ *then*

$$\Lambda = \left\{ \lambda \in \mathbb{C} : -\frac{\pi(\alpha+2)}{p\beta(\Omega)} < \mathtt{Re}\,\lambda < \frac{\pi(\alpha+2)}{p\nu(\Omega)} \right\}. \tag{1.14}$$

2. Preliminaries

2.1. **Bergman spaces.** For $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the (weighted) Bergman space A^p_α is the space of all holomorphic functions f in the unit disk for which

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha \, A(dz) < +\infty, \tag{2.1}$$

where A is the Lebesgue area measure on $\mathbb{D}$. It follows from the Hardy-Stein formula (see [16] and references therein) that

$$\|f\|^p_{A^p_\alpha} \asymp \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left(\log \frac{1}{|z|} \right)^{\alpha+2} A(dz) + |f(0)|^p. \tag{2.2}$$

Here and below, the notation $a \asymp b$ means that there exist constants $c, C > 0$ such that $c \leq a/b \leq C$. Similarly, the notation $a \lesssim b$ means that there exists a constant $C > 0$ such that $a \leq Cb$.

If f is holomorphic in $\mathbb{D}$ and $0 < r < 1$, we set

$$M(r) = M(r,f) = \max\{|f(re^{i\theta})| : \theta \in [0, 2\pi]\}. \tag{2.3}$$

By a theorem of Baernstein, Girela, and Peláez [3], modified by Pérez-González and Rättyä [16], a univalent holomorphic function f in $\mathbb{D}$ belongs to A^p_α if and only if

$$\int_0^1 M(r,f)^p (1-r^2)^{\alpha+1} dr < +\infty. \tag{2.4}$$

Lemma 1. *If* f *is a holomorphic mapping of* $\mathbb{D}$ *into the upper half-plane* $\mathbb{H} = \{w \in \mathbb{C} : \mathtt{Im}\, w > 0\}$*, then for every* $p < \alpha + 2$*,* $f \in A^p_\alpha$*.*

Proof. By subordination, we may assume that f is a conformal mapping of $\mathbb{D}$ onto $\mathbb{H}$. Then $M(r,f) \asymp (1-r)^{-1}$. Hence, if $p < \alpha + 2$, then

$$\int_0^1 M(r,f)^p (1-r^2)^{\alpha+1} dr \lesssim \int_0^1 (1-r)^{\alpha+1-p} dr < +\infty. \tag{2.5}$$

So $f \in A^p_\alpha$. ◻

2.2. **Harmonic measure and extremal length.** Let Ω be a planar domain with non-polar boundary; (for example, if $\Omega \subsetneq \mathbb{C}$ is simply connected then $\partial\Omega$ is non-polar). Let E be a Borel subset of $\partial\Omega$. The harmonic measure of E relative to Ω is the generalized (Perron) solution u of the Dirichlet problem for the Laplacian in Ω with boundary function equal to 1 on E and to 0 on $\partial\Omega \setminus E$. We will use the standard notation

$$u(z) = \omega(z, E, \Omega), \quad z \in \Omega. \tag{2.6}$$

For fixed $z \in \Omega$, $\omega(z, \cdot, \Omega)$ is a probability measure on $\partial\Omega$. The harmonic measure is conformally invariant and satisfies the Harnack and the maximum principles. We refer to [2], [11], [13], and [17] for presentations of the theory of harmonic measure.

If Ω is a planar domain and A, E are closed sets in the closure of Ω, we denote by $\lambda(A, E, \Omega)$ the extremal length of the family of the rectifiable curves in Ω joining A with E. We refer to [11] for the properties of extremal length and Beurling's estimates of harmonic measure in terms of extremal length.

2.3. **Green function.** Suppose that $\Omega \subsetneq \mathbb{C}$ is a domain. We denote by $g_\Omega(z, w)$ the Green function of Ω. We refer to [2], [13], [17] for the definition and main properties of the Green function.

Lemma 2. *Consider the strip*

$$S = \{z \in \mathbb{C} : 0 < \mathtt{Im}\, z < \pi\}.$$

For every $x > 0$,

$$g_S(i\pi/2, x + i\pi/2) \geq 2e^{-x}. \tag{2.7}$$

Proof. By the explicit expression of the Green function of the strip

$$g_S(i\pi/2, x + i\pi/2) = \log \frac{e^x + 1}{e^x - 1}. \tag{2.8}$$

The function $(1, +\infty) \ni r \mapsto r \log \frac{r+1}{r-1}$ is decreasing. Therefore

$$r \log \frac{r+1}{r-1} \geq \lim_{r \to +\infty} r \log \frac{r+1}{r-1} = 2. \tag{2.9}$$

□

2.4. **Growth of spiral-like functions.** We will prove two lemmas that give precise bounds for the growth of spiral-like functions.

Lemma 3. *Let $\sigma \in (-\pi/2, \pi/2)$ and let Ω be a σ-spiral-like domain. Suppose that the maximal angular opening of Ω is $\eta(\Omega) > 0$. Let f be a conformal mapping of $\mathbb{D}$ onto Ω. Then there exists a constant $C > 0$ such that for $0 < r < 1$,*

$$M(r, f) \geq \frac{C}{(1 - r)^{\eta(\Omega)(\cos^2 \sigma)/\pi}}. \tag{2.10}$$

Proof. Since $\eta = \eta(\Omega) > 0$, there exists a σ-spiral sector of the form

$$\Sigma = S_\sigma(\theta_o, \eta) = \left\{ e^{i\theta} e^{-te^{i\sigma}} : t \in \mathbb{R},\ |\theta - \theta_o| < \frac{\eta}{2} \right\}$$

contained in Ω; see [12]. The spiral curve (the bisector of Σ)

$$\gamma_o = \{e^{i\theta_o} e^{-te^{i\sigma}} : t \in \mathbb{R}\}$$

is a hyperbolic geodesic for Σ. Consider the point $w_o = e^{i\theta_o} \in \gamma_o \cap \partial\mathbb{D}$ and set $z_o = f^{-1}(w_o) \in \mathbb{D}$. Let f_o be the conformal mapping of $\mathbb{D}$ onto Σ such that $f_o(z_o) = w_o$ and the radial segment $(z_o, z_o/|z_o|)$ is mapped onto the hyperbolic geodesic segment $\{e^{i\theta_o}e^{-te^{i\sigma}} : t < 0\}$ in Σ. For $0 < r < 1$, set $z_r = rz_o/|z_o|$ and $w_r = f_o(z_r)$.

The exponential function maps conformally the strip

$$S = \left\{ x + iy \in \mathbb{C} : \ |y - (\theta_o - x\tan\sigma)| < \frac{\eta(\Omega)}{2} \right\}$$

onto Σ. Denote by L the inverse function (a branch of the logarithm) that maps Σ onto S. By the conformal invariance of the Green function, for every r close enough to 1,

$$1 - r \backsimeq \log\frac{1}{r} = g_{\mathbb{D}}(0, z_r) = g_\Sigma(f_o(0), w_r). \tag{2.11}$$

By the Harnack chain principle (see e.g. [2, Corollary 1.4.4]), there exists a constant $C_1 > 0$, independent of r, such that

$$g_\Sigma(f_o(0), w_r) \ge C_1 g_\Sigma(w_o, w_r). \tag{2.12}$$

By the conformal invariance of the Green function

$$g_\Sigma(w_o, w_r) = g_S(L(w_o), L(w_r)). \tag{2.13}$$

The points $L(w_o)$ and $L(w_r)$ lie on the midline of the strip S which has width $\eta\cos\sigma$. Moreover, $|L(w_o) - L(w_r)| = \log|w_r|/\cos\sigma$. Therefore, by Lemma 2,

$$g_S(L(w_o), L(w_r)) \ge \frac{2}{|w_r|^{\pi/(\eta\cos^2\sigma)}}. \tag{2.14}$$

Since $w_r = f_o(z_r)$ and $|z_r| = r$, we have $|w_r| \le M(r, f_o)$. Also, $\Sigma \subset \Omega$ and $f(z_o) = f_o(z_o) = w_o$. Hence [13, Theorem 2.20], f_o is subordinate to f. Therefore [13, Theorem 2.21] $M(r, f_o) \le M(r, f)$. So (2.14) implies that

$$g_S(L(w_o), L(w_r)) \ge \frac{2}{M(r, f)^{\pi/(\eta\cos^2\sigma)}}. \tag{2.15}$$

It follows from (2.11)-(2.13) and (2.15) that there exists a constant $C > 0$ such that for $0 < r < 1$,

$$M(r, f) \ge \frac{C}{(1 - r)^{\eta(\cos^2\sigma)/\pi}}. \tag{2.16}$$

□

Lemma 4. *Let f be a σ-spiral-like function with maximal angular opening $\eta \in [0, 2\pi]$. Let $\varepsilon > 0$. There exists a constant $C_\varepsilon > 0$ such that for every $r \in (0, 1)$,*

$$M(r, f) \le \frac{C_\varepsilon}{(1 - r)^{(\eta+\varepsilon)(\cos^2\sigma)/\pi}}. \tag{2.17}$$

Proof. It suffices to prove the inequality (2.17) for all sufficiently small $\varepsilon > 0$ and all $r \in (0, 1)$ that are sufficiently close to 1. Assume that $0 \le \eta < 2\pi$. (If $\eta = 2\pi$, the proof is similar; just put $\varepsilon = 0$ in the proof below). Let $\Omega = f(\mathbb{D})$. Take $\varepsilon > 0$ small enough so that $\eta + \varepsilon < 2\pi$. Let $r \in (0, 1)$. There exists a point z_r on the circle $\{|z| = r|\}$ such that $|f(z_r)| = M(r) = M(r, f)$; note that the function $r \mapsto M(r)$ is a strictly increasing continuous function. For the given $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that the component of $\Omega \cap \{|w| > M_\varepsilon\}$ containing $f(z_r)$ is contained in a spiral sector of the form $\Sigma_\varepsilon = S_\sigma(\theta_o, \eta + \varepsilon)$ (by the definition of η). Set $r_\varepsilon = M^{-1}(M_\varepsilon)$.

For every $\rho > 0$, let $I(\rho)$ be the component of $\Omega \cap \{|w| = \rho\}$ intersecting the spiral bisector of Σ_ε. Note that $f^{-1}(I(M(r))$ is a Jordan arc in $\mathbb{D}$ passing through z_r and that the circular projection of $f^{-1}(I(M(r))$ on the positive radius of $\mathbb{D}$ contains the interval $(r, 1)$. Therefore, the Beurling-Nevanlinna projection theorem (see e.g. [11] or [17]) and an elementary calculation of harmonic measure give

$$\begin{aligned} 1 - r \quad &\lesssim \quad \frac{1}{\pi} \cos^{-1} \frac{6r - 1 - r^2}{(1+r)^2} = \omega(0, [r, 1], \mathbb{D} \setminus [r, 1]) \\ &\leq \quad \omega(0, f^{-1}(I(M(r))), \mathbb{D} \setminus f^{-1}(I(M(r))). \end{aligned} \tag{2.18}$$

By the conformal invariance of harmonic measure,

$$\omega(0, f^{-1}(I(M(r))), \mathbb{D} \setminus f^{-1}(I(M(r))) = \omega(f(0), I(M(r)), \Omega \setminus I(M(r))). \tag{2.19}$$

By the Harnack chain principle (see e.g. [2, Corollary 1.4.4]), there exists a constant $C_\varepsilon > 0$ such that for $r > r_\varepsilon$,

$$\omega(f(0), I(M(r)), \Omega \setminus I(M(r))) \leq C_\varepsilon \omega(M_\varepsilon e^{i\theta_o}, I(M(r)), \Omega \setminus I(M(r))). \tag{2.20}$$

By Beurling's estimates connecting harmonic measure and extremal length and monotonicity properties of extremal length (see [11]),

$$\begin{aligned} \omega(M_\varepsilon e^{i\theta_o}, I(M(r)), \Omega \setminus I(M(r))) \quad &\lesssim \quad e^{-\pi \lambda(I(M_\varepsilon), I(M(r)), \Omega)} \\ &\lesssim \quad e^{-\pi \lambda(J(M_\varepsilon), J(M(r)), \Sigma_\varepsilon)}, \end{aligned} \tag{2.21}$$

where $J(\rho) = \Sigma_\varepsilon \cap \{|w| = \rho\}, \rho > 0$.

The exponential function maps conformally the strip

$$S_\varepsilon = \left\{ x + iy \in \mathbb{C} : -\frac{\eta + \varepsilon}{2} < |y - x \tan \sigma| < \frac{\eta + \varepsilon}{2} \right\}$$

which has width equal to $(\eta + \varepsilon) \cos \sigma$ onto Σ_ε. Let L be the inverse function (a branch of the logarithm). By the conformal invariance of extremal length,

$$\lambda(J(M_\varepsilon), J(M(r)), \Sigma_\varepsilon) = \lambda(L(J(M_\varepsilon)), L(J(M(r))), S_\varepsilon). \tag{2.22}$$

Note that $\lambda(L(J(M_\varepsilon))$ and $L(J(M(r)))$ are rectilinear orthogonal crosscuts of S_ε having distance to each other equal to $(\log M(r) - \log M_\varepsilon)/\cos \sigma$. Therefore,

$$\lambda(L(J(M_\varepsilon)), L(J(M(r))), S_\varepsilon) = \frac{\log M(r) - \log M_\varepsilon}{(\eta + \varepsilon) \cos^2 \sigma}. \tag{2.23}$$

It follows from (2.18)-(2.23) that there exists a constant $C_\varepsilon > 0$ such that for every $r \in (0, 1)$,

$$1 - r \leq C_\varepsilon \ \exp \left(\frac{-\pi \log M(r)}{(\eta + \varepsilon) \cos^2 \sigma} \right) \tag{2.24}$$

which is equivalent to (2.17). □

3. **Proof of Theorems 1, 2 and Corollary 1**

3.1. **Proof of Theorem 1.** Suppose first that $\eta = 0$. Let $\varepsilon > 0$. By Lemma 4, there exists a constant $C_\varepsilon > 0$ such that for every $r \in (0, 1)$,

$$M(r) = M(r, f) \leq \frac{C_\varepsilon}{(1 - r)^{\varepsilon (\cos^2 \sigma)/\pi}}. \tag{3.1}$$

Therefore, for every $p \in (0,+\infty)$ and every $\alpha \in (-1,+\infty)$

$$(3.2)\qquad \int_0^1 M(r)^p (1-r)^{\alpha+1}\,dr \le C_\varepsilon \int_0^1 (1-r)^{\alpha+1-p\varepsilon(\cos^2\sigma)/\pi}.$$

The last integral is finite when ε is sufficiently small. Hence (see subsection 2.1) $f \in A^p_\alpha$.

Suppose that $\eta > 0$. If $p < \frac{(\alpha+2)\pi}{\eta\cos^2\sigma}$, then we can find $\varepsilon > 0$ small enough so that

$$(3.3)\qquad p < \frac{(\alpha+2)\pi}{(\eta+\varepsilon)\cos^2\sigma}.$$

It follows as above (using Lemma 4) that

$$(3.4)\qquad \int_0^1 M(r)^p (1-r)^{\alpha+1}\,dr \le C_\varepsilon \int_0^1 (1-r)^{\alpha+1-p(\eta+\varepsilon)(\cos^2\sigma)/\pi} < +\infty.$$

Hence $f \in A^p_\alpha$.

Conversely, assume that $f \in A^p_\alpha$. Then by Lemma 3,

$$(3.5)\qquad +\infty > \int_0^1 M(r)^p (1-r)^{\alpha+1}\,dr \ge \int_0^1 (1-r)^{\alpha+1-p\eta(\cos^2\sigma)/\pi}\,dr.$$

Hence $p < \frac{(\alpha+2)\pi}{\eta\cos^2\sigma}$. □

3.2. **Proof of Corollary 1.** Let f be a σ-spiral-like function with maximal angular opening η. If $\eta = 0$ then f belongs to H^p and to A^p_0 for all $p \in (0,+\infty)$. Suppose that $\eta > 0$. By a theorem of Hansen [12], $f \in H^p$ if and only if $p < \pi/(\eta\cos^2\sigma)$. By Theorem 1 (with $\alpha = 0$) this condition is equivalent to $f \in A^{2p}_0$. □

3.3. **Proof of Theorem 2.** If Ω does not contain any spiral sector then $\eta = \eta(\Omega) = 0$. Therefore $h \in A^p_\alpha$ for every $p \in (0,+\infty)$ and every $\alpha \in (-1,+\infty)$. The same is true for the function h^k, $k = 0,1,2,\dots$. Hence

$$(3.6)\qquad \Lambda = \{kG'(0) : k = 0,1,2,\dots\}.$$

Assume that Ω contains a spiral sector. Then $\eta > 0$. If $k \in \{0,1,2,\dots\}$ and $kG'(0) \in \Lambda$, then $h^k \in A^p_\alpha$ and therefore $h \in A^{kp}_\alpha$. By Theorem 1,

$$(3.7)\qquad k < \frac{\pi(\alpha+2)}{p\,\eta\cos^2\sigma}.$$

Conversely, if k is a nonnegative integer satisfying (3.7), then by Theorem 1, $h \in A^{kp}_\alpha$ and therefore, $kG'(0) \in \Lambda$.

4. **Proof of Theorem 3**

We first prove two lemmas. Let Ω be a domain which is convex in the positive direction. Assume that Ω is contained in a horizontal strip and let

$$S = \left\{ w \in \mathbb{C} : y_o - \frac{\nu}{2} < \mathtt{Im}\, w < y_o + \frac{\nu}{2} \right\}, \quad y_o \in \mathbb{R},\ \nu > 0,$$

be the smallest horizontal strip containing Ω. There exists an $x_o > 0$ such that $x_o + iy_o \in \Omega$. For $x > x_o$, let $\ell(x)$ be the vertical line through x. Let $J(x)$ be the component of $\ell(x) \cap \Omega$ that intersects the midline of S and set $\theta(x) = |J(x)|$ (the length of $J(x)$). Since S is the smallest strip containing Ω, $\lim_{x\to+\infty}\theta(x) = \nu$.

Lemma 5. *Let Ω and S be as above. Let*

$$\partial\Omega_+ = \{\zeta \in \partial\Omega : \mathtt{Im}\,\zeta > y_o\}$$

Then

$$\lim_{x\to+\infty} \omega(x + iy_o, \partial\Omega_+, \Omega) = \frac{1}{2}. \tag{4.1}$$

Proof. The set $J(x)$ is a vertical interval of the form

$$J(x) = (x + iy_o - iy_-(x),\ x + iy_o + iy_+(x)).$$

Since Ω is convex in the positive direction, the functions $y_-(x)$ and $y_+(x)$ are increasing. Moreover, since S is the smallest strip containing Ω,

$$y_-(x) \to \nu/2 \quad \text{and} \quad y_+(x) \to \nu/2, \quad \text{as} \quad x \to +\infty. \tag{4.2}$$

Let $\varepsilon > 0$. Because of (4.2), there exists $x_\varepsilon > x_o$ such that for $x > x_\varepsilon$,

$$0 \le \nu/2 - y_-(x) < \varepsilon \quad \text{and} \quad 0 \le \nu/2 - y_+(x) < \varepsilon. \tag{4.3}$$

Consider the half-strip

$$\Pi_\varepsilon = \left\{w \in \mathbb{C} :\ \mathtt{Re}\,w > x_\varepsilon, \quad y_o - \frac{\nu}{2} - \varepsilon < \mathtt{Im}\,w < y_o + \frac{\nu}{2}\right\}$$

and let A_ε be its upper boundary half-line. By the maximum principle

$$\omega(x + iy_o, \partial\Omega_+, \Omega) \ge \omega(x + iy_o, A_\varepsilon, \Pi_\varepsilon). \tag{4.4}$$

It is easy to see that

$$\lim_{x\to+\infty} \omega(x + iy_o, A_\varepsilon, \Pi_\varepsilon) = \frac{(\nu-\varepsilon)/2}{\nu} = \frac{\nu-\varepsilon}{2\nu}. \tag{4.5}$$

Therefore,

$$\liminf_{x\to+\infty} \omega(x + iy_o, \partial\Omega_+, \Omega) \ge \frac{\nu-\varepsilon}{2\nu}. \tag{4.6}$$

Since this holds for every $\varepsilon > 0$, we infer that

$$\liminf_{x\to+\infty} \omega(x + iy_o, \partial\Omega_+, \Omega) \ge \frac{1}{2}. \tag{4.7}$$

Similarly, we prove that

$$\limsup_{x\to+\infty} \omega(x + iy_o, \partial\Omega_+, \Omega) \le \frac{1}{2}. \tag{4.8}$$

So (4.1) is proved. □

Let Ω and S be as above. By Lemma 5, there exists an $x^* > x_o$ such that for every $x > x^*$, $\theta(x) > 9\nu/10$ and $\omega(x + iy_o, \partial\Omega_+, \Omega) > 2/5$.

Lemma 6. *There exists a constant $C > 0$ such that for every*

$$w \in \left\{w \in \Omega : y_o - \frac{\nu}{5} < \mathtt{Im}\,w < y_o + \frac{\nu}{5}, \quad \mathtt{Re}\,w > x^*\right\}, \tag{4.9}$$

$$g_\Omega(x^* + iy_o, w) > C\ \exp\left(-\pi \int_{x_o}^{\mathtt{Re}\,w} \frac{dx}{\theta(x)}\right). \tag{4.10}$$

Proof. The horizontal line $\{w : \mathtt{Im}\, w = y_o\}$ determines two prime ends P_1, P_2 on the boundary of Ω. One of them, say P_2, is supported at ∞ and is determined by the crosscuts $J(x)$, $x > x^*$. Let ψ map Ω conformally onto the strip

$$\Sigma = \{\zeta \in \mathbb{C} : \; -\pi/2 < \mathtt{Im}\, \zeta < \pi/2\}$$

so that P_1 corresponds to the prime end of Σ at $-\infty$ and P_2 corresponds to the prime end of Σ at $+\infty$. By the conformal invariance of the Green function

$$g_\Omega(x^* + iy_o, w) = g_\Sigma(\psi(x^* + iy_o), \psi(w)). \tag{4.11}$$

Also, by the conformal invariance of harmonic measure, both $\psi(x^* + iy_o)$ and $\psi(w)$ lie in a horizontal strip with midline $\{\zeta : \mathtt{Im}\, \zeta = 0\}$ and width equal to a positive constant, smaller than π. It follows from the Harnack chain principle that there exists a constant (independent of w) such that

$$g_\Sigma(\psi(x^* + iy_o), \psi(w)) \geq C\, g_\Sigma(\mathtt{Re}\, \psi(x^* + iy_o), \mathtt{Re}\, \psi(w)). \tag{4.12}$$

Let

$$\sigma(x^*) = \psi(J(x^*)) \qquad \text{and} \qquad \sigma(w) = \psi(J(\mathtt{Re}\, w)).$$

These are crosscuts of Σ joining the boundary lines. Set

$$\xi_1(x^*) = \inf\{\mathtt{Re}\, \zeta : \zeta \in \sigma(x^*)\} \tag{4.13}$$

and

$$\xi_2(w) = \sup\{\mathtt{Re}\, \zeta : \zeta \in \sigma(w)\}. \tag{4.14}$$

Since $\xi_2(w) - \xi_1(x^*) \geq \mathtt{Re}\, \psi(w) - \mathtt{Re}\, \psi(x^* + iy_o)$, we infer that

$$g_\Sigma(\mathtt{Re}\, \psi(x^* + iy_o), \mathtt{Re}\, \psi(w)) \geq g_\Sigma(\xi_1(x^*), \xi_2(w)). \tag{4.15}$$

By Lemma 2,

$$g_\Sigma(\xi_1(x^*), \xi_2(w)) \geq 2e^{-(\xi_2(w) - \xi_1(x^*))}. \tag{4.16}$$

By Ahlfors' second fundamental inequality (see e.g. [14, Theorem 3]), there exists a constant $C > 0$ (independent of w) such that

$$\xi_2(w) - \xi_1(x^*) \leq \pi \int_{x^*}^{\mathtt{Re}\, w} \frac{dx}{\theta(x)} + C \leq \pi \int_{x_o}^{\mathtt{Re}\, w} \frac{dx}{\theta(x)} + C. \tag{4.17}$$

The inequality (4.10) follows from (4.11)-(4.17). □

Proof of Theorem 3

Step 1

Let (T_t) be the induced semigroup of composition operators on A^p_α. By [20, Lemma 1],

$$\begin{aligned} \|T_t\|_{A^p_\alpha \to A^p_\alpha} &\leq 4^{|\alpha|/p} \left(\frac{1 + |\phi_t(0)|}{1 - |\phi_t(0)|}\right)^{\frac{\alpha+2}{p}} \\ &\leq 4^{|\alpha|/p}\, 2^{(\alpha+2)/p} \frac{1}{(1 - |\phi_t(0)|)^{(\alpha+2)/p}}. \end{aligned} \tag{4.18}$$

By [8, Theorem 2.8], the orbit $t \mapsto \phi_t(0)$ tends to 1 as $t \to +\infty$ with a fixed slope and not tangentially. Therefore, there exists a positive constant $C > 0$ such that for every $t > 0$,

$$1 - |\phi_t(0)| \geq C|1 - \phi_t(0)|. \tag{4.19}$$

Also, by [5], for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for every $t > 0$

$$|1 - \phi_t(0)| \geq C_\varepsilon e^{-\pi t/(\nu-\varepsilon)}. \tag{4.20}$$

It follows from (4.18)-(4.20) that there exists a constant $C(\varepsilon, \alpha, p)$ such that for every $t > 0$,

$$\|T_t\|_{A^p_\alpha \to A^p_\alpha} \leq C(\varepsilon, \alpha, p) \exp\left(\frac{\pi t(\alpha+2)}{(\nu-\varepsilon)p}\right). \tag{4.21}$$

Thus we can estimate the growth bound (or type) of the semigroup (T_t):

$$\begin{aligned} \lim_{t\to+\infty} \frac{\log \|T_t\|_{A^p_\alpha \to A^p_\alpha}}{t} &\leq \lim_{t\to+\infty} \left[\frac{\log C(\varepsilon,\alpha,p)}{t} + \frac{(\alpha+2)\pi}{p(\nu-\varepsilon)}\right] \\ &= \frac{\pi(\alpha+2)}{(\nu-\varepsilon)p}. \end{aligned} \tag{4.22}$$

Since this holds for every $\varepsilon > 0$, we conclude that

$$\lim_{t\to+\infty} \frac{\log \|T_t\|_{A^p_\alpha \to A^p_\alpha}}{t} \leq \frac{\pi(\alpha+2)}{\nu p}. \tag{4.23}$$

By the general semigroup theory, the spectrum of Γ lies in the closed half-plane

$$\left\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq \frac{\pi(\alpha+2)}{\nu p}\right\}.$$

Step 2

Let h be the Koenigs function of the semigroup (ϕ_t). If $\lambda = 0$, then $e^{\lambda h} = 1 \in A^p_\alpha$. Hence $\lambda \in \Lambda$. Suppose that $\lambda \in \mathbb{R}$ and

$$0 < \lambda < \frac{\pi(\alpha+2)}{\nu p}. \tag{4.24}$$

It is clear that $e^{\lambda h} \in A^p_\alpha$ if and only if $e^{\pi h/\nu} \in A^{\lambda\nu p/\pi}_\alpha$. The function $e^{\pi h/\nu}$ maps $\mathbb{D}$ conformally into a half-plane. Therefore (see Lemma 1), (4.24) implies that $e^{\pi h/\nu} \in A^{\lambda\nu p/\pi}_\alpha$. We conclude that the interval $[0, \pi(\alpha+2)/(\nu p))$ is contained in Λ.

Step 3

Suppose that $\lambda < 0$ and $\beta = \beta(\Omega) > 0$. Then $e^{\lambda h} \in A^p_\alpha$ if and only if $e^{-\pi h/\nu} \in A^{|\lambda|\nu p/\pi}_\alpha$. The function $e^{-\pi h/\nu}$ is 0-spiral-like (namely, star-like) with maximal angular opening equal to $\beta\pi/\nu$. So Theorem 1 implies that $e^{-\pi h/\nu} \in A^{|\lambda|\nu p/\pi}_\alpha$ if and only if

$$\frac{|\lambda|\nu p}{\pi} < \frac{(\alpha+2)\pi}{\beta\pi/\nu} \tag{4.25}$$

which is equivalent to

$$\frac{-\pi(\alpha+2)}{\beta p} < \lambda < 0. \tag{4.26}$$

We conclude that

$$\Lambda \cap (-\infty, 0) = \left(-\frac{\pi(\alpha+2)}{\beta p},\ 0\right). \tag{4.27}$$

If $\beta(\Omega) = 0$, the above argument shows that $(-\infty, 0) \subset \Lambda$.

Step 4
Since $\Omega = h(\mathbb{D})$ is contained in a horizontal strip, the Koenigs function has bounded imaginary part, say $|\mathtt{Im}\, h| \le c$. Suppose that $\lambda \in \Lambda$ which means that $e^{\lambda h} \in A^p_\alpha$. For every $y \in \mathbb{R}$,

$$\left|e^{(\lambda+iy)h}\right| = e^{-y\mathtt{Im}h}|e^{\lambda h}| \le e^{|y|c}|e^{\lambda h}|. \tag{4.28}$$

Therefore, $e^{(\lambda+iy)h} \in A^p_\alpha$. So $\lambda + iy \in \Lambda$.

Step 5
It remains to study the case

$$\lambda = \frac{\pi(\alpha+2)}{\nu p}. \tag{4.29}$$

By (2.2), $e^{\lambda h} \in A^p_\alpha$ if and only if

$$\int_{\mathbb{D}} |e^{\lambda h(z)}|^{p-2}\, |(e^{\lambda h(z)})'|^2 \left(\log \frac{1}{|z|}\right)^{\alpha+2} A(dz) < +\infty. \tag{4.30}$$

By the change of variable $w = h(z)$ and the conformal invariance of Green function, the above integral is finite if and only if

$$\int_{\Omega} |e^{p\lambda w}|\, g_\Omega(0,w)^{\alpha+2}\, A(dw) < +\infty. \tag{4.31}$$

Therefore, the λ given by (4.29) belongs to Λ if and only if

$$\int_{\Omega} e^{\pi \mathtt{Re} w(\alpha+2)/\nu}\, g_\Omega(0,w)^{\alpha+2}\, A(dw) < +\infty. \tag{4.32}$$

Assume that $W_\alpha(\Omega) < +\infty$. Consider the vertical crosscuts $J(0)$ and $J(\mathtt{Re}\, w)$ of Ω. By Beurling's estimates (see [7, p.371] and [11]),

$$\begin{aligned} g_\Omega(0,w) &\lesssim e^{-\pi\lambda(J(0),J(\mathtt{Re} w),\Omega)} \\ &\lesssim \exp\left(-\pi \int_0^{\mathtt{Re}\, w} \frac{dx}{\theta(x)}\right). \end{aligned} \tag{4.33}$$

Combining this estimate with the assumption $W_\alpha(\Omega) < +\infty$, we see that (4.32) holds and hence $\lambda \in \Lambda$.

Conversely, assume that (4.32) is true. Let

$$\Omega_1 = \Omega \cap \{w \in \mathbb{C} : y_o - \nu/5 < \mathtt{Im}\, w < y_o + \nu/5\}.$$

By Lemma 6 and Harnack's principle,

$$
\begin{aligned}
(4.34) \qquad W_\alpha(\Omega) &= \int_1^\infty e^{\pi(\alpha+2)t/\nu}\, e^{-\pi(\alpha+2)\int_0^t \frac{dx}{\theta(x)}}\, dt \\
&\lesssim \int_{\Omega_1} e^{\pi(\alpha+2)\mathrm{Re}w/\nu}\, e^{-\pi(\alpha+2)\int_0^{\mathrm{Re}w} \frac{dx}{\theta(x)}}\, A(dw) \\
&\lesssim \int_{\Omega_1} e^{\pi(\alpha+2)\mathrm{Re}w/\nu}\, g_\Omega(x^* + iy_o, w)^{\alpha+2}\, A(dw) \\
&\lesssim \int_{\Omega_1} e^{\pi(\alpha+2)\mathrm{Re}w/\nu}\, g_\Omega(0, w)^{\alpha+2}\, A(dw) \\
&\lesssim \int_{\Omega} e^{\pi(\alpha+2)\mathrm{Re}w/\nu}\, g_\Omega(0, w)^{\alpha+2}\, A(dw) < +\infty.
\end{aligned}
$$

We conclude that $\lambda \in \Lambda$ if and only if $W_\alpha(\Omega) < +\infty$. □

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