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FOCAL SINGULARITIES FOR SPHEROIDAL STOKES EIGENFLOWS

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Abstract. The theory of Stokes flow in spheroidal geometry leads to an ordinary spectral theory, coming from separation of variables, and a generalized spectral theory coming from specific combinations of the ordinary eigensolutions. In the present work we demonstrate the steps we need to follow in order to identify the ultimate singularities of the generalized eigensolutions which, as it is expected, lie on the focal set of the corresponding spheroidal system, that is, on the focal line segment if the spheroid is prolate, or on the focal disc if the spheroid is oblate. This is done by expressing the velocity field generated by the chosen particular case in terms of harmonic functions via the Papkovich representation and then identify the focal singularities of the corresponding harmonic functions with the use of the classical Havelock's Theorem. Actually, it is not simple to work simultaneously for all such generalized eigensolution since there are several calculational steps that we have to follow and the formulae became increasingly complicated. Instead, we consider one particular non-trivial case and perform analytically all steps to demonstrate clearly the proposed procedure. That helps to understand the method and it provides the steps needed for any other case.

1. Introduction

Stokes flow characterizes slow motion of viscous fluids [1,2,4,6,9]. In particular, the velocity field for axisymmetric Stokes flows can be obtained either through the action of a first order differential operator on a stream function, or by the action of a different first order differential operator on a vector and a scalar harmonic function. We will refer to the first approach as the Stokes representation and to the second approach as the Papkovich representation.

The stream functions, which are involved in the Stokes representation, belong to the kernel of a fourth order differential operator while the four scalar harmonic functions needed for the Papkovich representation belong to the kernel of the Laplace operator which is of the second order. Hence, we need either one function that satisfies a fourth order equation, or four functions that satisfy a second order equation.

When the physical domain of the flow imposes the spheroidal geometry, the harmonic functions for the Papkovich representation are well known since Laplace's equation is separable in the spheroidal coordinate systems [3]. On the other hand, the stream functions could be vorticity free, in which case the stream eigensolutions are also well known, or they could involve vorticity, in which case the stream eigensolutions have a three-dimensional generalized structure.

The present work proposes an analytical method to utilize the equivalence between stream functions and harmonic functions, in the sense that they both generate the same velocity field [3], to identify the ultimate imaging system that gives rise to the vorticity depended stream functions. This is achieved by using the celebrated Havelock's theorem [5,7].

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In order to be clear and as simple as possible we decided to demonstrate the main steps of the procedure using a particular generic non-trivial stream eigensolution of the vorticity type. For any other eigensolution we can follow exactly the same steps with the appropriate calculational steps. As it is expected, the singularity system is supported on the focal set of the spheroid.

2. Spheroidal Stokes Flow

The stream function Ψ for axisymmetric Stokes flow satisfies the well known equation

$$E^4\Psi = 0. \tag{1}$$

In particular for prolate spheroidal geometry with meridian coordinates $\tau \in [1, +\infty)$and $\zeta \in [-1, +1]$ we have the form

$$E^4\Psi = E^2 \circ E^2\Psi = 0 \tag{2}$$

with

$$E^2 = \frac{1}{c^2\left(\tau^2 - \zeta^2\right)}\left[\left(\tau^2 - 1\right)\frac{\partial^2}{\partial\tau^2} + \left(1 - \zeta^2\right)\frac{\partial^2}{\partial\zeta^2}\right] \tag{3}$$

where c is the semifocal distance of the prolate spheroid.

In fact, the focal set along the x_3 -axis of rotation is $\{(0, 0, x_3) \mid -c \leq x_3 \leq c\}$.

For vanishing vorticity field equation (1) is reduced to

$$E^2\Psi(\tau, \zeta) = 0 \tag{4}$$

and this equation can be solved using separation of variables providing the vorticity free eigensolutions

$$\Theta_n^{(1)}(\tau, \zeta) = G_n(\tau)G_n(\zeta) \tag{5}$$

$$\Theta_n^{(2)}(\tau, \zeta) = G_n(\tau)H_n(\zeta) \tag{6}$$

$$\Theta_n^{(3)}(\tau, \zeta) = H_n(\tau)G_n(\zeta) \tag{7}$$

$$\Theta_n^{(4)}(\tau, \zeta) = H_n(\tau)H_n(\zeta) \tag{8}$$

where G_n and H_n stand for the Genenbauer functions of the first and second kind, respectively [4, 8].

However, equation (1) does not accept separation of variables. It actually offers a unique kind of spectral analytic case which leads to semiseparate solutions, that is solutions which are constructed as specific linear combinations of 3 by 3 Gegenbauer functions of different degree. These generalized solutions $\Omega_n^{(i)}$ are defined as solutions of the non-homogeneous equations

$$c^2E^2\Omega_n^{(i)}(\tau, \zeta) = \Theta_n^{(i)}(\tau, \zeta) \tag{9}$$

for $i = 1, 2, 3, 4$. For example, for $n \geq 4$

$$\begin{aligned}\Omega_n^{(1)} = &-\frac{a_n}{2(2n-3)}\left[G_{n-2}(\tau)G_n(\zeta) + G_n(\tau)G_{n-2}(\zeta)\right] \\ &+ \frac{\beta_n}{2(2n+1)}\left[G_{n+2}(\tau)G_n(\zeta) + G_n(\tau)G_{n+2}(\zeta)\right]\end{aligned} \tag{10}$$

with

$$a_n = \frac{(n-3)(n-2)}{(2n-3)(2n-1)} \tag{11}$$

and

$$\beta_n = \frac{(n+1)(n+2)}{(2n-1)(2n+1)}. \tag{12}$$

In fact it is very easy to use the recurrence relation

$$nG_n(x) = (2n-3)xG_{n-1}(x) - (n-3)G_{n-2}(x) \tag{13}$$

to prove the identity

$$G_{n+2}(x) = \frac{(2n-1)(2n+1)}{(n+1)(n+2)}x^2G_n(x) - \frac{(2n+1)(n-2)(n-3)}{(2n-3)(n+1)(n+2)}G_{n-2}(x) \tag{14}$$

with the use of which we can verify equation (9) for the case $i = 1$.

For the complete list of generalized eigensolutions $\Omega_n^{(i)}$ we refer to [1]. In what follows we will study in detail the generalized eigensolution $\Omega_3^{(3)}(\tau,\zeta)$ which involves both Gegenbauer functions G and H.

3. The Velocity Field via Stokes

Any stream function $\Psi(\tau,\zeta)$ in the prolate spheroidal system generates the velocity field $\boldsymbol{v}$ which lives in any meridian plane and is given by

$$\mathbf{v}(\tau,\zeta) = \frac{1}{c^2\sqrt{\tau^2-\zeta^2}}\left[\frac{\hat{\boldsymbol{\tau}}}{\sqrt{\tau^2-1}}\frac{\partial}{\partial\zeta} + \frac{\hat{\boldsymbol{\zeta}}}{\sqrt{1-\zeta^2}}\frac{\partial}{\partial\tau}\right]\Psi(\tau,\zeta), \tag{15}$$

where $\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\zeta}}$ form the meridian basis. In particular, inserting in (15) the stream function

$$\Psi(\tau,\zeta) = \Omega_3^{(3)}(\tau,\zeta) = \frac{2}{49}H_3(\tau)G_5(\zeta) + \frac{2}{49}H_5(\tau)G_3(\zeta) - \frac{1}{90}G_0(\tau)G_3(\zeta), \tag{16}$$

we obtain the velocity field corresponding to the stream function $\Omega_3^{(3)}(\tau,\zeta)$. Utilizing the relations

$$\frac{d}{dx}G_n(x) = -P_{n-1}(x), \quad n \geq 2, \tag{17}$$

$$\frac{d}{dx}H_n(x) = -Q_{n-1}(x), \quad n \geq 2, \tag{18}$$

$$\frac{d}{dx}G_0(x) = 0, \tag{19}$$

where P_n and Q_n are the Legendre functions of the first and second kind and zero order, we obtain

$$\begin{aligned}\boldsymbol{v}(\tau,\zeta) = &-\frac{2\hat{\boldsymbol{\tau}}}{49c^2\sqrt{\tau^2-\zeta^2}\sqrt{\tau^2-1}}\left[H_3(\tau)P_4(\zeta) + H_5(\tau)P_2(\zeta) - \frac{49}{180}G_0(\tau)P_2(\zeta)\right] \\ &- \frac{2\hat{\boldsymbol{\zeta}}}{49c^2\sqrt{\tau^2-\zeta^2}\sqrt{1-\zeta^2}}\left[Q_2(\tau)G_5(\zeta) + Q_4(\tau)G_3(\zeta)\right].\end{aligned} \tag{20}$$

With the help of the connection formulae

$$(2n-1)G_n(x) = P_{n-2}(x) - P_n(x), \tag{21}$$

$$(2n-1)H_n(x) = Q_{n-2}(x) - Q_n(x), \tag{22}$$

holding for $n \geq 2$, we can express the velocity field (20) in terms of Legendre functions alone. Hence,

$$\begin{aligned}\mathbf{v}(\tau,\zeta) = &-\frac{2\hat{\boldsymbol{\tau}}}{49c^2\sqrt{\tau^2-\zeta^2}\sqrt{\tau^2-1}}\\ &\cdot\left[-\frac{49}{180}P_2(\zeta)+\frac{1}{5}\left(Q_1(\tau)-Q_3(\tau)\right)P_4(\zeta)+\frac{1}{9}\left(Q_3(\tau)-Q_5(\tau)\right)P_2(\zeta)\right]\\ &-\frac{2\hat{\boldsymbol{\zeta}}}{49c^2\sqrt{\tau^2-\zeta^2}\sqrt{1-\zeta^2}}\\ &\cdot\left[\frac{1}{5}Q_4(\tau)P_1(\zeta)+\left(\frac{1}{9}Q_2(\tau)-\frac{1}{5}Q_4(\tau)\right)P_3(\zeta)-\frac{1}{9}Q_2(\tau)P_5(\zeta)\right].\end{aligned} \tag{23}$$

4. The Velocity Field via Papkovich

The velocity field for Stokes flow can also be represented in terms of harmonic functions through the following Papkovich representation

$$\mathbf{v}(\mathbf{r}) = \boldsymbol{\Phi}(\mathbf{r}) - \frac{1}{2}\nabla\left[\mathbf{r}\cdot\boldsymbol{\Phi}(\mathbf{r})+\Phi_0(\mathbf{r})\right] \tag{24}$$

where $\boldsymbol{\Phi}$ and Φ_0 are harmonic functions. Utilizing the equivalence relations between stream and harmonic function developed in [3] we arrive at the following representation of the velocity field

$$\begin{aligned}\mathbf{v}(\tau,\zeta) = &-\frac{\zeta\sqrt{\tau^2-1}\,\hat{\boldsymbol{\tau}}-\tau\sqrt{1-\zeta^2}\,\hat{\boldsymbol{\zeta}}}{60c^2\sqrt{\tau^2-\zeta^2}}\left[Q_1(\tau)P_1(\zeta)-Q_3(\tau)P_3(\zeta)\right]\\ &-\frac{\tau\zeta}{60c}\nabla\left[Q_1(\tau)P_1(\zeta)-Q_3(\tau)P_3(\zeta)\right]+\frac{1}{180c}\nabla\left[Q_0(\tau)P_0(\zeta)\right]\\ &-\frac{1}{196c}\nabla\left[Q_2(\tau)P_2(\zeta)\right]-\frac{2}{15\cdot 49c}\nabla\left[Q_4(\tau)P_4(\zeta)\right].\end{aligned} \tag{25}$$

In view of the operator

$$\nabla = \frac{\sqrt{\tau^2-1}}{\sqrt{\tau^2-\zeta^2}}\frac{\hat{\boldsymbol{\tau}}}{c}\frac{\partial}{\partial\tau}-\frac{\sqrt{1-\zeta^2}}{\sqrt{\tau^2-\zeta^2}}\frac{\hat{\boldsymbol{\zeta}}}{c}\frac{\partial}{\partial\zeta} \tag{26}$$

as well as of the expressions

$$Q_0'(\tau) = \frac{1}{1-\tau^2}, \tag{27}$$

$$Q_k'(\tau) = \frac{k}{1-\tau^2}\left[Q_{k-1}(\tau)-\tau Q_k(\tau)\right], \tag{28}$$

for $k = 1,2,3,4$, and

$$\begin{aligned}&P_0'(\zeta) = 0,\\ &P_1'(\zeta) = 1,\\ &P_k'(\zeta) = \frac{k}{1-\zeta^2}\left[P_{k-1}(\zeta)-\zeta P_k(\zeta)\right],\end{aligned} \tag{29}$$

for $k = 2, 3, 4$, where

$$P_0(\zeta) = 1, \tag{30}$$
$$P_1(\zeta) = \zeta, \tag{31}$$
$$P_2(\zeta) = \frac{1}{2}\left(3\zeta^2 - 1\right), \tag{32}$$
$$P_3(\zeta) = \frac{1}{2}\left(5\zeta^3 - 3\zeta\right), \tag{33}$$
$$P_4(\zeta) = \frac{1}{8}\left(35\zeta^4 - 30\zeta^2 + 3\right), \tag{34}$$

and

$$Q_0(\tau) = \frac{1}{2}\ln\frac{\tau+1}{\tau-1}, \tag{35}$$
$$Q_1(\tau) = P_1(\tau)Q_0(\tau) - 1, \tag{36}$$
$$Q_2(\tau) = P_2(\tau)Q_0(\tau) - \frac{3}{2}P_1(\tau), \tag{37}$$
$$Q_3(\tau) = P_3(\tau)Q_0(\tau) - \frac{5}{3}P_2(\tau) - \frac{1}{6}P_0(\tau), \tag{38}$$
$$Q_4(\tau) = P_4(\tau)Q_0(\tau) - \frac{7}{4}P_3(\tau) - \frac{1}{3}P_1(\tau), \tag{39}$$

we can show that

$$\begin{aligned}
\mathbf{v}(\tau,\zeta) = &- \frac{2\hat{\boldsymbol{\tau}}}{49c^2\sqrt{\tau^2-\zeta^2}\sqrt{\tau^2-1}} \\
&\cdot \left[\frac{49}{120}\left(1-\tau^2\right)Q_1(\tau)\zeta P_1(\zeta) + \frac{49}{3\cdot 120} - \frac{49}{120}\left(1-\tau^2\right)Q_3(\tau)\zeta P_3(\zeta)\right. \\
&- \frac{49}{120}\tau\left(Q_0(\tau) - \tau Q_1(\tau)\right)\zeta P_1(\zeta) - \frac{1}{4}\left(Q_1(\tau) - \tau Q_2(\tau)\right)P_2(\zeta) \\
&\left.-\frac{4}{15}\left(Q_3(\tau) - \tau Q_4(\tau)\right)P_4(\zeta) + \frac{49}{40}\tau\left(Q_2(\tau) - \tau Q_3(\tau)\right)\zeta P_3(\zeta)\right] \\
&- \frac{2\hat{\boldsymbol{\zeta}}}{49c^2\sqrt{\tau^2-\zeta^2}\sqrt{\tau^2-1}} \\
&\cdot \left[\frac{49}{40}\tau Q_3(\tau)\zeta P_2(\zeta) - \frac{49}{40}\tau Q_3(\tau)\zeta^2 P_3(\zeta)\right. \\
&- \frac{49}{120}\tau Q_3(\tau)\left(1-\zeta^2\right)P_3(\zeta) - \frac{1}{4}Q_2(\tau)\left(P_1(\zeta) - \zeta P_2(\zeta)\right) \\
&\left.-\frac{4}{15}Q_4(\tau)\left(P_3(\zeta) - \zeta P_4(\zeta)\right)\right].
\end{aligned} \tag{40}$$

Using the identities

$$\zeta P_1(\zeta) = \frac{1}{3} + \frac{2}{3} P_2(\zeta), \tag{41}$$

$$\zeta P_3(\zeta) = \frac{3}{7} P_2(\zeta) + \frac{4}{7} P_4(\zeta), \tag{42}$$

$$\tau Q_3(\tau) = \frac{3}{7} Q_2(\tau) + \frac{4}{7} Q_4(\tau), \tag{43}$$

and

$$P_5(\zeta) = \frac{1}{8}\left[63\zeta^5 - 70\zeta^3 + 15\zeta\right], \tag{44}$$

$$Q_5(\tau) = P_5(\tau) Q_0(\tau) - \frac{63}{8}\tau^4 + \frac{49}{8}\tau^2 - \frac{8}{15}, \tag{45}$$

we can show, after some long and tedious calculations that the expression (40) for the velocity field in terms of harmonic functions is identical with the expression (23) that was calculated using stream functions. Therefore, we have demonstrated that the velocity field generated by $\Omega_3^{(3)}(\tau, \zeta)$ is the same with the one obtained from a particular set of harmonic functions.

5. The Focal Singularities

It is straightforward to identify the vector harmonic function $\mathbf{\Phi}$ and the scalar harmonic function Φ_0 from equations (24) and (25). These are

$$\mathbf{\Phi}(\tau, \zeta) = \frac{\hat{\mathbf{x}}_3}{30c^2}\left[Q_1(\tau)P_1(\tau) - Q_3(\tau)P_3(\tau)\right], \tag{46}$$

$$\Phi_0(\tau, \zeta) = -\frac{1}{90c} Q_0(\tau)P_0(\zeta) + \frac{1}{98c} Q_2(\tau)P_2(\zeta) + \frac{4}{15 \cdot 49c} Q_4(\tau)P_4(\zeta). \tag{47}$$

The functions (46), (47) give exactly the same velocity field, via the Papkovich representation (24), with the function $\Omega_3^{(3)}$, via the Stokes representation (15). In the prolate spheroidal system

$$x_1 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}\cos\varphi, \tag{48}$$

$$x_2 = c\sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}\sin\varphi, \tag{49}$$

$$x_3 = c\tau\zeta, \tag{50}$$

where $\tau \in [1, +\infty), \zeta \in [-1, 1], \varphi \in [0, 2\pi)$, the Laplace operator reads

$$\Delta = \frac{1}{c^2\left(\tau^2 - \zeta^2\right)}\left[\partial_\tau\left(\tau^2 - 1\right)\partial_\tau + \partial_\zeta\left(1 - \zeta^2\right)\partial_\zeta\right] + \frac{1}{c^2\left(\tau^2 - 1\right)\left(1 - \zeta^2\right)}\partial_{\varphi\varphi}. \tag{51}$$

The corresponding exterior harmonics are the functions $Q_n^m(\tau)P_n^m(\zeta)e^{\pm im\varphi}$ with $n = 0, 1, 2, \ldots$ and $m = -n, -n+1, \ldots, n$, and P_n^m, Q_n^m denote the Legendre functions of the first and the second kind, respectively.

Havelock's Theorem [5] provides the ultimate distribution of singularities on the focal segment that give rise to each exterior prolate spherical harmonic. More precisely, for the spheroidal system which is based on the prolate spheroid

$$\frac{x_1^2 + x_2^2}{c^2\left(\tau^2 - 1\right)} + \frac{x_3^2}{c^2\tau^2} = 1, \quad \tau > 1, \tag{52}$$

Havelock's representation is

$$Q_n^m(\tau)P_n^m(\zeta)e^{im\varphi} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)\int_{-c}^{m+c} \frac{\left(c^2 - t^2\right)^{\frac{m}{2}} P_n^m\left(\frac{t}{c}\right)}{\sqrt{x_1^2 + x_2^2 + (x_3 - t)^2}} dt. \tag{53}$$

In particular, the above formula for axisymmetric geometries, where $m = 0$, reduces to

$$Q_n(\tau)P_n(\zeta) = \frac{1}{2}\int_{-c}^{+c} \frac{P_n\left(\frac{t}{c}\right)}{\sqrt{x_1^2 + x_2^2 + (x_3 - t)^2}} dt. \tag{54}$$

The integrals (53) and (54) are regular at any Cartesian point (x_1, x_2, x_3) that does not belong to the focal segment

$$F = \{(0, 0, x_3) \mid -c \le x_3 \le c\}. \tag{55}$$

Formula (54) expresses the exterior harmonic $Q_n(\tau)P_n(\zeta)$ as the potential generated by a continuous distribution of monopoles along the focal segment F with strength equal to

$$\frac{1}{2}P_n^m\left(\frac{t}{c}\right)$$

for each $t \in [-c, +c]$. In particular, at the focus $t = c$ the strength of the corresponding monopole is

$$\frac{1}{2}P_n(1) = \frac{1}{2} \tag{56}$$

while at the focus $t = -c$ the strength is

$$\frac{1}{2}P_n(-1) = \frac{1}{2}(-1)^n. \tag{57}$$

Inserting the representation (54) in the potentials (46) and (47) we obtain the focal representations

$$\mathbf{\Phi}(\tau, \zeta) = \frac{\hat{\mathbf{x}}_3}{60c^2}\int_{-c}^{+c} \left[P_1\left(\frac{t}{c}\right) - P_3\left(\frac{t}{c}\right)\right]\frac{dt}{r(t)} \tag{58}$$

and

$$\Phi_0(\tau, \zeta) = \frac{1}{2c}\int_{-c}^{+c} \left[-\frac{1}{90}P_0\left(\frac{t}{c}\right) + \frac{1}{98}P_2\left(\frac{t}{c}\right) + \frac{4}{15 \cdot 49}P_4\left(\frac{t}{c}\right)\right]\frac{dt}{r(t)}, \tag{59}$$

where

$$r(t) = \sqrt{x_1^2 + x_2^2 + (x_3 - t)^2}. \tag{60}$$

Consequently, the singularity systems for $\mathbf{\Phi}$ and Φ_0 are two continuous distributions of monopoles along the focal segment with densities

$$d\left(\frac{t}{c}\right) = \frac{1}{60c^2}\left[P_1\left(\frac{t}{c}\right) - P_3\left(\frac{t}{c}\right)\right] \tag{61}$$

and

$$d_0\left(\frac{t}{c}\right) = \frac{1}{2c}\left[-\frac{1}{90}P_0\left(\frac{t}{c}\right) + \frac{1}{98}P_2\left(\frac{t}{c}\right) + \frac{4}{15 \cdot 49}P_4\left(\frac{t}{c}\right)\right] \tag{62}$$

respectively.

Finally, by the equivalence of $\{\mathbf{\Phi}, \Phi_0\}$ with $\Omega_3^{(3)}$, it follows that (61) and (62) provide the focal singularity system of $\Omega_3^{(3)}$ as well. As we mentioned in the beginning, this logic can be followed for each one of the stream functions $\Theta_n^{(i)}, \Omega_n^{(i)}$ with $n = 0, 1, 2, \ldots$

Remark. One can argue that the stream function $\Omega_3^{(3)}$, given by (16), can be expressed in terms of the Legendre functions P_n and Q_n via formulae (21) and (22), and then we can use Havelock's theorem to arrive at the singularity system. However, the above reduction will lead to terms of the form $Q_n(\tau)P_{n'}(\zeta)$ with $n \neq n'$ and this product is not covered by Havelock's theorem. Hence, we need to follow the proposed method that ends up with products where $n = n'$.

Actually, Havelock's representation holds for harmonic functions, where always $n = n'$, while the combination of products defining $\Omega_3^{(3)}$ represents an element in the $\ker E^4$ and not in the $\ker \Delta$.

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