BULLETIN OF THE
HELLENIC MATHEMATICAL SOCIETY
Volume 61, 2017 (55–72)

THE CLASS OF $(1,3)$-GROUPS WITH A HOMOCYCLIC REGULATOR QUOTIENT OF EXPONENT p^5 IS OF FINITE REPRESENTATION TYPE.

DAVID M. ARNOLD, ADOLF MADER, OTTO MUTZBAUER, AND EBRU SOLAK

Abstract. The class of almost completely decomposable groups with a critical typeset of type $(1,3)$ and a homocyclic regulator quotient of exponent p^5 is shown to be of bounded representation type, i.e., in particular, a Remak-Krull-Schmidt class of torsion-free abelian groups. There are precisely 36 near-isomorphism classes of indecomposables all of rank ≤ 9.

1. Introduction

Kaplansky once observed, in essence, that anything can happen in torsion-free abelian groups even if the groups have finite rank. Thus to obtain results one has to consider subclasses and, in addition, a weakening of the isomorphism concept proved to be essential. A suitable, non-trivial, yet amenable class is the class of almost completely decomposable groups first introduced by E.L. Lady, [10]. Every torsion-free abelian group of finite rank is the direct sum of indecomposable groups, but even in the case of almost completely decomposable groups such decompositions can be notoriously "pathological". This problem is avoided by restricting the "regulator index" to be a power of a single prime and by employing a modest weakening of isomorphism, also due to E.L. Lady, called "near-isomorphism", [11]. In this way one obtains a Remak-Krull-Schmidt category and achieves a classification up to near-isomorphism as soon as the indecomposable groups in the class are found. As was shown in [3] most of these classes contain indecomposable groups of arbitrarily large rank in which case it is hopeless to try to describe all near-isomorphism classes of indecomposable groups. This leaves some special subclasses that may have a finite number of near-isomorphism classes of indecomposable groups. The class considered in this paper is shown to be such a class and the indecomposables are determined.

Any torsion-free abelian group G is an additive subgroup of a $\mathbb{Q}$-vector space V. The $\mathbb{Q}$-subspace of V generated by G is denoted $\mathbb{Q}\,G$ and $\dim(\mathbb{Q}\,G)$ is the *rank* of G. A torsion-free abelian group R of finite rank is *completely decomposable* if R is the direct sum of rank-1 groups. A *type* is an isomorphism class $[X]$ of a rank-one group X and $\tau = [X]$ is the type of X. Given a completely decomposable group R, we get a decomposition $R = \bigoplus_{\rho\in \mathrm{T}_{\mathrm{cr}}(R)} R_\rho$ where R_ρ is obtained by combining the rank-1 summands of type ρ of R into a summand $R_\rho(\neq 0)$. The set $\mathrm{T}_{\mathrm{cr}}(R)$ is the *critical typeset of R*.

A group G is *almost completely decomposable* if G contains a completely decomposable subgroup of finite index. An almost completely decomposable group G contains a well-understood fully invariant completely decomposable subgroup of finite index, the *regulator* $\mathrm{R}(G)$, [8]. The critical typeset of G is the critical typeset of R, $\mathrm{T}_{\mathrm{cr}}(G) = \mathrm{T}_{\mathrm{cr}}(R)$.

Date: Received 11 July 2017. Accepted 16 December 2017.
1991 *Mathematics Subject Classification.* 20K15, 20K25, 20K35, 15A21, 16G60.
Key words and phrases. Remak-Krull-Schmidt, almost completely decomposable group, indecomposable, bounded representation type.
Research of the third author supported by the DFG (Deutsche Forschungsgemeinschaft), Mu 628/10-2.

A type τ is *p-locally free* if $pX \neq X$ for any rank-1 group X with $[X] = \tau$. Given a finite poset S of p-locally free types, an almost completely decomposable group G is an *(S, p^k)-group* if $S = \mathrm{T}_{\mathrm{cr}}(G)$ and the exponent of the *regulator quotient* $G/\mathrm{R}(G)$ is p^k, i.e., $\exp(G/\mathrm{R}(G)) = p^k$. In the article [6] we used a more general definition of (S, p^k)-groups, namely $\mathrm{T}_{\mathrm{cr}}(G) \subset S$ and $p^k G \subset \mathrm{R}(G)$, so that the class (S, p^k) is closed under direct summands. Our approach here is motivated by obtaining a complete list of indecomposables. Two (S, p^k)-groups G and H are *nearly isomorphic* if there is an integer n relatively prime to p and homomorphisms $f : G \to H$ and $g : H \to G$ with $fg = n$ and $gf = n$. The group G is indecomposable if and only if G is nearly isomorphic to an indecomposable group, [1]. Moreover, an almost completely decomposable G with $G/\mathrm{R}(G)$ p-primary is, up to near-isomorphism, uniquely a direct sum of indecomposable groups, [9], [12, Corollary 10.4.6]. Consequently, a classification of all indecomposable (S, p^k)-groups up to near isomorphism results in a classification of all (S, p^k)-groups up to near isomorphism. Hence, for almost completely decomposable groups G with $G/\mathrm{R}(G)$ p-primary, the main question is to determine the near-isomorphism classes of indecomposable (S, p^k)-groups.

There is an interesting connection between almost completely decomposable groups and representations of finite partially ordered sets. (For a definition see below.) This is where the terms “bounded representation type” and “unbounded representation type” originate. Details are in [6] that also contains a complete survey of the known and open problems in the subject. The present paper settles one of these open problems.

We denote by $(1, n)$ the poset $\{\tau_0, \tau_1 < \cdots < \tau_n\}$ where τ_0 is incomparable to any one of the other elements. In this paper we study homocyclic $((1, 3), p^5)$-groups, where $G \in ((1, 3), p^5)$ is *homocyclic* if $G/\mathrm{R}(G)$ is a direct sum of cyclic groups all of the same order $p^5 = \exp(G/\mathrm{R}(G))$. We present a complete collection of near-isomorphism types of indecomposable homocyclic groups in $((1, 3), p^5)$. There are precisely 36 near-isomorphism classes, and all have rank ≤ 9. The proof includes finding a normal form for coordinate matrices of $((1, 3), p^5)$-groups, see Section 3.

We remark that there are infinitely many indecomposable $((1, 3), p^5)$-groups when the regulator quotient is not required to be homocyclic ([6]). It is therefore to be expected that the boundedness proof in the homocyclic case will be delicate.

2. Matrices

We deal with integer matrices. A *line* of a matrix is a row or a column. *Transformations* of matrices are successive applications of elementary transformations. Matrices are simplified by making entries equal to 0. While annihilating an entry, other entries that were originally zero may become nonzero; such entries are called *fill-ins* and must be removed, i.e., the original 0 must be restored. There is a fixed exponent p^k and entries may be changed modulo p^k, in particular $p^h = 0$ if $h \geq k$. A *unit* in our context is an integer that is relatively prime to p. An integer matrix $A = [a_{i,j}]$ is called *p-reduced (modulo p^k)* if

(1) there is at most one 1 in a line and all other entries are in $p\mathbb{Z}$,
(2) if an entry 1 of A is at the position (i_s, j_s), then $a_{i_s,j} = 0$ for all $j > j_s$ and $a_{i,j_s} = 0$ for all $i < i_s$, and $a_{i_s,j}, a_{i,j_s} \in p\mathbb{Z}$ for all $j < j_s$ and all $i > i_s$.

Thus in a p-reduced matrix, the entries left of and below an entry 1 are in $p\mathbb{Z}$.

Lemma 1. (cf. [5, Lemma 1]) *Let A be an integer matrix.*

(1) *The matrix A can be transformed into a p-reduced matrix by multiplying lines by units, by elementary row transformations upward and by elementary column transformations to the right.*
(2) *If in addition row transformations down are allowed, then the matrix A can be transformed into a p-reduced matrix where all entries are* 0 *below a* 1.

3. Homocyclic $((1,n),p^k)$-Groups and Coordinate Matrices

The following terminology is used in this paper. Details, equivalent formulations, and confirmation of assertions can be found in [1] or [12]. For a general treatment of (S,p^k)-groups see [6].
Let G be an almost completely decomposable group. The isomorphism types of the regulator $\mathrm{R}(G)$ and the regulator quotient $G/\mathrm{R}(G)$ are near-isomorphism invariants of G. In particular, the rank r of the regulator quotient is an invariant of G. Given a prime p, G is *p-reduced* if the localization $G_{(p)}$ of G at p is a free $\mathbb{Z}_{(p)}$-module, or, equivalently, if each type $\tau \in \mathrm{T_{cr}}(G)$ is *p-locally free*. An almost completely decomposable group without summands of rank 1 is called *clipped*.
Let R be a completely decomposable subgroup of G with G/R a finite p-group. A coordinate matrix of G is obtained by means of bases of R and G/R. Write $R = S_1x_1 \oplus \cdots \oplus S_mx_m$ with $x_i \in R$, $S_i = \{s \in \mathbb{Q} : sx_i \in R\}$, and $p^{-1} \notin S_i$. In this case, $\{x_1, ..., x_m\}$ is called a *p-basis* of R. Since we focus on homocyclic $((1,n),p^k)$-groups, the general notation (see [6]) is not used here and we simply define a coordinate matrix of the almost completely decomposable group G relative to a completely decomposable subgroup R with homocyclic quotient G/R of exponent p^k.
A matrix $M = [m_{i,j}]$ is a *coordinate matrix* of G modulo R if M is integral, there is a basis $(\gamma_1,\ldots,\gamma_r)$ of G/R, there are representatives $g_i \in G$ of γ_i, and there is a p-basis $\{x_1,\ldots,x_m\}$ of R such that

$$g_i = p^{-k}(\textstyle\sum_{j=1}^m m_{i,j}x_j) \quad \text{where} \quad \langle\gamma_i\rangle \cong \mathbb{Z}_{p^k}, \quad k = \exp(G/R).$$

A coordinate matrix M of G is of size $r \times m$ and coordinate matrices are determined only up to congruence modulo p^k. This means that coordinate matrices that are congruent modulo p^k are considered equal.
Since $(\gamma_1,\ldots,\gamma_r)$ is a basis of G/R, a coordinate matrix M of size $r \times m$ has (p-)rank r, i.e., the r rows of M are linearly independent modulo p^k. Each column of a coordinate matrix corresponds to a type. So we speak of the type of a column and of τ-columns of M. The number $r_\tau(G)$ of τ-columns of M equals $\mathrm{rank}(R_\tau)$ and is called the *τ-rank of G*. These *type ranks* are near-isomorphism invariants of G.
A matrix M is said to be *decomposable* if there are permutation matrices X, Y, such that $XMY = M_1 \oplus M_2$. The following lemma is a well-known fact and we include the simple argument for the convenience of the reader.

Lemma 2. (cf. [6, Lemma 3.1]) *The almost completely decomposable group G is decomposable if and only if there exists a decomposable coordinate matrix of G.*
In particular, if G has a decomposable coordinate matrix M, i.e., $XMY = M_1 \oplus M_2$ with permutation matrices X, Y, then $G = G_1 \oplus G_2$ where G_i has the coordinate matrix M_i.
Clearly, a 0*-column of M displays a summand of rank* 1*, i.e., G is not clipped if M contains a* 0*-column.*

Proof. Assume for the coordinate matrix M of G relative to the regulator $\mathrm{R}(G)$ that $XMY = M_1 \oplus M_2$. The coordinate matrix is obtained by means of a p-basis B of $\mathrm{R}(G)$. Each column of M

corresponds to a basis element and the columns of the M_i determine a partition $B = B_1 \cup B_2$ of the p-basis and there is a corresponding direct decomposition $\mathrm{R}(G) = R_1 \oplus R_2$. It is easy to see that $G = G_1 \oplus G_2$ where $G_i = \langle R_i \rangle_*$, the purification of R_i in G.
Conversely, if $G = G_1 \oplus G_2$, then the coordinate matrix of G is the direct sum of the coordinate matrices of G_1 and of G_2. □

Henceforth let G be a homocyclic $((1,n),p^k)$-group of rank m with regulator $R = \mathrm{R}(G)$ and critical typeset $\mathrm{T}_{\mathrm{cr}}(G) = \{\tau_0, \tau_1 < \cdots < \tau_n\}$, where $\mathrm{T}_{\mathrm{cr}}(G)$ is a poset of p-locally free types, and G/R is a homocyclic group of rank r and of exponent p^k, and we write $M = \left[M_{\tau_0} \,\middle\|\, M_{\tau_1} \,\middle|\, \cdots \,\middle|\, M_{\tau_n}\right]$ where M_{τ_i} contains all τ_i-columns.
We call transformations of rows and of columns of a coordinate matrix of G *allowed* if the transformed coordinate matrix is the coordinate matrix of a near-isomorphic group. In particular, transformations are allowed if they are due to changes of the two bases involved. The following transformations are allowed in our case (see [6, page 471], [2, Theorem 12])

(a),(b) Add an integer multiple of a row of M to any other row of M (this is because our groups are homocyclic);
(c) multiply a row of M by a unit modulo p^k;
(d) interchange any two rows of M;
(e) for $j \geq i$, add an integer multiple of a column of M_{τ_i} to a column of M_{τ_j};
(f) multiply a column of M by a unit modulo p^k;
(g) interchange any two columns of M_{τ_i}.

If the coordinate matrix M is formed with respect to the regulator R, the submatrices of M formed by all τ_0-columns and the rest matrix both have rank equal to the rank r of the regulator quotient. Conversely, if the coordinate matrix M is formed with respect to a completely decomposable subgroup R of finite index and M satisfies the stated rank conditions, then R is the regulator ([12, Theorem 8.1.10], [2, Lemma 13]). These rank conditions are called the *Regulator Criterion*.
For clipped groups the τ_0-columns of a coordinate matrix always can be transformed to the identity matrix without any change of the rest, because of the Regulator Criterion. By Lemma 1 the part $\left[M_{\tau_1} \,\middle|\, \cdots \,\middle|\, M_{\tau_n}\right]$ of a coordinate matrix M can be changed into p-reduced form, cf. Proposition 3.

4. Standard Coordinate Matrices

We establish a standard form for coordinate matrices of homocyclic $((1,n),p^k)$-groups. If $A = [A_{i,j}]$ is a block matrix, then we denote by $A_{*,j}$ and by $A_{i,*}$ the jth block column and the ith-block row of A, respectively. Integer entries that are prime to p are called units.
Our main technique is forming Smith Normal Forms and variations thereof. Two matrices A, B are said to be *equivalent* if there are invertible matrices X, Y such that $B = XAY$. It is well known that every integral matrix is equivalent to a matrix in *Smith Normal Form*, $\operatorname{diag}(a_{1,1}, \ldots, a_{k,k}, 0, \ldots, 0)$ where $a_{i,i}$ divides $a_{i+1,i+1}$. Here we consider integer (coordinate) matrices and deal with them modulo p^k, because in our setting matrices that are congruent modulo p^k describe the same group. So we have the *(modified) Smith Normal Form*

$$\begin{bmatrix} I & & & & \\ & pI & & & \\ & & \ddots & & \\ & & & p^{k-1}I & \\ & & & & 0 \end{bmatrix}$$

where the empty space indicates 0-blocks and I stands for identity matrices of various sizes. We call $\begin{bmatrix} p^h I & 0 \\ 0 & p^{h+1}X \end{bmatrix}$ the *(partial) Smith Normal Form.* In our case $p^5 = 0$, and the possible (partial) Smith Normal Forms are $\begin{bmatrix} I & 0 \\ 0 & pX \end{bmatrix}$, $\begin{bmatrix} pI & 0 \\ 0 & p^2X \end{bmatrix}$, $\begin{bmatrix} p^2I & 0 \\ 0 & p^3X \end{bmatrix}$, $\begin{bmatrix} p^3I & 0 \\ 0 & p^4X \end{bmatrix}$, $\begin{bmatrix} p^4I & 0 \\ 0 & 0 \end{bmatrix}$, where lines may be absent.

There is a basic problem for block matrices when forming Smith Normal Forms of sub-blocks, namely if some sub-block is replaced by its (partial) Smith Normal Form, then the whole matrix gets bigger in the sense that there are more block lines.

A block matrix $A = [A_{i,j}]$ with blocks $A_{i,j}$ is said to be *completely reduced* if all blocks are either 0 or of the form $p^l I$, l some nonnegative integer. A block $A_{i,j}$ is either reduced or it is a *placeholder* with changing values.

Let $[A_1 \big| A_2 \big| \dots \big| A_r]$ be a sequence of blocks in a block row of a block matrix M. Assume that arbitrary row transformations in the block rows can be applied, arbitrary column transformations can be applied in each A_i-column and all column transformations to the right can be done. Then we form the partial Smith Normal Form of A_1 and annihilate, assuming that this is possible, with $I \subset A_1$ to the right in all $A_2, A_3, \dots$ and downward in A_1. This splits $A_2 = \begin{bmatrix} 0 \\ X_2 \end{bmatrix}$. Actually $A_i = \begin{bmatrix} 0 \\ X_i \end{bmatrix}$ for all $i \geq 2$. Then form the Smith Normal Form of $X_2 = \begin{bmatrix} I & 0 \\ 0 & pX \end{bmatrix}$ and annihilate with $I \subset X_2$ in $A_3, A_4, \dots$ and downward in A_2. Continuing through the sequence of the A_i's, we obtain the so-called *iterated (partial) Smith Normal Form* of the sequence $[A_1 \big| A_2 \big| \dots]$ *starting with* A_1. The iterated Smith Normal Form looks like:

$$[A_1 \big| A_2 \big| A_3 \big| \cdots] = \left[\begin{array}{c|c|c|c} I \; 0 & 0 \; 0 & 0 \; 0 & \cdots \\ \hline 0 \; pX & I \; 0 & 0 \; 0 & \cdots \\ \hline 0 \; pX & 0 \; pX & I \; 0 & \cdots \\ \hline 0 \; pX & 0 \; pX & 0 \; pX & \cdots \end{array}\right].$$

If a block $I \subset A_i$ does not exist, then the corresponding block row and block column are absent. There is an obvious variant for columns instead of rows and the start is from below. In general, if we form the iterated Smith Normal Form we tacitly assure that the already obtained "reduced blocks" of the whole coordinate matrix can be reestablished.

Proposition 3. ([7, Proposition 3]) *Let n be a natural number and let p be a prime and $(1, n) = (\tau_0, \tau_1 < \cdots < \tau_n)$. A homocyclic $((1, n), p^k)$-group without summands of rank ≤ 3 has a coordinate matrix of the form*

$$(1) \quad [I \| pA \| I] = \left[\begin{array}{c||ccccc||ccccc} & pA_{2,1} & 0 & 0 & \cdots & 0 & I(\tau_2) & 0 & \cdots & \cdots & 0 \\ & pA_{3,1} & pA_{3,2} & 0 & \cdots & 0 & 0 & I(\tau_3) & 0 & \cdots & 0 \\ I & \vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & \ddots & & \vdots \\ & \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & \ddots & \\ & pA_{n,1} & pA_{n,2} & \cdots & \cdots & pA_{n,n-1} & 0 & 0 & \cdots & \cdots & I(\tau_n) \\ \hline \tau_0 & \tau_1 & \tau_2 & \tau_3 & \cdots & \tau_{n-1} & \tau_2 & \tau_3 & \cdots & \cdots & \tau_n \end{array}\right] \equiv \left[\begin{array}{c||cccccccc} & pA_{2,1} & I & 0 & 0 & \cdots & \cdots & 0 & 0 \\ & pA_{3,1} & 0 & pA_{3,2} & I & \cdots & \cdots & 0 & 0 \\ I & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ & pA_{n,1} & 0 & pA_{n,2} & 0 & \cdots & \cdots & pA_{n,n-1} & I \\ \hline \tau_0 & \tau_1 & \tau_2 & \tau_2 & \tau_3 & \cdots & \cdots & \tau_{n-1} & \tau_n \end{array}\right]$$

The block matrix $A = [A_{i,j}]$ is lower triangular as a block matrix and A is p-reduced. For the block $pA_{n,1}$ there is a matrix D such that $pA_{n,1} = p^2D$. The blocks $pA_{i,j}$ are of the form

$$pA_{i,j} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & pI & 0 \\ 0 & 0 & p^2A'_{i,j} \end{array}\right], \text{ where lines may be absent.}$$

If G is a homocyclic $((1,3),p^5)$-group, then its coordinate matrix has the general form

$$(2) \qquad \left[\begin{array}{c||cc|cccc|c} I000 & pI & 0 & 0 & 0 & I0 & 00 \\ 0I00 & 0 & p^2A & 0 & 0 & 0I & 00 \\ \hline\hline 00I0 & p^2B & p^2C & pI & 0 & 00 & I0 \\ 000I & p^2D & p^2E & 0 & p^2F & 00 & 0I \\ \hline \tau_0 & \tau_1 & & & \tau_2 & & \tau_3 \end{array}\right].$$

We omit the identity matrix in front formed by the τ_0-columns, and τ_i in the last column indicates the location of the $I(\tau_i)$-matrix in the back. A coordinate matrix as in (2) is called *standard* for homocyclic $((1,3),p^5)$-groups.
Note that we use all blocks pI in the coordinate matrix to annihilate, so there remain only blocks of the form p^2X. Moreover, we use partitioning lines to separate types.

A coordinate matrix of G as in Proposition 3 displays near-isomorphism invariants of G.

Proposition 4. (cf. [5, Proposition 5]) *Let G be a homocyclic $((1,n),p^k)$-group with a coordinate matrix as in* Proposition 3. *Then the size of the $I(\tau_i)$'s, the size of the blocks $A_{i,j}$ and the numbers of entries p in a block $pA_{i,j}$ are near-isomorphism invariants of G for all i,j.*

5. Indecomposable Groups in the Class of Homocyclic $((1,3),p^5)$-Groups

$(1,3)$-groups are of rank ≥ 4 because the critical typeset consists of 4 types.
Below we present a list of homocyclic $((1,3),p^5)$-groups G of ranks between 4 and 9 given by their coordinate matrices. They are ordered by their ranks, by their type ranks and lexicographically. Later we show that this list contains all indecomposable homocyclic $((1,3),p^5)$-groups up to near-isomorphism and that any two groups in this list are not near-isomorphic.

List of indecomposable homocyclic $((1,3),p^5)$-groups

Groups of rank 4

$\left[1\|p^4|p^3|1\right] = [4.1]$, $\left[1\|p^4|p^2|1\right] = [4.2]$, $\left[1\|p^4|p|1\right] = [4.3]$, $\left[1\|p^3|p^2|1\right] = [4.4]$,
$\left[1\|p^3|p|1\right] = [4.5]$, $\left[1\|p^2|p|1\right] = [4.6]$.

Groups of rank 5

$\left[\begin{array}{c||c|c|c} 10 & p^3 & 1 & 0 \\ 01 & p^4 & 0 & 1 \end{array}\right] = [5.1]$, $\left[\begin{array}{c||c|c|c} 10 & p^2 & 1 & 0 \\ 01 & p^4 & 0 & 1 \end{array}\right] = [5.2]$, $\left[\begin{array}{c||c|c|c} 10 & p & 1 & 0 \\ 01 & p^4 & 0 & 1 \end{array}\right] = [5.3]$, $\left[\begin{array}{c||c|c|c} 10 & p^2 & 1 & 0 \\ 01 & p^3 & 0 & 1 \end{array}\right] = [5.4]$,
$\left[\begin{array}{c||c|c|c} 10 & p & 1 & 0 \\ 01 & p^3 & 0 & 1 \end{array}\right] = [5.5]$, $\left[\begin{array}{c||c|c|c} 10 & p & 1 & 0 \\ 01 & p^2 & 0 & 1 \end{array}\right] = [5.6]$.

Groups of rank 6

$$\left[\begin{array}{cc||cc|c|c} 1&0&p^2&0&1&0\\ 0&1&p^3&p^4&0&1\end{array}\right]=[6.1],\quad \left[\begin{array}{cc||cc|c|c} 1&0&p&0&1&0\\ 0&1&p^3&p^4&0&1\end{array}\right]=[6.2],\quad \left[\begin{array}{cc||cc|c|c} 1&0&p&0&1&0\\ 0&1&p^2&p^4&0&1\end{array}\right]=[6.3],\quad \left[\begin{array}{cc||cc|c|c} 1&0&p&0&1&0\\ 0&1&p^2&p^3&0&1\end{array}\right]=[6.4];$$

$$\left[\begin{array}{cc||c|cc|c} 1&0&p^3&0&1&0\\ 0&1&p^4&p^3&0&1\end{array}\right]=[6.5],\quad \left[\begin{array}{cc||c|cc|c} 1&0&p^3&0&1&0\\ 0&1&p^4&p^2&0&1\end{array}\right]=[6.6],\quad \left[\begin{array}{cc||c|cc|c} 1&0&p^2&0&1&0\\ 0&1&p^4&p^3&0&1\end{array}\right]=[6.7],\quad \left[\begin{array}{cc||c|cc|c} 1&0&p^2&0&1&0\\ 0&1&p^3&p^2&0&1\end{array}\right]=[6.8];$$

$$\left[\begin{array}{cc||c|c|cc} 1&0&p^3&p^2&1&0\\ 0&1&p^4&0&0&1\end{array}\right]=[6.9],\quad \left[\begin{array}{cc||c|c|cc} 1&0&p^3&p&1&0\\ 0&1&p^4&0&0&1\end{array}\right]=[6.10],\quad \left[\begin{array}{cc||c|c|cc} 1&0&p^2&p&1&0\\ 0&1&p^4&0&0&1\end{array}\right]=[6.11],\quad \left[\begin{array}{cc||c|c|cc} 1&0&p^2&p&1&0\\ 0&1&p^3&0&0&1\end{array}\right]=[6.12].$$

Groups of rank 7

$$\left[\begin{array}{cc||c|cc|cc} 1&0&p^3&p&0&1&0\\ 0&1&p^4&0&p^3&0&1\end{array}\right]=[7.1],\quad \left[\begin{array}{cc||cc|c|cc} 1&0&p^2&0&p&1&0\\ 0&1&p^3&p^4&0&0&1\end{array}\right]=[7.2],\quad \left[\begin{array}{cc||cc|cc|c} 1&0&p^2&0&0&1&0\\ 0&1&p^3&p^4&p^2&0&1\end{array}\right]=[7.3].$$

Groups of rank 8

$$\left[\begin{array}{ccc||c|cc|cc} 1&0&0&p^2&0&1&0&0\\ 0&1&0&p^3&p^2&0&1&0\\ 0&0&1&p^4&0&0&0&1\end{array}\right]=[8.1],\quad \left[\begin{array}{ccc||cc|c|cc} 1&0&0&p&p^2&1&0&0\\ 0&1&0&p^2&0&0&1&0\\ 0&0&1&0&p^4&0&0&1\end{array}\right]=[8.2],\quad \left[\begin{array}{ccc||cc|cc|c} 1&0&0&p&0&1&0&0\\ 0&1&0&0&p^3&0&1&0\\ 0&0&1&p^3&p^4&0&0&1\end{array}\right]=[8.3].$$

Groups of rank 9

$$\left[\begin{array}{ccc||cc|cc|cc} 1&0&0&p^3&0&0&1&0&0\\ 0&1&0&0&p^2&p&0&1&0\\ 0&0&1&p^4&p^3&0&0&0&1\end{array}\right]=[9.1],\quad \left[\begin{array}{ccc||cc|cc|cc} 1&0&0&p&0&0&1&0&0\\ 0&1&0&0&p^3&p&0&1&0\\ 0&0&1&p^3&p^4&0&0&0&1\end{array}\right]=[9.2],\quad \left[\begin{array}{ccc||cc|cc|cc} 1&0&0&p^2&0&0&1&0&0\\ 0&1&0&0&p^3&p^2&0&1&0\\ 0&0&1&p^4&p^4&0&0&0&1\end{array}\right]=[9.3],$$

$$\left[\begin{array}{ccc||cc|cc|cc} 1&0&0&p&0&0&1&0&0\\ 0&1&0&0&p^4&p^3&0&1&0\\ 0&0&1&p^2&p^3&0&0&0&1\end{array}\right]=[9.4],\quad \left[\begin{array}{ccc||cc|cc|cc} 1&0&0&p^2&0&0&1&0&0\\ 0&1&0&0&p^4&p^2&0&1&0\\ 0&0&1&p^3&p^4&0&0&0&1\end{array}\right]=[9.5],\quad \left[\begin{array}{ccc||cc|cc|cc} 1&0&0&p^2&0&0&1&0&0\\ 0&1&0&p^3&p^4&p^2&0&1&0\\ 0&0&1&p^4&0&0&0&0&1\end{array}\right]=[9.6].$$

Proposition 5. *The groups in the list all are indecomposable and pairwise not near-isomorphic.*

Proof. The groups in the list are pairwise not near-isomorphic. By [5, Proposition 5], if G and H are near-isomorphic, then also p^lG+R and p^lH+R are near-isomorphic. Let p^k be the exponent of the regulator quotient of G. By [5, Proposition 4] the regulator of G is also the regulator of p^lG+R. So the coordinate matrix of p^lG+R simply is the coordinate matrix of G considered modulo p^{k-l}. The type ranks are near-isomorphism invariants. So it is enough to compare groups that have the same type ranks. The groups p^lG+R decompose as a rule and by Faticoni-Schultz [9] these decompositions are unique up to near–isomorphism. Moreover, we know all indecomposables in the classes $((1,3),p^l)$ where $l\leq 4$, cf. [5]. Hence an easy check clarifies that all groups in the list are not near-isomorphic.

We exemplify this in the most difficult case, namely for the groups of rank 9. We consider for those groups G_i of type [9.i] the decomposition of the respective subgroups pG_i+R. The decompositions of the groups pG_i+R are as follows:

pG_1+R decomposes in indecomposable summands of rank 3 and 6,
pG_2+R decomposes in indecomposable summands of rank 4 and 5,
pG_3+R decomposes in indecomposable summands of rank 2, 3 and 4,
pG_4+R decomposes in indecomposable summands of rank 3 and 6,
pG_5+R decomposes in indecomposable summands of rank 1, 3 and 5,
pG_6+R decomposes in indecomposable summands of rank 1, 2 and 6.

The summands of rank 3 in the decompositions of pG_1+R and pG_4+R are not near-isomorphic. In fact, the coordinate matrix of $pG_1 + R$ is

$$\left[\begin{array}{c||c|c|c} 100 & p^3\ 0 & 01 & 00 \\ 010 & 0\ p^2 & p0 & 10 \\ 001 & 0\ p^3 & 00 & 01 \end{array}\right] = \left[\begin{array}{c||c|c|c} 1 & p^3 0 & 1 & 0 \end{array}\right] \oplus \left[\begin{array}{c||c|c|c} 10 & p^2 & p & 10 \\ 01 & p^3 & 0 & 01 \end{array}\right]$$

and the coordinate matrix of $pG_4 + R$ is

$$\left[\begin{array}{c||c|c|c} 100 & p\ \ 0 & 0\ 1 & 00 \\ 010 & 0\ \ 0 & p^3 0 & 10 \\ 001 & p^2 p^3 & 0\ 0 & 01 \end{array}\right] = \left[\begin{array}{c||c|c|c} 1 & 0 & p^3 & 1 \end{array}\right] \oplus \left[\begin{array}{c||c|c|c} 10 & p\ \ 0 & 1 & 0 \\ 01 & p^2 p^3 & 0 & 1 \end{array}\right].$$

Thus groups of rank 9 in the list are pairwise not near-isomorphic.

To show that the groups in the list are indecomposable we utilize the connection with representations. We cover the situation at hand by letting $S := (\tau_0, \tau_1 < \cdots < \tau_n)$ where the τ_i are p-free types.

Let G be an almost completely decomposable group with $\mathrm{T_{cr}}(G) = S$ and $p^k = \exp(G/\mathrm{R}(G))$ with regulator $R := \mathrm{R}(G) = \bigoplus_{\rho \in S} R_\rho$. Define $\overline{} : R \to R/p^kR : \overline{x} = x + p^kR$, so $\overline{R} = R/p^kR$. The *(anti)-representation* U_G *of* G is given by $U_G = (\overline{R}, \overline{R(\sigma)}, \overline{p^kG} : \sigma \in S)$. Let G' be another group with $\mathrm{T_{cr}}(G') = S$, $p^k = \exp(g/\mathrm{R}(G))$, regulator R' and representation $U_{G'}$. A homomorphism $f : U_G \to U'_G$ is a $\mathbb{Z}_{p^k} = \mathbb{Z}/p^k\mathbb{Z}$-module homomorphism $f : \overline{R} \to \overline{R'}$ such that $\forall\, \sigma \in S : f(\overline{R(\sigma)}) \subset \overline{R'(\sigma)}$ and $f(\overline{p^kG}) \subset \overline{p^kG'}$. The following facts are well-known and can be found in [1], [12] and [4].

- $\overline{R}$ is a finite free $\mathbb{Z}_{p^k}$-module.
- In general, $\overline{p^kG}$ is a finite $\mathbb{Z}_{p^k}$-module, and in the homocyclic case it is free.
- $\overline{p^kG} \cong G/R$.
- G is nearly isomorphic with G' if and only if $\mathrm{R}(G) \cong \mathrm{R}(G')$ and U_G, $U_{G'}$ are isomorphic. In practice, we always have $\mathrm{R}(G) = \mathrm{R}(G')$.
- G is indecomposable if and only if $\operatorname{End} U_G$ contains no idempotents other than 0 and 1.

For simplicity, we assume in the following that our groups are homocyclic. We associate with a representation U_G a $\mathbb{Z}_{p^k}$-matrix that encodes $\overline{p^kG}$. Let $\{x_1, \dots, x_m\}$ be a p-basis of R and let $\{g_1, \dots, g_r\}$ be a basis of $\overline{p^kG}$. Then $\{\overline{x_1}, \dots, \overline{x_m}\}$ is a basis of the free module $\overline{R}$. Expressing the generators g_i in terms of the basis $\{\overline{x_1}, \dots, \overline{x_m}\}$ we obtain the *representing matrix* of G. This matrix (in the homocyclic case) is identical with the earlier (integral) coordinate matrix except that instead of "working modulo p^k", the entries are considered elements of $\mathbb{Z}_{p^k}$.

Exemplarily we show that one of the group G of rank 8 is indecomposable. Its representing matrix is

$$M = \left[\begin{array}{c||c|c|c} 100 & p\ \ 0 & 1 & 00 \\ 010 & p^2 p^3 & 0 & 10 \\ 001 & 0\ p^4 & 0 & 01 \end{array}\right].$$

The representing matrix comes with basis of $\overline{R}$ and we view the elements of $\overline{R}$ as coordinate vectors with respect to this basis. In particular, $\overline{p^5G}$ is simply the row space of M. To show that G is indecomposable we show that any idempotent endomorphism of $U_G = (\overline{R}, \overline{R(\sigma)}, \overline{p^3G} : \sigma \in \{\tau_0, \tau_1, \tau_2, \tau_3\})$ is either the zero map 0 or the identity 1. In terms of coordinates an endomorphism of $\overline{R}$ is a Z_{p^5}-matrix that acts by right multiplication on the elements

$(x_{01}, x_{02}, x_{03}, x_{11}, x_{12}, x_{21}, x_{22}, x_3)$ of $\overline{R}$. The requirement that $f(\overline{R(\sigma)}) \subset \overline{R(\sigma)}$ implies that f, as a matrix, is of the form

$$f = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ 0 & 0 & 0 & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} \\ 0 & 0 & 0 & 0 & 0 & c_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & d_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{21} & d_{22} \end{bmatrix}.$$

The additional requirement that $f(\overline{p^5G}) \subset \overline{p^5G}$, i.e., that the row space of M is invariant under right multiplication by f, has a very handy description due to the fact that M has the right inverse $M^* = \begin{bmatrix} 100 \\ 010 \\ 001 \\ 000 \\ 000 \\ 000 \\ 000 \\ 000 \end{bmatrix}$. In fact, [4, Theorem 4.3], $f(\overline{p^3G}) \subset \overline{p^3G}$ if and only if $Mf = MfM^*M$.

Using a computer algebra program we find that (in our example) $Mf = MfM^*M$ if and only if

$$pb_{11} = pa_{11} + p^2a_{12} \Rightarrow b_{11} \equiv a_{11} \mod p \tag{3}$$
$$pb_{12} = p^4a_{13} + p^3a_{12} \Rightarrow p^3b_{12} = 0 \tag{4}$$
$$p^2b_{11} + p^3b_{21} = p^2a_{22} + pa_{21} \Rightarrow p^4b_{11} = p^4a_{22} + p^3a_{21} \tag{5}$$
$$p^2b_{12} + p^3b_{22} = p^3a_{22} + p^4a_{23} \Rightarrow p^3b_{12} + p^4b_{22} = p^4a_{22} \tag{6}$$
$$p^4b_{21} = p^2a_{32} + pa_{31} \tag{7}$$
$$p^4b_{22} = p^4a_{33} + p^3a_{32} \tag{8}$$
$$c_{11} + pb_{13} = a_{11} \Rightarrow c_{11} \equiv a_{11} \mod p \tag{9}$$
$$b_{23}p^3 + b_{13}p^2 = a_{21} \Rightarrow p^3a_{21} = 0 \tag{10}$$
$$b_{24}p^3 + b_{14}p^2 + d_{11} = a_{22} \Rightarrow d_{11} \equiv a_{22} \mod p \tag{11}$$
$$p^4b_{23} = a_{31} \Rightarrow pa_{31} = 0 \tag{12}$$
$$b_{24}p^4 + d_{21} = a_{32} \Rightarrow p^3d_{21} = p^3a_{32} \tag{13}$$
$$b_{25}p^4 + d_{22} = a_{33} \Rightarrow d_{22} \equiv a_{33} \mod p. \tag{14}$$

In this listing some equations are omitted because they are of no consequence and some immediate consequences are listed. The immediate goal is to show that $f \mod p$ is trivial, i.e. $f \equiv 0 \mod p$ or $f \equiv 1 \mod p$. As f is idempotent it then follows that $f = 0$ or $f = 1$. We now use the previous conclusions to draw further conclusions.

(15) $$(3)\Rightarrow p^4b_{11} = p^4a_{22} + p^3a_{21} \overset{(8)}{\Rightarrow} b_{11} \equiv a_{22} \mod p$$

(16) $$(4)\Rightarrow p^3b_{12} + p^4b_{22} = p^4a_{22} \overset{(2)}{\Rightarrow} b_{22} \equiv a_{22} \mod p$$

(17) $$(5)\Rightarrow p^4b_{21} = p^2a_{32} + pa_{31} \overset{(10)}{\Rightarrow} p^4b_{21} = p^2a_{32} \Rightarrow p^3a_{32} = 0$$

(18) $$(6)\Rightarrow p^4b_{22} = p^4a_{33} + p^3a_{32} \overset{(15)}{\Rightarrow} b_{22} \equiv a_{33} \mod p$$

(19) $$(11)\Rightarrow p^3d_{21} = p^3a_{32} \overset{(15)}{\Rightarrow} p^3d_{21} = 0$$

We now have, with $\alpha \equiv a_{11} \mod p$, that

$$f \equiv \begin{bmatrix} \alpha & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & a_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & b_{13} & b_{14} & b_{15} \\ 0 & 0 & 0 & b_{21} & \alpha & b_{23} & b_{24} & b_{25} \\ 0 & 0 & 0 & 0 & 0 & \alpha & c_{12} & c_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha & d_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix} \mod p.$$

As $f^2 = f$ it follows that $f \equiv 0, 1 \mod p$ and hence $f = 0, 1$.
All other indecomposability proofs are similar and straightforward. □

6. The Class of Homocyclic $((1,3), p^5)$-Groups is Bounded

Theorem 6. *There are precisely* 36 *near-isomorphism classes of indecomposable homocyclic* $((1,3), p^5)$*-groups as in the list.*

Proof. We assume that the group G is an indecomposable homocyclic $((1,3), p^5)$-group with coordinate matrix in standard form, cf. (2), i.e., we deal with block matrices. If a summand is displayed, it is either contradictory or we check its class in the list and assume furthermore that G is not in this class. So we will find all indecomposable homocyclic $((1,3), p^5)$-groups. In particular, the occurrence of a summand of rank ≤ 3 or a summand that is no $(1,3)$-group is contradictory.
We start successively forming (partial) Smith Normal Form's of sub-blocks to split off the parts p^2I and annihilate with those p^2I's. Then the remaining placeholder blocks are of the form p^3X. Continuing this way we get an overview of the p-powers that divide the entries.
Recall that forming Smith Normal Forms of blocks refines the block structure and hence increases the number of block lines. For example if the partial Smith Normal Form of a block X is $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then the number of block rows and the number of block columns increase each by 1. If we know ahead, that the block X cannot have a 0-row, then the Smith Normal Form is $[I|0]$ and only the number of block columns increases by 1. Whenever possible, we argue ahead that a certain block has no 0-row, or no 0-column or both, to restrict the upcoming Smith Normal Form. This technique is of extreme importance to keep the size of the created block matrices manageable. Our argument is as follows:

> *If an assumed* 0*-row (or* 0*-column) in some block* X *leads to a summand, then this is caused by the exclusive connection of this row (or column) to already obtained reduced blocks. Such a summand is either contradictory or we may find*

it in the list and assume that G is not in this class. So this 0-row (or 0-column) does not occur. Forming the Smith Normal Form of X very well can destroy the reduced form of connected blocks. But we only form the Smith Normal Form of such a block X, if all involved (formerly reduced) blocks can be reestablished, preserving the Smith Normal Form of X. Thus, if X has no 0-row (or 0-column) also the Smith Normal Form of X has no 0-row (or 0-column).

We annihilate p^2B with pI above, i.e., $p^2B = 0$. A $p^2 \in p^2E$ allows to annihilate in its row and its column and leads to a summand of rank 3. So we write p^3E.
We form the Smith Normal Forms of p^2C, p^2D and annihilate with $p^2I \subset p^2C$ in p^2A and with $p^2I \subset p^2D$ in p^2F. Thereafter we form the Smith Normal Form of the respective rest of p^2F and of p^2A. With $p^2I \subset p^2A$ we annihilate in p^3C, with $p^2I \subset p^2D$ we annihilate in p^3E (below $p^2I \subset p^2A$ and below $p^2I \subset p^2C$). The resulting coordinate matrix is

$$(20) \qquad \left[\begin{array}{cc|ccc||cc|cc|c}
pI & 0 & 0 & 0 & 0 & & & & & \\
0 & pI & 0 & 0 & 0 & & & & & \tau_2 \\
\cline{1-5}
0 & 0 & 0 & p^2I & 0 & & & & & \\
0 & 0 & 0 & 0 & p^3A & & & & & \\
\hline\hline
0 & 0 & p^2I & 0 & 0 & pI & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & p^3C & 0 & pI & 0 & 0 & \\
\cline{1-9}
p^2I & 0 & 0 & 0 & p^3E_1 & 0 & 0 & 0 & 0 & \tau_3 \\
\cline{1-9}
0 & p^3D_1 & p^3E_2 & p^3E_3 & p^3E_4 & 0 & 0 & p^2I & 0 & \\
0 & p^3D_2 & p^3E_5 & p^3E_6 & p^3E_7 & 0 & 0 & 0 & p^3F & \\
\hline
\multicolumn{5}{c||}{\tau_1} & \multicolumn{4}{c|}{\tau_2} &
\end{array}\right]$$

The block p^3D_1 can be annihilated by pI above and p^3E_2 can be annihilated by p^2I above. An entry $p^3 \in p^3E_7$ leads to a summand of rank 3, so we write p^4E_7. We form the iterated Smith Normal Form of $\begin{bmatrix} p^3A \\ p^3C \\ p^3E_1 \\ p^3E_4 \end{bmatrix}$ starting with p^3E_4, and annihilate with $p^3I \subset p^3E_4$ in p^3E_3. We obtain the new coordinate matrix

$$
(21)\qquad
\left[\begin{array}{ccc|cc|ccccc||ccc|ccc|c}
pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \tau_2 \\
\cline{1-10}
0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^3I & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^4A & & & & & & & \\
\hline\hline
0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p^3I & 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^4C_1 & p^4C_2 & 0 & 0 & pI & 0 & 0 & 0 & \\
\cline{1-16}
p^2I & 0 & 0 & 0 & 0 & 0 & p^3I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_3 \\
0 & p^2I & 0 & 0 & 0 & 0 & 0 & p^4E_1^1 & p^4E_1^2 & p^4E_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\cline{1-16}
0 & 0 & 0 & 0 & 0 & p^3I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2I & 0 & 0 & \\
0 & 0 & 0 & 0 & p^3E_3 & 0 & p^4E_4^1 & p^4E_4^2 & p^4E_4^3 & p^4E_4^4 & 0 & 0 & 0 & 0 & p^2I & 0 & \\
0 & 0 & p^3D_2 & p^3E_5 & p^3E_6 & p^4E_7^1 & p^4E_7^2 & p^4E_7^3 & p^4E_7^4 & p^4E_7^5 & 0 & 0 & 0 & 0 & 0 & p^3F & \\
\hline
\multicolumn{10}{c||}{\tau_1} & \multicolumn{6}{c|}{\tau_2} &
\end{array}\right]
$$

The blocks $p^4E_1^1, p^4E_1^2$ can be annihilated by p^2I on the left. The blocks $p^4C_1, p^4E_4^1, p^4E_4^2$ can be annihilated by the respective p^3I above.

Then we form the iterated Smith Normal Form of $[p^3D_2|p^3E_5|p^3E_6]$ starting with p^3E_6. With $p^3I \subset p^3E_6$ we annihilate in $[p^4E_7^1|p^4E_7^2|p^4E_7^3|p^4E_7^4|p^3F]$ and in p^3E_3. With $p^3I \subset p^3E_5$ we annihilate in $[p^4E_6|p^4E_7^1|p^4E_7^2|p^4E_7^3|p^3F]$. With $p^3I \subset p^3D_2$ we annihilate in $[p^4E_5|p^4E_6|p^3F]$. Then we form the Smith Normal Form of the rest of p^3F.

Note that all non-zero entries in p^3E_3 are in $(p^3\,\mathbb{Z}\setminus p^4\,\mathbb{Z})$ by p^2I above. So we obtain the (partial) Smith Normal Form $\left[\begin{smallmatrix} p^3I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ of the rest of p^3E_3. With $p^3I \subset p^3E_3$ we annihilate in $p^4E_4^3$. After all of these transformations the coordinate matrix contains only placeholders of type p^4X.

pI	0	0	0	0	0	0	0	0	0	0	0	0	0										
0	pI	0	0	0	0	0	0	0	0	0	0	0	0										
0	0	pI	0	0	0	0	0	0	0	0	0	0	0										
0	0	0	pI	0	0	0	0	0	0	0	0	0	0										
0	0	0	0	0	0	p^2I	0	0	0	0	0	0	0										τ_2
0	0	0	0	0	0	0	p^2I	0	0	0	0	0	0										
0	0	0	0	0	0	0	0	p^2I	0	0	0	0	0										
0	0	0	0	0	0	0	0	0	0	0	0	p^3I	0										
0	0	0	0	0	0	0	0	0	0	0	0	0	p^4A										
0	0	0	0	p^2I	0	0	0	0	0	0	0	0	0	pI	0	0	0	0	0	0	0	0	
0	0	0	0	0	p^2I	0	0	0	0	0	0	0	0	0	pI	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	p^3I	0	0	0	0	pI	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	p^4C_2	0	0	0	pI	0	0	0	0	0	
p^2I	0	0	0	0	0	0	0	0	0	p^3I	0	0	0	0	0	0	0	0	0	0	0	0	
0	p^2I	0	0	0	0	0	0	0	0	0	0	0	$p^4E_1^3$	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	p^3I	0	0	0	0	0	0	0	0	p^2I	0	0	0	0	τ_3
0	0	0	0	0	0	0	p^3I	0	0	0	0	0	$p^4E_4^4(1)$	0	0	0	0	0	p^2I	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	$p^4E_4^3$	$p^4E_4^4(2)$	0	0	0	0	0	0	p^2I	0	0	
0	0	0	0	0	0	p^3I	0	0	0	0	0	0	$p^4E_7^5(1)$	0	0	0	0	0	0	0	0	0	
0	0	0	0	p^3I	0	0	0	0	0	0	0	$p^4E_7^4(1)$	$p^4E_7^5(2)$	0	0	0	0	0	0	0	0	0	
0	0	p^3I	0	0	0	0	0	0	$p^4E_7^1(1)$	$p^4E_7^2(1)$	$p^4E_7^3(1)$	$p^4E_7^4(2)$	$p^4E_7^5(3)$	0	0	0	0	0	0	0	0	0	
0	0	0	$p^4D_2^1$	0	$p^4E_5^1$	0	$p^4E_6^1$	$p^4E_6^2$	$p^4E_7^1(2)$	$p^4E_7^2(2)$	$p^4E_7^3(2)$	$p^4E_7^4(3)$	$p^4E_7^5(4)$	0	0	0	0	0	0	0	p^3I	0	
0	0	0	$p^4D_2^2$	0	$p^4E_5^2$	0	$p^4E_6^3$	$p^4E_6^4$	$p^4E_7^1(3)$	$p^4E_7^2(3)$	$p^4E_7^3(3)$	$p^4E_7^4(4)$	$p^4E_7^5(5)$	0	0	0	0	0	0	0	0	p^4F	
							τ_1											τ_2					

All blocks $p^4E_7^j(i)$ are 0, except of $p^4E_7^1(3), p^4E_7^5(1)$

The blocks $p^4E_7^1(1), p^4E_7^2(1), p^4E_7^1(2), p^4E_6^1, p^4E_5^1$ can be annihilated by the respective p^3I's above, and $p^4D_2^1$ can be annihilated by pI above.

A $p^4 \in p^4E_7^5(5)$ allows to annihilate in its whole row and in its whole column, displaying a summand of rank 3. Thus $p^4E_7^5(5) = 0$. A $p^4 \in p^4E_7^5(4)$ allows to annihilate in its whole row, except of p^3I to the right, and in its whole column, displaying a summand [4.1]. Thus $p^4E_7^5(4) = 0$. A $p^4 \in p^4E_7^5(3)$ allows to annihilate in its whole row, except of p^3I to the left, and in its whole column, displaying a summand [6.2]. Thus $p^4E_7^5(3) = 0$. With a $p^4 \in p^4E_7^5(2)$ we annihilate in $p^4E_7^4(1)$ and in its whole column. This displays a summand of [7.2]. Thus $p^4E_7^5(2) = 0$.
A $p^4 \in p^4E_7^4(4)$ allows to annihilate in its whole column, except of p^3I above, and in its whole row, displaying a summand [5.1]. Thus $p^4E_7^4(4) = 0$. A $p^4 \in p^4E_7^4(3)$ allows to annihilate in its whole column, except of p^3I above, and in its whole row, except of p^3I to the right, displaying a summand [6.5]. Thus $p^4E_7^4(3) = 0$. With a $p^4 \in p^4E_7^4(2)$ we annihilate in $p^4E_7^3(1), p^4E_7^4(1)$. This displays a summand [8.3]. Thus $p^4E_7^4(2) = 0$. A $p^4 \in p^4E_7^4(1)$ allows to annihilate in $p^4E_4^3$, displaying a summand [9.1]. Thus $p^4E_7^4(1) = 0$.
A $p^4 \in p^4E_7^3(3)$ allows to annihilate in its whole column, except of p^3I above, and in its whole row, displaying a summand [6.10]. Thus $p^4E_7^3(3) = 0$. A $p^4 \in p^4E_7^3(2)$ allows to annihilate in $p^4E_7^3(1), p^4E_7^2(2), p^4E_6^2$, displaying a summand [7.1]. Thus $p^4E_7^3(2) = 0$. A $p^4 \in p^4E_7^3(1)$ displays a summand [9.2]. Thus $p^4E_7^3(1) = 0$.

A $p^4 \in p^4E_7^2(3)$ allows to annihilate in $p^4E_7^2(2)$ and in its whole row, displaying a summand [8.2]. Thus $p^4E_7^2(3) = 0$. A $p^4 \in p^4E_7^2(2)$ allows to annihilate in $p^4E_6^2$, displaying a summand [9.4]. Thus $p^4E_7^2(2) = 0$.
A $p^4 \in p^4E_6^4$ allows to annihilate in $p^4E_6^2$, so there is a 0-column below a $p^4 \in p^4E_6^2$. Then a $p^4 \in p^4E_6^2$ allows to annihilate to the left, and to the right only p^3I remains. This displays a summand [6.7]. Thus $p^4E_6^2 = 0$. We obtained lots of 0-blocks.

pI	0	0	0	0	0	0	0	0	0	0	0	0	0										
0	pI	0	0	0	0	0	0	0	0	0	0	0	0										
0	0	pI	0	0	0	0	0	0	0	0	0	0	0										
0	0	0	pI	0	0	0	0	0	0	0	0	0	0										
0	0	0	0	0	0	p^2I	0	0	0	0	0	0	0										τ_2
0	0	0	0	0	0	0	p^2I	0	0	0	0	0	0										
0	0	0	0	0	0	0	0	p^2I	0	0	0	0	0										
0	0	0	0	0	0	0	0	0	0	0	0	p^3I	0										
0	0	0	0	0	0	0	0	0	0	0	0	0	p^4A										
0	0	0	0	p^2I	0	0	0	0	0	0	0	0	0	pI	0	0	0	0	0	0	0	0	
0	0	0	0	0	p^2I	0	0	0	0	0	0	0	0	0	pI	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	p^3I	0	0	0	0	pI	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	p^4C_2	0	0	0	pI	0	0	0	0	0	
p^2I	0	0	0	0	0	0	0	0	0	p^3I	0	0	0	0	0	0	0	0	0	0	0	0	
0	p^2I	0	0	0	0	0	0	0	0	0	0	0	$p^4E_1^3$	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	p^3I	0	0	0	0	0	0	0	0	p^2I	0	0	0	0	τ_3
0	0	0	0	0	0	0	p^3I	0	0	0	0	0	$p^4E_4^4(1)$	0	0	0	0	0	p^2I	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	$p^4E_4^3$	$p^4E_4^4(2)$	0	0	0	0	0	0	p^2I	0	0	
0	0	0	0	0	0	p^3I	0	0	0	0	0	0	$p^4E_7^5(1)$	0	0	0	0	0	0	0	0	0	
0	0	0	0	p^3I	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	p^3I	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	p^3I	0	
0	0	0	$p^4D_2^2$	0	$p^4E_5^2$	0	$p^4E_6^3$	$p^4E_6^4$	$p^4E_7^1(3)$	0	0	0	0	0	0	0	0	0	0	0	0	p^4F	
τ_1														τ_2									

The rows 1, 14 and the columns 1, 11 display a summand [6.4], and are omitted. The rows 3, 21 and the column 3 display a summand [5.5], and are omitted. The rows 10, 20 and the columns 5, 15 display a summand [6.12], and are omitted. The row 12 and the columns 12, 17 display a summand [4.5], and are omitted. The row 22 and the column 22 display a summand of rank 3, and are not present. So we obtain a new coordinate matrix.

$$
(22)\qquad
\left[\begin{array}{cc|cccc|ccc||cc|cccc|c}
pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
\cline{1-9}
0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 & & & & & & & \tau_2 \\
0 & 0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p^3I & 0 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^4A & & & & & & & \\
\hline\hline
0 & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^4C_2 & 0 & pI & 0 & 0 & 0 & 0 & \\
\cline{1-15}
p^2I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^4E_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & \\
\cline{1-15}
0 & 0 & 0 & 0 & 0 & 0 & p^3I & 0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & p^3I & 0 & 0 & 0 & p^4E_4^4(1) & 0 & 0 & 0 & p^2I & 0 & 0 & \tau_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & p^4E_4^3 & p^4E_4^4(2) & 0 & 0 & 0 & 0 & p^2I & 0 & \\
0 & 0 & 0 & p^3I & 0 & 0 & 0 & 0 & p^4E_7^5(1) & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & p^4D_2^2 & p^4E_5^2 & 0 & p^4E_6^3 & p^4E_6^4 & p^4E_7^1(3) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^4F & \\
\hline
\multicolumn{9}{c||}{\tau_1} & \multicolumn{6}{c|}{\tau_2} &
\end{array}\right]
$$

Complete Reduction of $[p^4D_2^2|p^4E_5^2|p^4E_6^3|p^4E_6^4|p^4E_7^1(3)|p^4F]$, non-existence of blocks

A 0-column of $p^4E_7^1(3)$ displays a summand [4.4]. So we may assume that $p^4E_7^1(3)$ has no 0-column and its Smith Normal Form is $\left[\begin{smallmatrix} p^4I \\ 0 \end{smallmatrix}\right]$.

We deal with the submatrix $[p^4D_2^2|p^4E_5^2|p^4E_6^3|p^4E_6^4|p^4E_7^1(3)|p^4F]$. The Smith Normal Form of $p^4E_7^1(3)$ splits this submatrix horizontally in $\left[\begin{smallmatrix} Y \\ X \end{smallmatrix}\right]$, where Y contains $p^4I \subset p^4E_7^1(3)$ and X contains the 0-rows of $p^4E_7^1(3)$. A 0-column of $p^4D_2^2$ displays a summand of rank 3. A 0-column of $p^4E_5^2$ displays a summand [4.6]. Hence we may assume that $[p^4D_2^2|p^4E_5^2]$ has no 0-column. Note that a 0-row of $X \cap [p^4D_2^2|p^4E_5^2|p^4E_6^3|p^4E_6^4]$ leads to a summand of rank ≤ 3. With these informations we form the iterated Smith Normal Form of $X \cap [p^4D_2^2|p^4E_5^2|p^4E_6^3|p^4E_6^4]$, starting with $X \cap p^4E_6^4$. Then we annihilate with $p^4I \subset p^4E_7^1(3)$ in $[p^4D_2^2|p^4E_5^2|p^4E_6^3|p^4F]$. We cannot annihilate in $p^4E_6^4$. Thus we obtain

$$
\left[p^4D_2^2(1)\,\middle|\,p^4E_5^2(1)\,\middle|\,p^4E_6^3(1)\,\middle|\,p^4E_6^4\,\middle|\,p^4E_7^1(3)\,\middle|\,p^4F\right] = \begin{bmatrix} Y \\ X \end{bmatrix} =
\left[\begin{array}{c|c|cc|cc|c|c}
0 & 0 & 0 & 0 & p^4E_6^4(1) & p^4E_6^4(2) & p^4I & 0 \\
\hline
0 & 0 & 0 & 0 & p^4I & 0 & 0 & p^4F_1 \\
0 & 0 & p^4I & 0 & 0 & 0 & 0 & p^4F_2 \\
0 & p^4I & 0 & 0 & 0 & 0 & 0 & p^4F_3 \\
p^4I & 0 & 0 & 0 & 0 & 0 & 0 & p^4F_4
\end{array}\right].
$$

With the p^4I's in $X \cap [p^4D_2^2|p^4E_5^2|p^4E_6^3|p^4E_6^4]$ we annihilate all p^4F_i's and with $p^4I \subset X \cap p^4E_6^4$ we annihilate $p^4E_6^4(1)$. So we get the contradiction $p^4F = 0$, and the F-column is not present. A 0-row of $E_6^4(2)$ displays a summand [6.9]. Moreover, $p^4E_6^4$ has no 0-column, to avoid a summand of rank 3. So the Smith Normal Form of $E_6^4(2)$ is p^4I. Hence, in particular, Y has no 0-row. Thus we obtained the claimed complete reduced form:

$$\left[p^4D_2^2(1)\,\middle|\,p^4E_5^2(1)\,\middle|\,p^4E_6^3(1)\,\middle|\,p^4E_6^4\,\middle|\,p^4E_7^1(3)\,\middle|\,p^4F\right] = \left[\frac{Y}{X}\right] = \left[\begin{array}{c|c|cc|cc|c|c} 0 & 0 & 0 & 0 & 0 & p^4I & p^4I & 0 \\ \hline 0 & 0 & 0 & 0 & p^4I & 0 & 0 & 0 \\ 0 & 0 & p^4I & 0 & 0 & 0 & 0 & 0 \\ 0 & p^4I & 0 & 0 & 0 & 0 & 0 & 0 \\ p^4I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A (non-zero) row of Y displays a summand [9.3]. Thus the block row Y is not present and this creates the contradiction $p^4E_7^1(3) = 0$, because we know already that $E_7^1(3)$ has no 0-column. So row 11 and the columns $7, 12$ are not present.

A $p^4 \in X \cap E_6^4$ displays a summand [5.2]. Thus we get the contradiction $p^4E_6^4 = 0$, because we know already that E_6^4 has no 0-column. So row 5 and column 6 are not present. Moreover, a $p^4 \in p^4D_2^2$ displays a summand [5.3]. Thus we get the contradiction $p^4D_2^2 = 0$, because we know already that D_2^2 has no 0-column. So row 2 and column 2 are not present. A $p^4 \in p^4E_5^2$ displays a summand [6.11]. Thus we get the contradiction $p^4E_5^2 = 0$, because we know already that E_5^2 has no 0-column. So row 8 and the columns $3, 10$ are not present. The new coordinate matrix is

$$(23)\qquad \left[\begin{array}{c|ccc|cc||c|ccc|c}
pI & 0 & 0 & 0 & 0 & 0 & & & & & \\ \cline{1-6}
0 & p^2I & 0 & 0 & 0 & 0 & & & & & \\
0 & 0 & p^2I & 0 & 0 & 0 & & & & & \tau_2 \\
0 & 0 & 0 & p^2I & 0 & 0 & & & & & \\
0 & 0 & 0 & 0 & p^3I & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & p^4A & & & & & \\ \hline\hline
0 & 0 & 0 & 0 & 0 & p^4C_2 & pI & 0 & 0 & 0 & \\ \cline{1-10}
p^2I & 0 & 0 & 0 & 0 & p^4E_1^3 & 0 & 0 & 0 & 0 & \\ \cline{1-10}
0 & 0 & p^3I & 0 & 0 & p^4E_4^4(1,1) & 0 & p^2I & 0 & 0 & \\
0 & 0 & 0 & p^3I & 0 & p^4E_4^4(1,2) & 0 & 0 & p^2I & 0 & \tau_3 \\
0 & 0 & 0 & 0 & p^4E_4^3 & p^4E_4^4(2) & 0 & 0 & 0 & p^2I & \\
0 & p^3I & 0 & 0 & 0 & p^4E_7^5(1) & 0 & 0 & 0 & 0 & \\
0 & 0 & p^4I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \hline
 & & & \tau_1 & & & & & \tau_2 & &
\end{array}\right]$$

Complete Reduction of $p^4E_4^3$ and of the $p^4E_7^5(1)$-column

We deal with $p^4E_4^3$ and the $p^4E_7^5(1)$-column. A 0-row of p^4A displays a summand of rank 2. A

0-row of p^4C_2 displays a summand of rank 3. A 0-row of $p^4E_1^3$ displays a summand [5.6]. A 0-row of $p^4E_4^4(1,1)$ displays a summand [8.1]. A 0-row of $p^4E_4^4(1,2)$ displays a summand [6.8]. A 0-row of $p^4E_7^5(1)$ displays a summand [5.4].
The $p^4E_7^5(1)$-block column has no 0-column. The Smith Normal Form of $p^4E_7^5(1)$ is $[p^4I\big|0]$. This splits the $p^4E_7^5(1)$-column vertically $[Y\big|X]$, where Y continues the nonzero columns and X continues the 0-columns of $p^4E_7^5(1)$. With $X\cap p^4E_4^4(2)$ we start the iterated Smith Normal Form upward till $X\cap p^4A$. Then we annihilate with $p^4I\subset p^4E_7^5(1)$ upward, except of $p^4E_4^4(2)$.
This iterated Smith Normal Form has the five blocks p^4A, p^4C_2, $p^4E_1^3$, $p^4E_4^4(1,1)$, $p^4E_4^4(1,2)$. We discuss these blocks to 0. Consequently those block rows are not present, because we already know that these blocks have no 0-rows.
A $p^4\in p^4A$ causes a summand of rank 3. So the p^4A-row is not present. A $p^4\in p^4C_2$ causes a summand [4.3]. So the p^4C_2-row and column 7 are not present. A $p^4\in p^4E_1^3$ causes a summand [6.3]. So the $p^4E_1^3$-row, row 1 and column 1 are not present. A $p^4\in p^4E_4^4(1,1)$ causes a summand [9.6]. So the $p^4E_4^4(1,1)$-row, the rows $3,13$ and column 8 are not present. A $p^4\in p^4E_4^4(1,2)$ causes a summand [7.3]. So the $p^4E_4^4(1,2)$-row, row 4 and the columns $4,9$ are not present. We obtain the new coordinate matrix

$$\left[\begin{array}{c|ccc||cc|c} p^2I & 0 & 0 & 0 & & & \tau_2 \\ 0 & p^3I & 0 & 0 & & & \\ \hline\hline 0 & p^4E_4^3(1) & Y\cap p^4E_4^4(2,1) & p^4I & p^2I & 0 & \\ 0 & p^4E_4^3(2) & Y\cap p^4E_4^4(2,2) & 0 & 0 & p^2I & \tau_3 \\ p^3I & 0 & p^4I & 0 & 0 & 0 & \\ \hline & & \tau_1 & & \tau_2 & & \end{array}\right]$$

With p^4I in row 3 we annihilate $p^4E_4^3(1), Y\cap p^4E_4^4(2,1)$. This displays a summand [4.2]. Thus the $p^4E_4^3(1)$-row and the columns $4,5$ are not present. The matrix $[p^4E_4^3(2)\big|Y\cap p^4E_4^4(2,2)]$ has no 0-row to avoid a summand of rank 3. The block $p^4E_4^3(2)$ has no 0-column to avoid a summand of rank 3 and a 0-column in the block $Y\cap p^4E_4^4(2,2)$ leads to a summand [6.1]. We form the iterated Smith Normal Form of $[p^4E_4^3(2)\big|Y\cap p^4E_4^4(2,2)]$ starting with $Y\cap p^4E_4^4(2,2)$, i.e., $[p^4E_4^3(2)\big|Y\cap p^4E_4^4(2,2)]=\left[\begin{smallmatrix}0 & p^4I\\ p^4I & 0\end{smallmatrix}\right]$ and get the new coordinate matrix

$$\left[\begin{array}{c|cc||cc|c} p^2I & 0 & 0 & & & \tau_2 \\ 0 & p^3I & 0 & & & \\ \hline\hline 0 & 0 & p^4I & p^2I & 0 & \\ 0 & p^4I & 0 & 0 & p^2I & \tau_3 \\ p^3I & 0 & p^4I & 0 & 0 & \\ \hline & \tau_1 & & \tau_2 & & \end{array}\right]$$

This coordinate matrix displays summands [6.6] and [9.5], respectively.
We obtained all indecomposable homocyclic $((1,3),p^5)$-groups, and it is shown that the list is complete. □

References

[1] Arnold, David M. *Abelian Groups and Representations of Partially Ordered Sets*, CMOS Advanced Books in Mathematics, Springer-Verlag, New York, 2000.

[2] Arnold, David M., Mader, Adolf, Mutzbauer, Otto, and Solak, Ebru. $(1,3)$-Groups, *Czech. Math. J.* **63** (2013), 307-355.

[3] Arnold, David M., Mader, Adolf, Mutzbauer, Otto, and Solak, Ebru. Almost completely decomposable groups of unbounded representation type, *J. Alg.* **349** (2012), 50-62.

[4] Arnold, David M., Mader, Adolf, Mutzbauer, Otto, and Solak, Ebru. Representations of posets and rigid almost completely decomposable groups, Proceedings of the Balikesir Conference 2013, Palestine Journal of Mathematics, **3** (2014), 320-341.

[5] Arnold, David M., Mader, Adolf, Mutzbauer, Otto, and Solak, Ebru. The class of $(1,3)$-groups with homocyclic regulator quotient of exponent p^4 has bounded representation type, *J. Alg.* **400** (2014), 43-55.

[6] Arnold, David M., Mader, Adolf, Mutzbauer, Otto, and Solak, Ebru. Representations of finite posets over the ring of integers modulo a prime power, *J. Comm. Alg.* **8** (2016).

[7] Arnold, David M., Mader, Adolf, Mutzbauer, Otto, and Solak, Ebru. A Remak-Krull-Schmidt Class of Torsion-Free Abelian Groups, Proceedings Volume of the Mülheim Conference 2016.

[8] Burkhardt, Rolf. On a special class of almost completely decomposable groups I, in Abelian Groups and Modules, Proceedings of the Udine Conference 1984, CISM Courses and Lecture Notes, vol. 287, Springer Verlag, 1984, 141-150.

[9] Faticoni, Ted, and Schultz, Phill. Direct decompositions of almost completely decomposable groups with primary regulating index, *Abelian Groups and Modules*, Marcel Dekker, New York (1996), 233-242.

[10] Lady, E. L. Almost completely decomposable torsion free abelian groups, *Proc. AMS* **45** (1974), 41-47.

[11] Lady, E. L. Nearly isomorphic torsion free abelian groups, *J. Alg.* **35** (1975), 235-238.

[12] Mader, Adolf. Almost Completely Decomposable Groups, Gordon Breach, 2000.

Dept. of Math., Baylor University, Waco, Texas 76798-7328
E-mail address: David_Arnold@baylor.edu

Dept. of Math., University of Hawaii, 2565 McCarthy Mall, Honolulu, Hawaii 96822
E-mail address: adolf@math.hawaii.edu

Universität Würzburg, Math. Inst., Emil-Fischer-Str. 30, 97074 Würzburg, Germany
E-mail address: mutzbauer@mathematik.uni-wuerzburg.de

Dept. of Math., Middle East Technical University, Inönü Bulvari, 06531 Ankara, Turkey
E-mail address: esolak@metu.edu.tr