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ANGLE CHAINS AND PINNED VARIANTS

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ABSTRACT. We study a variant of the Erdős unit distance problem, concerning angles between successive triples of points chosen from a large finite point set. Specifically, given a large finite set of n points E, and a sequence of angles $(\alpha_1, \ldots, \alpha_k)$, we give upper and lower bounds on the maximum possible number of tuples of distinct points $(x_1, \ldots, x_{k+2}) \in E^{k+2}$ satisfying $\angle(x_j, x_{j+1}, x_{j+2}) = \alpha_j$ for every $1 \leq j \leq k$ as well as pinned analogues.

1. INTRODUCTION

1.1. **Background.** In [5], Erdős introduced two popular problems in discrete geometry, the unit distance problem and the distinct distances problem. Given a finite point set in the plane, the unit distance problem asks how often a single distance can occur between pairs of points, while the distinct distances problem asks how many distinct distances must be determined by pairs of points. See [4, 7] for surveys of these and related problems. The distinct distances problem was resolved in 2010 by Guth and Katz in [9] in the plane, but remains open in higher dimensions. By contrast, the unit distance problem has not seen any progress since the work of Spencer, Szemerédi, and Trotter [18] in $\mathbb{R}^2$, while recent progress has been made in $\mathbb{R}^3$ by Zahl [20]. In $\mathbb{R}^d$ with $d \geq 4$ the unit distance problem becomes trivial without further restrictions due to the celebrated Lenz example, presented below (Theorem F). A variant of these problems involves fixing one of the points from which the distances are counted. These are referred to as pinned variants. The same bounds are conjectured for the pinned Erdős distinct distances problem as the unpinned version with the best partial results obtained by Katz and Tardos in [10] in the plane. The pinned version of the unit distance problem is trivial since all but one of the points can be placed on a circle around the remaining point.

Another variant of the family of problems proposed by Erdős that is important to this paper involves point configurations determined by distances measured between more than two points. One of the most commonly studied configurations is a $(k+1)$-tuple of points where each distance between the k consecutive pairs of points is specified. In the context of the distinct distances problem this was solved in the style of the Guth-Katz argument by Misha Rudnev in [16] and Jonathan Passant [15]. Analogous problems have been considered in both finite fields [2] as well as the continuous setting for both pinned and unpinned Falconer type problems for chains [3, 12]. The study of such problems in the unit distance setting was initiated by the first two authors and Sheffer in [14] and their results were improved upon by Frankl and Kupavskii [6]. These chain variants of the original problems include the original problems as special cases, so it was surprising in the unit distance setting that sharp results were obtained for many types of chains, while the remaining cases only miss by as much as the best results currently available for

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the unit distance problem itself. This unexpected development has brought attention to these types of problems, with some follow up work done both on more complicated relationships, as well as replacing distances by dot products [8, 11].

1.2. **Angles.** The final variant of the family of problems proposed by Erdős important to this paper is replacing the distance between two points by the angle made by three points. The important paradigm shift here is that the base configuration now depends on three points as opposed to only two points as in the case of other widely-studied objects, such as distances, dot products, and directions. This poses a unique challenge in that the incidence results ubiquitous in this area tend to look at level sets with respect to a single point, such as circles which encode a fixed distance to a given point. However, in the case of angles, the level set of a single point is already rather complicated, and looking at how level sets of how multiple pairs of points interact is even more so.

To keep track of the various parameters, we borrow notation from related problems on distances in [3, 14]. Specifically, if we fix a k-tuple of real numbers, $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in (0, \pi)^k$, then a k**-chain** of that type is a $(k+2)$-tuple of points, $(x_1, x_2, \ldots, x_{k+2})$, such that for all $j = 1, \ldots, k$, we have $\angle(x_j, x_{j+1}, x_{j+2}) = \alpha_j$. For example, if we fix a triple of real numbers, (α, β, γ), a 3-chain of that type will be a set of five points, where the angle determined by the first three points is α, the angle determined by the middle three points is β, and the angle determined by the last three points is γ. Given a large finite point set E and a k-tuple of angles, $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, we denote the set of k-chains determined by $(k+2)$-tuples of points in E by

$$\Lambda_k(E; \alpha_1, \alpha_2, \ldots, \alpha_k) := \left\{(x_1, x_2, \ldots, x_{k+2}) \in E^{k+2} : \angle(x_j, x_{j+1}, x_{j+2}) = \alpha_j\right\}.$$

When context is clear, we suppress the angles and just write $\Lambda_k(E)$. In some cases, we will have all of the α_j equal to a fixed α, and will refer to a chain as an α angle k-chain. Also, we assume that angles are not integer multiples of π, as then we could just arrange points along a line and get n^{k+2} instances of a k-chain whose angles are multiple of π. Moreover, we will also assume that k is like a constant compared to the number of points in a given set. If two quantities, $X(n)$ and $Y(n)$, vary with respect to some natural number parameter, n, then we write $X(n) \lesssim Y(n)$ if there exist constants, C and N, both independent of n, such that for all $n > N$, we have $X(n) \leq CY(n)$. If $X(n) \lesssim Y(n)$ and $Y(n) \lesssim X(n)$, we write $X(n) \approx Y(n)$.

In [13], Pach and Sharir gave the following upper bound on the size of $\Lambda_1(E)$, the number of triples of points determining a fixed angle. They also showed that their result is sharp for some angles, so we cannot expect to do better in general.

Theorem A. *Given a large finite point set E of n points in the plane,*

$$|\Lambda_1(E)| \lesssim n^2 \log n.$$

This work was continued in higher dimensions by Apfelbaum and Sharir [1]. They proved the following two results in three and four dimensions. Here and below, the best known constants are independent of the choice of angle, unless stated otherwise.

Theorem B. *Given a large finite point set E of n points in $\mathbb{R}^3$,*

$$|\Lambda_1(E)| \lesssim n^{\frac{7}{3}}.$$

This estimate is sharp in the case that the angle in question is $\frac{\pi}{2}$. To convey the four-dimensional bound, we use the function $\beta(n)$ which grows extremely slowly, as it is defined using the inverse Ackermann function. To be completely rigorous, we can write $\beta(n) \lesssim n^\epsilon$ for any $\epsilon > 0$.

Theorem C. *Given a large finite point set E of n points in $\mathbb{R}^4$, for $\alpha \neq \frac{m\pi}{2}$ for any integer m,*

$$|\Lambda_1(E;\alpha)| \lesssim n^{\frac{5}{2}}\beta(n).$$

In the case that the angle under consideration is $\frac{\pi}{2}$, there is a construction that yields $\approx n^3$ triples that determine a right angle. This is not surprising, as the angle $\frac{\pi}{2}$ is related to points with dot product zero, which exhibit some distinct behavior in their own right. Therefore, without special assumptions on either the point set or the angles, the question becomes trivial in higher dimensions. We discuss this in greater detail in Section 5.

2. Main results

In this note, we extend the aforementioned results to k-chains of angles, pinned angles and pinned k-chains of angles.

2.1. Angle chains in $\mathbb{R}^2$.

We first obtain the following bounds on angle chains in the plane. The bounds are tight up to constants in for even-length chains, but there is a gap of a factor of a logarithm for odd-length chains.

Theorem 2.1. *Given a large finite point set E of n points in the plane, and a k-tuple of angles $(\alpha_1, ..., \alpha_k)$,*

$$|\Lambda_k(E)| \leq \begin{cases} 2^{\frac{k-1}{2}} n^{\frac{k-1}{2}+2}\log n, & k \text{ odd} \\ 2^{\frac{k}{2}} n^{\frac{k}{2}+2}, & k \text{ even} \end{cases}$$

Moreover, there exists a set of n points in the plane that has $\gtrsim n^{\lfloor \frac{k}{2} \rfloor + 2}$ angle k-chains of type $(\alpha_1, \ldots, \alpha_k)$.

For the upper bound, we treat even- and odd-length chains differently. However, we could not find a way to capitalize on this difference in the lower bound, so we use the same construction for both. The proof of the lower bound shows that the constant buried in the asymptotic estimate is at least $(k+2)^{-\lceil \frac{k}{2} \rceil - 2}$. This is proved in Section 3.

2.2. Angle chains in $\mathbb{R}^3$.

In contrast to the planar case, a wide open problem reveals itself for 2-chains of right angles. We currently have no nontrivial bounds for chains of non-right angles in three dimensions. In general we are only able to obtain the trivial upper bound of $\lesssim n^{\frac{10}{3}}$ obtained by observing there are $\lesssim n^{\frac{7}{3}}$ choices for the first three points using the result of Apfelbaum and Sharir in Theorem B, and then n choices for the fourth and final point. In Subsection 4.2 we obtain many improvements on this result under further conditions on our point set. Collectively, these partial results point towards the following conjecture.

Conjecture 2.2. *Given a large finite point set $E \subset \mathbb{R}^3$ of n points we have that for right angles,*

$$|\Lambda_2(E)| \lesssim n^3.$$

Note that this conjecture matches the lower bound that we get by embedding the appropriate lower bound 2-chain construction from Theorem 2.1 in $\mathbb{R}^3$. In Subsection 4.3 we obtain non-trivial results for right angle k-chains with $k \geq 3$ and develop an induction mechanism that generates bounds for right angle chains of arbitrary length. If the conjecture above were to be confirmed, or even if progress is made toward it in the general case, it would immediately lead to improved bounds on some cases of longer right angle chains. We now summarize our results for point sets in $\mathbb{R}^3$, which build on Theorem B.

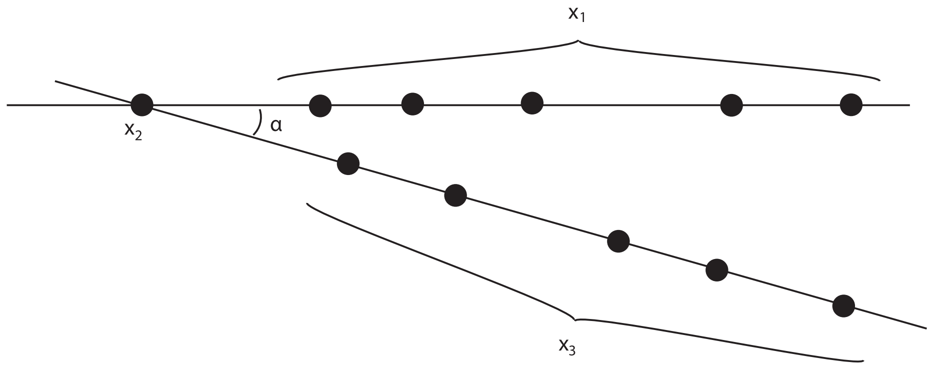

FIGURE 1. We have chosen x_1 and x_2 in the first triple, and x_4 and x_5 in the second triple. The choices of x_3 are given by the dotted circles.

Theorem 2.3. *Given a large finite point set $E \subset \mathbb{R}^3$ of n points, we have that for right angles,*

$$|\Lambda_1(E)| \lesssim n^{\frac{7}{3}}, |\Lambda_2(E)| \lesssim n^{\frac{10}{3}}, |\Lambda_3(E)| \lesssim n^4, |\Lambda_4(E)| \lesssim n^{\frac{13}{3}}, |\Lambda_5(E)| \lesssim n^5,$$

$$\text{and } |\Lambda_k(E)| \lesssim n^{\frac{1}{3}\left(12+\left\lfloor \frac{7(k-3)}{4} \right\rfloor\right)}, \text{ for } k \geq 6.$$

Proofs of these results are found in Section 4.

2.3. **Higher dimensions and pinned variants.** In Section 5 we recall the Lenz example, as it provided inspiration for many of the other higher dimensional results. We then prove that the angle chain question becomes trivial without significant further restrictions in $\mathbb{R}^d$ with $d \geq 6$. We also give some partial results in five dimensions, where the problem does not appear to be trivial. Of course, if a given construction exists in some dimension d, then it can be embedded into a higher dimensional space.

Section 6 contains a number of estimates when we fix or "pin" one of the points in question. For pinned variants we first observe that different behavior is possible depending on which point is pinned, unlike in the single distance case where the roles are symmetric. While we focus on pinning the first point for most of our estimates, a straightforward construction (shown in Figure 1) makes this distinction quite explicit when compared to the other pinned results we consider. This shows that the trivial bound of n^2 can be achieved in the case of angles with a pinned middle point.

Proposition 2.4. *In $\mathbb{R}^d$, with $d \geq 2$, for any large, finite n, there exists a set of n points with $\approx n^2$ triples of points determining any angle α, that share the same middle point.*

In contrast to the previous result, we get much different results in the plane by pinning the first point.

Theorem 2.5. *For any n points in $\mathbb{R}^2$, and any fixed angle $0 < \alpha < \pi$, there are at most $\lesssim n^{4/3}$ triples of points with angle α starting from the origin, and this is sharp.*

For angle chains in $\mathbb{R}^2$ pinned at the endpoint we establish sharp bounds up to logarithmic terms, similar to the unpinned setting. By modifying the proof of Theorem 2.1, we obtain the following.

Theorem 2.6. *For any n point set in $\mathbb{R}^2$, integer $k \geq 2$ and angles $(\alpha_1, ..., \alpha_k)$, the number of k-chains of type $(\alpha_1, ..., \alpha_k)$ starting from the origin is*

$$\leq \begin{cases} 2^{\frac{k-1}{2}} n^{\frac{k-1}{2}+1} \log n, & k \text{ odd} \\ 2^{\frac{k}{2}} n^{\frac{k}{2}+1}, & k \text{ even} \end{cases}$$

Moreover, there is a set of $\approx n$ points in $\mathbb{R}^2$ forming $\gtrsim n^{\lfloor \frac{k}{2} \rfloor +1}$ instances of k-angle chains of type $(\alpha_1, ..., \alpha_k)$ starting at the origin.

As with Theorem 2.1, the proof of the lower bound shows that for $k > 23$, the constant buried in the asymptotic estimate is at least $2^k k^{-k}$. In $\mathbb{R}^3$ we show that for a single right angle the problem is already trivial without further restrictions. We also show that the pinned problem is trivial for longer chains in $\mathbb{R}^6$. The precise statements and proofs are in Section 6.

3. Angle chains in $\mathbb{R}^2$

We first recall the statement of Theorem 2.1.

Theorem 2.1. *Given a large finite point set E of n points in the plane, and a k-tuple of angles $(\alpha_1, ..., \alpha_k)$,*

$$|\Lambda_k(E)| \leq \begin{cases} 2^{\frac{k-1}{2}} n^{\frac{k-1}{2}+2} \log n, & k \text{ odd} \\ 2^{\frac{k}{2}} n^{\frac{k}{2}+2}, & k \text{ even} \end{cases}$$

Moreover, there exists a set of n points in the plane that has $\gtrsim n^{\lfloor \frac{k}{2} \rfloor +2}$ angle k-chains of type $(\alpha_1, \dots, \alpha_k)$.

Proof. We begin by proving the upper bounds. Given a large finite point set E, and a type of k-chain, $(\alpha_1, \alpha_2, \dots, \alpha_k)$, we seek to bound the number of $(k+2)$-tuples of points from E, $(x_1, \dots, x_{k+2})$ with the property that $\angle(x_i, x_{i+1}, x_{i+2}) = \alpha_i$. We handle the cases of k even and k odd separately.

Even k: Pick x_1 and x_2. We have $n(n-1)$ such choices. Now pick x_4 from one of the remaining points. If x_1, x_2, and x_4 are collinear, then there are two possible choices (see Figure 2) for x_3 so that $\angle(x_1, x_2, x_3) = \alpha_1$ and $\angle(x_2, x_3, x_4) = \alpha_2$. If x_1, x_2, and x_4 are not collinear, then there is a unique location for x_3 so that $\angle(x_1, x_2, x_3) = \alpha_1$ and $\angle(x_2, x_3, x_4) = \alpha_2$. In either case, there are no more than two choices for x_3 once we have fixed x_1, x_2, and x_4. Continuing inductively, there are $\leq n$ choices for each subsequent even indexed point, x_{2j}. Each such choice, will fix x_{2j-1} to be one or two points, as we have already chosen x_{2j-3} and x_{2j-2}. This yields an upper bound of $2^{\frac{k}{2}} n^{\frac{k}{2}+2}$.

Odd k: We use Theorem A to get a bound of $n^2 \log n$ choices for the first triple of points, (x_1, x_2, x_3) so that $\angle(x_1, x_2, x_3) = \alpha_1$. Then we proceed in a manner similar to the even case. That is, continuing inductively, there are $\leq n$ choices for each subsequent odd indexed point, x_{2j+1}. Each such choice, will limit possible locations of x_{2j} to one or two points, as we have already chosen x_{2j-2} and x_{2j-1}. This yields an upper bound of $2^{\frac{k-1}{2}} n^{\frac{k-1}{2}+2} \log n$.

We now prove the lower bound. For $k = 1$, we use a construction similar to the one shown in Figure 1. Select two lines that meet at angle α_1, place the point x_2 at their intersection, and arrange $\lfloor \frac{n-1}{2} \rfloor$ other points on each line. This gives us $n - 1$ choices of points to be x_1, as it

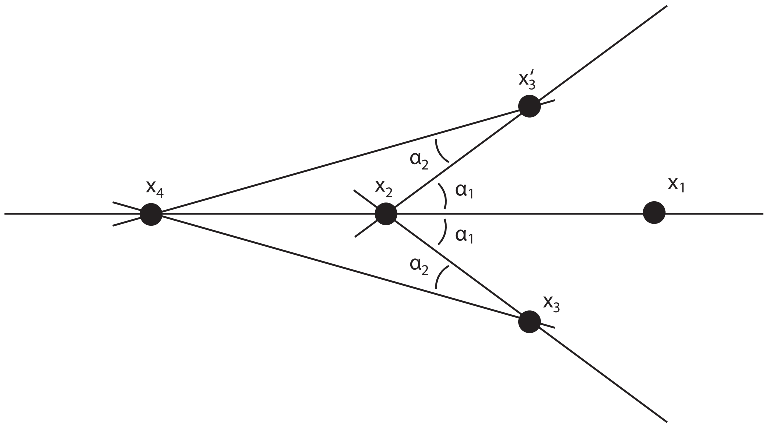

FIGURE 2. Here, x_1, x_2, and x_4 are collinear, so there are exactly two possible points, x_3 and x_3', that give a 2-chain with angles α_1 and α_2.

cannot be the same as x_2. Once we have chosen x_1, we have about $\frac{n}{2}$ points on the line that x_1 is not on, each of which is a potential choice for x_3. This gives us a total of $\geq \frac{(n-1)(n-1)}{2}$ choices.

Now, for $k \geq 2$, given $(\alpha_1, \ldots, \alpha_k)$, set $q = \left\lceil \frac{k}{2} \right\rceil + 1$, and $m = \lfloor n/q \rfloor$. Arrange m points, $p_{1,1}, \ldots, p_{1,m}$, in order away from the origin, along the x-axis, which we will call ℓ_1. Next, draw m parallel lines that make an angle of α_1 with ℓ_1. The line $L_{1,i}$ will intersect ℓ_1 at the point $p_{1,i}$. Now draw the line ℓ_2 so that it intersects each of these parallel lines $L_{1,i}$ at an angle of α_2. For each $L_{1,i}$, let the intersection with ℓ_2 be called $p_{2,i}$. Repeat the process by drawing another set of parallel lines $L_{2,i}$ through each of the points $p_{2,i}$ forming the angle α_3 with ℓ_2, and draw ℓ_3 so that it makes an angle of α_4 with each of the lines $L_{2,i}$, and so on, until you have drawn all of the n points on the lines $\ell_1, \ell_2, \ldots \ell_q$.

To simplify the exposition, we assume that m is even. We will see that if m is not even, then our count will only be off by a small amount. Notice that we can form the desired number of angle k-chains by picking x_1 from the left half of ℓ_1, and x_2 from the right half. This choice of x_2 will fix x_3 on the right half of ℓ_2, so we pick x_4 from the left half of ℓ_2. This choice will fix x_5 on the left half of ℓ_3, and continue in a similar manner. To be precise, we choose x_1 from $\{p_{1,1}, p_{1,2}, \ldots, p_{1,(m/2)}\}$, and x_2 from $\{p_{1,(m/2)+1}, p_{1,(m/2)+2}, \ldots, p_{1,m}\}$, then a fixed point x_3, determined by x_2, from ℓ_2, then choosing x_4 from $\{p_{2,1}, p_{2,2}, \ldots, p_{2,(m/2)}\}$, on ℓ_2 so that $\angle(x_2, x_3, x_4) = \alpha_2$, and so on, for a total of q lines. See Figure 3 for an illustration of this construction for $k = 2$.

To see the constant's dependence on larger k, that are still like a constant compared to n, notice that there are $m/2$ choices for x_1, then $m/2$ choices for x_2. Our choice of x_3 is fixed, but we again have $m/2$ choices for x_4. The pattern continues that each odd subscript point is fixed, while each even subscript point has $m/2$ choices. Recall that we are counting $(k+2)$-tuples of points, so we have at least

$$\left(\frac{m}{2}\right) \cdot \left(\frac{m}{2}\right) \underbrace{\cdot 1 \cdot \left(\frac{m}{2}\right) \cdot 1 \cdot \left(\frac{m}{2}\right) \cdots \left(\frac{m}{2}\right)}_{k \text{ factors}}$$

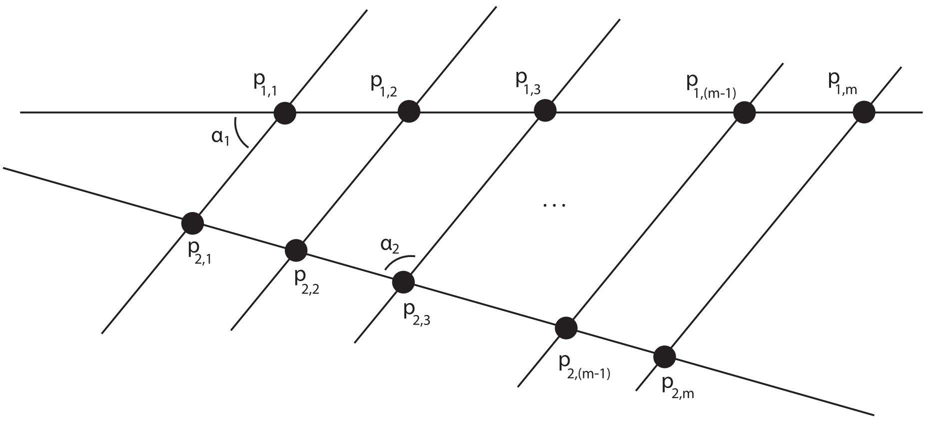

FIGURE 3. This shows a set of points arranged on two lines that determine $\gtrsim n^3$ instances of 2-chains of the form $(p_{1,i}, p_{1,j}, p_{2,j}, p_{2,k})$ of some non-right angles, where $i < j$ and $k < j$.

where from the k factors, at least half are $\left(\frac{m}{2}\right)$ and the rest are 1. So, by recalling the definitions of q and m, we get a lower bound of

$$\geq \left(\frac{m}{2}\right)^{q+1} = \left(\frac{n}{2q}\right)^{q+1} = \left(2\left(\left\lceil\frac{k}{2}\right\rceil + 1\right)\right)^{-\left(\left\lceil\frac{k}{2}\right\rceil+2\right)} n^{\left\lceil\frac{k}{2}\right\rceil+2}.$$

So the constant in the lower bound is roughly $(k+2)^{-\left\lceil\frac{k}{2}\right\rceil-2}$.

□

4. ANGLE CHAINS IN $\mathbb{R}^3$

4.1. **Point-line incidences.** As noted in the Introduction, Theorem B tells us that the maximum number of triples defining a given angle determined by a large, finite set of n points in $\mathbb{R}^3$ is $\approx n^{7/3}$. Despite considerable effort, we could not improve the upper bound $|\Lambda_2(E)| \lesssim n^{\frac{10}{3}}$ for general sets $E \subset \mathbb{R}^3$ of n points. While there are many results bounding points and various algebraic varieties (Adam Sheffer has an extensive exposition in [17]), we were unable to control incidences of points and planes sufficiently to beat the trivial estimate in general.

Here we record some partial results toward Conjecture 2.2. We first state the celebrated Szemerédi-Trotter point-line incidence estimate from [19].

Theorem D. *Given a large finite set of n points and m lines in $\mathbb{R}^2$, the number of point-line incidences is bounded above by*

$$\lesssim n^{\frac{2}{3}} m^{\frac{2}{3}} + n + m.$$

One consequence of this estimate is the following.

Theorem E. *Given a large, finite set of n points in $\mathbb{R}^d$, with $d \geq 2$, and a number $r \geq 2$, the number of lines with at least r points from the set on them is*

$$\lesssim \frac{n}{r} + \frac{n^2}{r^3}.$$

This holds in higher dimensions because with a finite set of points, one can always safely project points to some plane, and apply Theorem D there.

4.2. **Counting 2-chains in $\mathbb{R}^3$.** The following results show that with many different types of additional hypotheses, we are able to show that the number of right angle 2-chains is no more than n^3. We include these to show support for Conjecture 2.2, that n^3 is the correct upper bound.

Proposition 4.1. *Consider a set E of n points in $\mathbb{R}^3$. If every plane contains at most p points of E, then the number of right angle 2-chains is $\lesssim pn^{7/3}$.*

Proof. Consider a right angle 2-chain of points (x_1, x_2, x_3, x_4). By Theorem B we can choose (x_1, x_2, x_3) in $n^{7/3}$ ways. Since $\angle(x_2, x_3, x_4)$ is a right angle, x_4 can lie in the plane containing x_3 with normal vector $\overrightarrow{x_2x_3}$. Since each plane has at most p points of E, the total number of right angle 2-chains is $\lesssim pn^{7/3}$.

□

In [1] they show the construction obtaining the asymptotically tight bound of $\approx n^{7/3}$ is the lattice cube $[1, ..., n^{1/3}]^3$. Since the number of points in any plane is at most $n^{2/3}$, we get the following corollary:

Corollary 4.2. *The lattice cube $[1, ..., n^{1/3}]^3$ has at most $\lesssim n^3$ instances of right angle 2-chains.*

Proposition 4.3. *The number of right angle 2-chains of the form (x_1, x_2, x_3, x_4) where the $\overline{x_1x_2}$ line is parallel to the $\overline{x_3x_4}$ line is $\lesssim n^3$.*

Proof. We can choose x_1, x_2 in $\approx n^2$ ways. This choice determines the line $\overline{x_1x_2}$, and therefore fixes the direction for $\overline{x_3x_4}$. Notice that in order to form a right angle 2-chain, x_4 cannot lie on $\overline{x_1x_2}$. Still, we may have as many as $\approx n$ choices for x_4. However, after we have chosen x_4, the line $\overline{x_3x_4}$ is completely determined, as it must be parallel to $\overline{x_1x_2}$. Now, x_3 must lie in the plane P that is normal to the vector $\overrightarrow{x_2x_1}$. Because $\overline{x_3x_4}$ is parallel to $\overline{x_1x_2}$, we have that $\overline{x_3x_4}$ is perpendicular to the plane P. Therefore, there is only one possible choice for x_3 at the intersection of this plane and the line $\overline{x_3x_4}$. See Figure 4

□

Proposition 4.4. *For a set of n points in $\mathbb{R}^3$, suppose a right angle 2-chain is formed where the $\overline{x_1x_2}$ line has at least j points, the $\overline{x_3x_4}$ line has at least k points, and these lines are not parallel. The number of ways to choose these 2-chains is*

$$\lesssim \left(\frac{n^2}{j^2} + n\right)\left(\frac{n^2}{k^2} + n\right).$$

In particular, if j and k are both greater than $n^{\frac{1}{4}}$ or one of them is greater than $n^{\frac{1}{2}}$, then this term is $\lesssim n^3$.

Proof. By Theorem E, the number of ways to choose a line with at least i points is $\lesssim \left(\frac{n^2}{i^3} + \frac{n}{i}\right)$. We can choose the two lines in $\lesssim \left(\frac{n^2}{j^3} + \frac{n}{j}\right)\left(\frac{n^2}{k^3} + \frac{n}{k}\right)$ ways, plus choosing x_1 and x_4 in $j \cdot k$

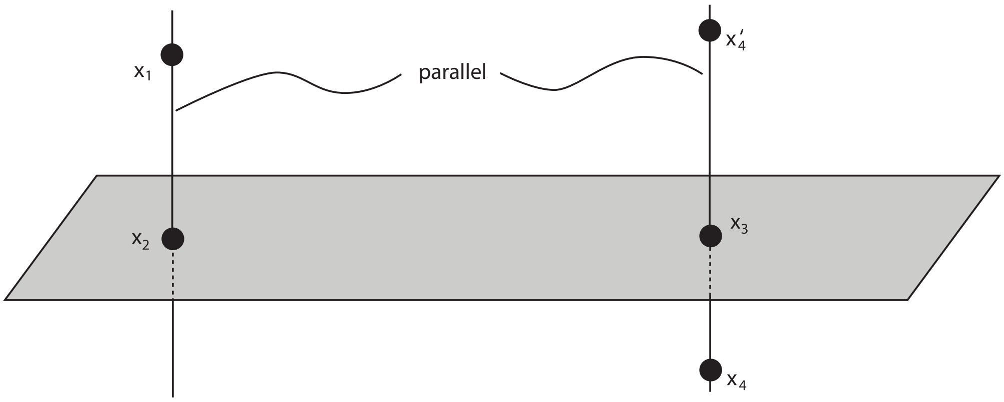

FIGURE 4. This shows how the choices of x_1, x_2, and x_4 fix the possible location of x_3, even with a different choice of x_4 (shown here as x_4') on the same line.

ways. Since the $\overline{x_1x_2}$ line is not parallel to the $\overline{x_3x_4}$ line, there is at most one choice of x_2 and x_3 to form a right angle 2-chain. So the number of right angle 2-chains is $\lesssim \left(\frac{n^2}{j^2}+n\right)\left(\frac{n^2}{k^2}+n\right)$. □

Proposition 4.5. *For a set of n points in $\mathbb{R}^3$, suppose a right angle 2-chain is formed where the $\overline{x_2x_3}$ line has at least m points, and the lines $\overline{x_1x_2}$ and $\overline{x_3x_4}$ are not parallel. The number of ways to choose these 2-chains is*

$$\lesssim \left(\frac{n^4}{m^3}+\frac{n^3}{m}\right).$$

In particular, if $m \geq n^{1/3}$, then the number of 2-chains is $\lesssim n^3$.

Proof. The number of ways to choose a $\overline{x_2x_3}$ line with at least m points is $\lesssim \left(\frac{n^2}{m^3}+\frac{n}{m}\right)$. Having chosen this line, we can choose x_1 and x_4 in $\approx n^2$ ways. Since the lines $\overline{x_1x_2}$ and $\overline{x_3x_4}$ are not parallel, there are is at most one choice of x_2 and x_3. So the number of these 2-chains is $\lesssim \left(\frac{n^4}{m^3}+\frac{n^3}{m}\right)$.

□

4.3. **Longer chains in $\mathbb{R}^3$.** While the case of right angle 2-chains remains open, we were able to prove nontrivial bounds on right angle k-chains for higher k.

Theorem 4.6. *Given a large finite point set E of n points in $\mathbb{R}^3$, the number of right angle 3-chains is no more than*

$$|\Lambda_3(E)| \lesssim n^4.$$

Proof. First we choose the points x_1, x_2, x_4, and x_5. To form right angles, x_3 must lie in the plane containing x_2 with normal vector $\overrightarrow{x_1x_2}$, as well as the plane containing x_4 with normal vector $\overrightarrow{x_4x_5}$. These two planes must intersect to contain x_3.

Case 1: If they are not the same plane, then they intersect on a line. In order for $\angle(x_2, x_3, x_4)$ to be a right angle, x_3 must lie on the sphere with antipodal points x_2 and x_4. The intersection of the two planes and the sphere is two points, which is the number of choices of x_3. We can choose x_1, x_2, x_4, and x_5 in n^4 ways, so the total number of choices is $\lesssim n^4$.

Case 2: If they are the same plane, observe that x_1, x_2, x_4, and x_5 will form a 2-chain, with the $\overline{x_1x_2}$ line parallel to the $\overline{x_4x_5}$ line. Therefore, (x_1, x_2, x_4, x_5) will form a 2-chain of the type handled by Proposition 4.3, so there are $\lesssim n^3$ choices of these points. With n choices of x_3, this makes a total of $\lesssim n^4$ choices.

□

Theorem 4.7. *Given a large finite point set E of n points in $\mathbb{R}^3$,*

$$|\Lambda_4(E)| \lesssim n^{\frac{13}{3}}.$$

Proof. First we choose the points x_1, x_2, x_4, x_5, x_6. To form right angles, x_3 must lie in the plane containing x_2 with normal vector $\overrightarrow{x_1x_2}$, as well as the plane containing x_4 with normal vector $\overrightarrow{x_4x_5}$. These two planes must intersect to contain x_3.

Case 1: If they are not the same plane, then they intersect on a line. In order for $\angle(x_2, x_3, x_4)$ to be a right angle, x_3 must lie on the sphere with antipodal points x_2 and x_4. The intersection of the two planes and the sphere is two points, which is the number of choices of x_3. We can choose x_4, x_5, and x_6 in $n^{7/3}$ ways by Theorem B, and x_1 and x_2 in n^2 ways. So the total number of choices is $\lesssim n^{\frac{13}{3}}$.

Case 2: If they are the same plane, then we can first choose x_4, x_5, and x_6 in $n^{7/3}$ ways by Theorem B. Since the two planes to be equal, $\overrightarrow{x_1x_2}$ is parallel to $\overrightarrow{x_4x_5}$. So if we choose x_1 in n ways, this fixes the line where x_2 could lie. Moreover, x_2 must lie in the common plane, so together with the line from x_1 there is at most one choice for x_2. We can choose x_3 in n ways, so the total number of 4-chains is $\lesssim n^{\frac{13}{3}}$.

□

The lower bound from Theorem 2.1 gives us that the number of 4-chains in $\mathbb{R}^2$ is at least $\gtrsim n^4$. This construction can also be embedded in $\mathbb{R}^3$, so the multiplicative gap between the lower and upper bounds is $n^{1/3}$.

For longer chains we have the following technical result. It is proved by bounding the number of right angle k-chains formed by two different families. Because we do not know a priori which family will dominate, we are only guaranteed whichever bound is the weakest. So these upper bounds are presented as the sum of two terms.

Theorem 4.8. *Given a large finite point set E of n points in $\mathbb{R}^3$, for $k \geq 5$,*

$$|\Lambda_k(E)| \lesssim n|\Lambda_{k-2}(E)| + n^{7/3}|\Lambda_{k-4}(E)|.$$

Proof. We will split it into two types of k-chains: either $\overrightarrow{x_2x_3}$ is parallel to $\overrightarrow{x_5x_6}$ or it is not.

Case 1: If they are not parallel, then we can choose x_1, x_2, and x_3 in $n^{7/3}$ ways, and $x_5, ..., x_k, x_{k+1}, x_{k+2}$ in $|\Lambda_{k-4}(E)|$ ways. Now, x_4 must lie in the plane with normal vector $\overrightarrow{x_2x_3}$ containing x_3, as well as the plane with normal vector $\overrightarrow{x_5x_6}$ containing x_5. Since the normal vectors are not parallel, the planes intersect in a line. Finally, x_4 must lie on the sphere with x_3 and x_5 as antipodal points. The sphere intersects the line in at most two places, so there are at most two choices for x_4. Hence the bound for $|\Lambda_k(E)|$ is $2 \cdot n^{7/3} \cdot |\Lambda_{k-4}(E)|$.

Case 2: If they are parallel, then we can first choose $x_3, ..., x_k, x_{k+1}, x_{k+2}$ in $|\Lambda_{k-2}(E)|$ ways and x_1 in n ways. The point x_2 must lie on the line through x_3 and parallel to $\overrightarrow{x_5x_6}$. It must also lie on the sphere with x_1 and x_3 as antipodal points. The line intersects the sphere in at most two places, so there are at most two choices for x_2. Hence the bound for $|\Lambda_k(E)|$ is $2 \cdot n \cdot |\Lambda_{k-2}(E)|$.

□

Here we present the upper bounds for some values of k given by the two terms in Theorem 4.8. Because the upper bound is a sum, we have to take the larger of the two terms as our bound. For example, when $k = 5$, Theorem 4.8 gives us an upper bound of

$$|\Lambda_5(E)| \lesssim n|\Lambda_3(E)| + n^{7/3}|\Lambda_1(E)| \lesssim n^5 + n^{\frac{16}{3}},$$

which is dominated by the second term. To see how the next few values of k work out, we include the following table of upper bounds given by each term in the statement of Theorem 4.8.

k	5	6	7	8	9
upper bound of: $n\|\Lambda_{k-2}(E)\|$	$n^{\frac{15}{3}}$	$n^{\frac{16}{3}}$	$n^{\frac{18}{3}}$	$n^{\frac{20}{3}}$	$n^{\frac{22}{3}}$
upper bound of: $n^{7/3}\|\Lambda_{k-4}(E)\|$	$n^{\frac{14}{3}}$	$n^{\frac{17}{3}}$	$n^{\frac{19}{3}}$	$n^{\frac{20}{3}}$	$n^{\frac{22}{3}}$

Sometimes the first term dominates, and sometimes the second term dominates, until we get to sufficiently large values of k. We make this explicit in the proof of Theorem 2.3, which is a corollary of several of the results mentioned above. For convenience, we recall the statement here.

Theorem 2.3. *Given a large finite point set $E \subset \mathbb{R}^3$ of n points, we have that for right angles,*

$$|\Lambda_1(E)| \lesssim n^{\frac{7}{3}}, |\Lambda_2(E)| \lesssim n^{\frac{10}{3}}, |\Lambda_3(E)| \lesssim n^4, |\Lambda_4(E)| \lesssim n^{\frac{13}{3}}, |\Lambda_5(E)| \lesssim n^5,$$

$$\text{and } |\Lambda_k(E)| \lesssim n^{\frac{1}{3}\left(12+\left\lfloor \frac{7(k-3)}{4} \right\rfloor\right)}, \text{ for } k \geq 6.$$

Proof. Again, we let $|\Lambda_k(E)|$ be the maximum number of right angle k-chains for a set E of n points in $\mathbb{R}^3$. By Theorem B, we have that $|\Lambda_1(E)| \lesssim n^{\frac{7}{3}}$. For the case $k = 2$, we again apply Theorem B to get a bound on the number of triples of points that form a right angle. Then by choosing the fourth point freely, we get that $|\Lambda_2(E)| \lesssim n|\Lambda_1(E)| \lesssim n^{\frac{10}{3}}$. Theorem 4.6 and Theorem 4.7 give us $|\Lambda_3(E)| \lesssim n^4$ and $|\Lambda_4(E)| \lesssim n^{\frac{13}{3}}$, respectively. We then appeal to Theorem 4.8 to get that $|\Lambda_5(E)| \lesssim n^5$. For $6 \leq k \leq 9$, we can use Theorem 4.8 to verify that we have

$$n|\Lambda_{k-2}(E)| \lesssim n^{\frac{7}{3}}|\Lambda_{k-4}(E)|. \tag{1}$$

Now we show that (1) holds for larger values of k by induction. To see this, suppose that (1) is true for all $6 \leq k \leq m$, for some $m \geq 9$. Now consider $|\Lambda_{m+1}(E)|$. By Theorem 4.8, we have that

$$\begin{aligned}|\Lambda_{m+1}(E)| &\lesssim n|\Lambda_{(m+1)-2}(E)| + n^{\frac{7}{3}}|\Lambda_{(m+1)-4}(E)| \\ &= n|\Lambda_{m-1}(E)| + n^{\frac{7}{3}}|\Lambda_{m-3}(E)|.\end{aligned}$$

Now, because (1) holds for all values of k between 6 and m, it applies to $|\Lambda_{m-1}(E)|$, so we get that

$$(n|\Lambda_{m-1}(E)|) + n^{\frac{7}{3}}|\Lambda_{m-3}(E)| \lesssim \left(n^{\frac{7}{3}}|\Lambda_{m-3}(E)|\right) + n^{\frac{7}{3}}|\Lambda_{m-3}(E)|,$$

and by induction we have shown that (1) holds for $k \geq 6$.

To finish, we just notice that this recurrence implies that, starting from the value of $\frac{17}{3}$ when $k = 6$, every four consecutive exponents will increase with k by $\frac{2}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{2}{3}$, before repeating. So we need the exponents to increase by $\frac{7}{3}$ in four discrete steps. One way to express this is setting, for any $k \geq 6$,

$$|\Lambda_k(E)| \lesssim n^{\frac{1}{3}\left(12+\left\lfloor \frac{7(k-3)}{4} \right\rfloor\right)}.$$

□

5. Angle chains in higher dimensions

Many discrete geometry problems become trivial in general when we consider them in higher dimensions. One motivation for much of the work in this section is the classical Lenz example for distances. See [4] for more on the subject. We now present the Lenz example, as it provides motivation for many of the constructions to follow.

Theorem F. *For $d \geq 4$, there exists a set of n points in $\mathbb{R}^d$ with $\approx n^2$ pairs of points that define the same distance.*

Proof. We construct the set in four dimensions, and it can easily be embedded in higher dimensional spaces. The basic idea is to evenly distribute points along two circles: one in the first two dimensions, and another in the next two dimensions. To this end, define

$$E := \left\{(\cos a_1, \sin a_1, 0, 0) : a_1 = 1, \ldots, \left\lceil \frac{n}{2} \right\rceil \right\},$$
$$F := \left\{(0, 0, \cos a_2, \sin a_2) : a_2 = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Notice that any point in E is at a distance $\sqrt{2}$ to any point in F. So the union of E and F is a set of n points, with $\gtrsim n^2$ pairs of points that each determine the same distance.

□

One key feature to take away from this construction is that sharpness examples for these kinds of questions often rely on low dimensional intersections of varieties in higher dimensions. Getting control on examples like this within general point sets can guide one to better upper bounds, but we believe that they are interesting in their own right. So for the following constructions, we are again looking for low dimensional intersections of high dimensional varieties. We begin with a result from [4], which is also described in [1].

Theorem G. *There is a set of n points in $\mathbb{R}^4$ forming $\approx n^3$ right angles.*

Proof. For the upper bound, there are at most n^3 choices of triples of points. The lower bound comes from considering

$$x_1 = (-1, 0, a_1, 0)$$
$$x_2 = (\cos(a_2), \sin(a_2), 0, 0)$$
$$x_3 = (1, 0, 0, a_3)$$

where $a_1, a_2, a_3 \in \{1, ..., \lfloor n/3 \rfloor\}$. The triple (x_1, x_2, x_3) is a right angle for any choice of (a_1, a_2, a_3), so there are $\gtrsim n^3$ of these chains.

□

Expanding on this construction shows that without further hypotheses, the right angle 2-chain question is trivial in dimension five.

Theorem 5.1. *There is a set of n points in $\mathbb{R}^5$ forming $\approx n^4$ right angle 2-chains.*

Proof. For the upper bound, there are trivially at most n^4 choices of quadruples of n points. The lower bound is attained by setting

$$\begin{aligned} x_1 &= (-1, 0, a_1, 0, 1) \\ x_2 &= (\cos(a_2), \sin(a_2), 0, 0, 1) \\ x_2 &= (1, 0, 0, \cos(a_3), \sin(a_3)) \\ x_4 &= (1, 0, a_4, 0, -1), \end{aligned}$$

where $a_1, a_2, a_3, a_4 \in \{1, ..., \lfloor n/4 \rfloor\}$. The quadruple (x_1, x_2, x_3, x_4) is a right angle 2-chain for any choice of (a_1, a_2, a_3, a_4), so there are $\gtrsim n^4$ of these chains.

□

By following this general idea, we can show that such questions about longer chains are trivial in higher dimensions. In fact, without further restrictions, there exists a set with asymptotically as many right angle k-chains as possible in $\mathbb{R}^6$, for any k that is like a constant compared to n.

Theorem 5.2. *For any positive integer k, there is a set of n points in $\mathbb{R}^6$ forming $\approx n^{k+2}$ right angle k-chains.*

Proof. For the upper bound, there are at most n^{k+2} choices of points. Consider the following set of points, in this order:

$$\begin{aligned} x_1 &= (-1, 0, 1, 0, \cos(a_1), \sin(a_1)) \\ x_2 &= (\cos(a_2), \sin(a_2), 1, 0, 1, 0) \\ x_3 &= (1, 0, \cos(a_3), \sin(a_3), 1, 0) \\ x_4 &= (1, 0, -1, 0, \cos(a_4), \sin(a_4)) \\ x_5 &= (\cos(a_5), \sin(a_5), -1, 0, -1, 0) \\ x_6 &= (-1, 0, \cos(a_6), \sin(a_6), -1, 0) \end{aligned}$$

where the a_i are positive integers between 1 and $n/6$.

For every ordered triple (x_i, x_{i+1}, x_{i+2}), where $i - 1 \in \mathbb{Z}/6\mathbb{Z}$, these points form a right angle. Therefore, for any positive integer k, there are $n/6$ choices of the form x_1, and $n/6$ of x_2, and then of x_3, x_4, x_5, x_6, and then another $n/6$ going back to x_1, and so on. This forms a total of $\gtrsim n^{k+2}$ instances of k-chains.

□

The following construction is directly motivated by the Lenz example. The basic idea is that using a Lenz-type construction, we can construct a set with many prescribed distances between points. We can then verify that the points form the the desired angles by computing the dot products of the vectors between them. In contrast to many other results in this section, it holds for a range of angles, rather than just right angles.

Theorem 5.3. *For any choice of k positive acute angles $(\alpha_1, ..., \alpha_k)$, with $\cot \alpha_i < \tan \alpha_{i-1}$, there is a set of n points in $\mathbb{R}^6$ forming $\approx n^{k+2}$ instances of a k-chain with angles $(\alpha_1, ..., \alpha_k)$.*

Proof. Consider the following $k+2$ families of points, where the a_i are positive integers between 1 and $n/(k+2)$, and the c_i are positive real numbers.

$$\begin{aligned}
x_1 &= (c_1\cos(a_1), c_1\sin(a_1), 0, 0, 0, 0)\\
x_2 &= (0, 0, c_2\cos(a_2), c_2\sin(a_2), 0, 0)\\
x_3 &= (0, 0, 0, 0, c_3\cos(a_3), c_3\sin(a_3))\\
x_4 &= (c_4\cos(a_4), c_4\sin(a_4), 0, 0, 0, 0)\\
x_5 &= (0, 0, c_5\cos(a_5), c_5\sin(a_5), 0, 0)\\
x_6 &= (0, 0, 0, 0, c_6\cos(a_6), c_6\sin(a_6))\\
&\vdots\\
&x_{k+2}.
\end{aligned}$$

Here x_i has $c_i\cos(a_i)$ in the $2i-1 \pmod 6$ entry, $c_i\sin(a_i)$ in the entry that is $2i \pmod 6$, and 0 in the other entries. We claim that any combination of the $n/(k+2)$ choices for each x_i will be an angle k-chain of type $(\alpha_1, \alpha_2, \dots, \alpha_k)$, where these angles α_i depend on the choices of the c_i, but not the a_i. We recall the following elementary formula relating the dot product of two vectors to their magnitudes and angle between them.

$$\overrightarrow{u}\cdot\overrightarrow{v} = \|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta.$$

By applying this to our points, regardless of the values of the a_i, we get

$$\alpha_i = \angle(x_i, x_{i+1}, x_{i+2}) = \arccos\left(\frac{\overrightarrow{x_{i+1}x_i}\cdot\overrightarrow{x_{i+1}x_{i+2}}}{\|\overrightarrow{x_{i+1}x_i}\|\|\overrightarrow{x_{i+1}x_{i+2}}\|}\right).$$

Computing the magnitudes and dot product of the appropriate vectors, gives

$$\alpha_i = \arccos\left(\frac{c_{i+1}^2}{\sqrt{c_i^2+c_{i+1}^2}\sqrt{c_{i+2}^2+c_{i+1}^2}}\right). \tag{2}$$

We now describe an algorithm for picking the constants c_i based on the given set of angles α_1. The basic idea is to fix a value for c_1, then use this to determine some bounds on the values of the subsequent c_i. After we get lower bounds on all of the c_i where $i = 2, \dots, (k+1)$, we follow a similar scheme to get upper bounds. We then put these together to choose explicit values of c_i.

Notice that (2) can be rearranged to yield

$$c_{i+2} = c_{i+1}\sqrt{\frac{c_{i+1}^2}{(c_i^2+c_{i+1}^2)\cos^2\alpha_i} - 1}.$$

For c_{i+2} to be real and positive, we need that the radicand above to be positive. This happens precisely when

$$\begin{gathered}
c_{i+1}^2 > c_i^2\cos^2\alpha_i + c_{i+1}^2\cos^2\alpha_i\\
c_{i+1}^2 - c_{i+1}^2\cos^2\alpha_i > c_i^2\cos^2\alpha_i\\
c_{i+1}^2 > c_i^2\frac{\cos^2\alpha_i}{1-\cos^2\alpha_i},
\end{gathered}$$

which gives us

$$c_{i+1} > c_i \cot \alpha_i. \tag{3}$$

Also, (2) can also be rewritten as

$$c_i = c_{i+1}\sqrt{\frac{c_{i+1}^2}{(c_{i+2}^2 + c_{i+1}^2)\cos^2 \alpha_i} - 1}.$$

By similar reasoning, we get

$$c_{i+1} > c_{i+2} \cot \alpha_i.$$

Keeping in mind that this holds for all $i > 2$, we can deduce that

$$c_i > c_{i+1} \cot \alpha_{i-1}. \tag{4}$$

Combining (3) and (4), keeping in mind that the $\alpha_i \in (0, \pi/2)$, gives us

$$c_i \tan \alpha_{i-1} > c_{i+1} > c_i \cot \alpha_i. \tag{5}$$

Now, we pick $c_1 = 1$, and any $c_2 > \cot \alpha_1$, so we can satisfy (3). For $i > 2$, we pick any value of c_i satisfying (5). Notice that this is only possible if $\tan \alpha_{i-1} > \cot \alpha_i$.

□

Example 5.4. *As an example of this theorem, we can choose $c_1 = c_2 = \cdots c_{k+2} = 1$, which makes $\alpha_1 = \cdots = \alpha_k = \pi/3$.*

6. Pinned results

6.1. Proof of Proposition 2.4.

Proof. This follows by the same construction used in the proof of the lower bound of Theorem 2.1 for $k = 1$, illustrated in Figure 1. Find two rays that determine the angle α. Call their shared endpoint x_2, and let it be in our point set. Then arrange half of the remaining points along one ray, and the rest along the other. So there are roughly $n/2$ points on the first ray, giving us $\approx n$ choices for x_1, and roughly $n/2$ points on the second ray, giving us $\approx n$ choices for x_3. By the symmetry of the choice of x_1 and x_3, we have $\geq \frac{(n-1)(n-2)}{2}$ triples of the form (x_1, x_2, x_3) that determine our angle α with the same choice of middle point.

□

6.2. **Proof of Theorem 2.5.** Henceforth, we focus on pinned results where the choice of x_1 remains fixed. For pinned angles in $\mathbb{R}^2$ we first recall the statement of Theorem 2.5.

Theorem 2.5. *For any n points in $\mathbb{R}^2$, and any fixed angle $0 < \alpha < \pi$, there are at most $\lesssim n^{4/3}$ triples of points with angle α starting from the origin O, and this is sharp.*

Proof. Let P be a set of n points in $\mathbb{R}^2 \setminus \{O\}$. Pick a point $x \in P$. There are exactly 2 lines $\ell_x^{(1)}, \ell_x^{(2)}$ through x such that the line through O and x forms an angle of α with each of the lines $\ell_x^{(1)}, \ell_x^{(2)}$. (When $\alpha = \pi/2$, there will be only one such line ℓ_x.) Call L the collection of these lines over all points $x \in P$. Since each of these n choices of x has 2 associated lines, then $|L|$ is at most $2n$, since it is possible the same line could be associated with 2 different choices of x.

For each $x \in P$, every other point $p \in P$ satisfying $\angle Oxp = \alpha$ must lie on one of two lines $\ell_x^{(1)}, \ell_x^{(2)}$.[1] So the number of times this occurs is at most the number of incidences between the

[1]We further can say that for a line ℓ_x that each point p lying on ℓ_x satisfying $\angle Oxp = \alpha$ can be on only one side of the line through x and the origin, but this does not affect the asymptotics.

points P and lines L. Since $|P| = n$ and $|L| \leq 2n$, we can appeal to Theorem D to conclude that the number of these incidences is $\lesssim n^{4/3}$.

We now turn our attention to the sharpness, which is realized with the following construction, motivated by classical sharpness examples for the Theorem D:

$$P = \left\{(a,b) \in \mathbb{Z}^2 : 1 \leq a \leq n^{1/3}, 1 \leq b \leq n^{2/3}\right\}$$
$$L = \left\{y = ax + b : (a,b) \in \mathbb{Z}^2, 1 \leq a \leq n^{1/3}, 1 \leq b \leq \frac{n^{2/3}}{2}\right\}$$

Note that $|P| = n$, $|L| = n/2$, and each line of L is incident to $n^{1/3}$ points. For each line in L, there are two pivot points (one point if $\alpha = \pi/2$) on the line (not necessarily coming from P) forming an angle of α with origin. Combined through these two pivot points on L, there are $n^{1/3}$ points on the line intersecting P forming an α angle. Call Q the collection of these points from each of these lines. Since $|Q| = n$ (or $n/2$ if $\alpha = \pi/2$), then $|P \cup Q| \approx n$. So we have a collection of $\approx n$ points of $P \cup Q$ forming $|L|n^{1/3} \gtrsim n^{4/3}$ instances of α-angles. Thus the upper bound of the theorem is asymptotically sharp.

□

For pinned chains in $\mathbb{R}^2$ we now recall the statement of Theorem 2.6. The proof uses the same ideas as the proof of Theorem 2.1.

Theorem 2.6. *For any set E of n points in $\mathbb{R}^2$, integer $k \geq 2$ and angles $(\alpha_1, ..., \alpha_k)$, the number of k-chains of type $(\alpha_1, ..., \alpha_k)$ starting from the origin is*

$$\leq \begin{cases} 2^{\frac{k-1}{2}} n^{\frac{k-1}{2}+1} \log n, & k \text{ odd} \\ 2^{\frac{k}{2}} n^{\frac{k}{2}+1}, & k \text{ even} \end{cases}$$

Moreover, there is a set of $\approx n$ points in $\mathbb{R}^2$ forming $\gtrsim n^{\lfloor \frac{k}{2} \rfloor + 1}$ instances of k-angle chains of type $(\alpha_1, ..., \alpha_k)$ starting at the origin.

Proof. For $k = 2$, we can choose points x_3 and x_4 in $\lesssim n^2$ ways. There are at most two lines through x_3 that form an angle of α_2 with the $\overline{x_3 x_4}$. So x_2 must lie on one of these lines. In order for $\angle(x_1, x_2, x_3) = \alpha_1$, there is only one choice of x_2 on each of the at most two lines. Since $x_1 = (0,0)$ is fixed, there are a total of $\lesssim n^2$ choices.

For $k \geq 3$, we can choose an k-chain (unpinned) of type $(\alpha_3, ..., \alpha_k)$ in $\lesssim n^{\frac{k-1}{2}+1} \log(n)$ ways if k is odd and $\lesssim n^{\frac{k}{2}+1}$ ways if k is even. By the same argument above, there are at most two choices of x_2, and x_1 is fixed. This concludes the proof of the upper bound.

The proof of the lower bound follows by the same argument used for to prove the lower bound of Theorem 2.1, but fixing x_1 as the origin.

□

In $\mathbb{R}^3$ we show that already for a single right angle the problem is trivial without further restrictions.

Theorem 6.1. *There is a set of $\approx n$ points in $\mathbb{R}^3$ forming $\gtrsim n^2$ right angles starting at the origin.*

For a set of $\approx n$ points, such a right angle chain comprises of some 2 points together with the origin. So there are $\lesssim n^2$ ways to choose these 2 points, which this theorem shows is asymptotically tight.

Proof. Let P be the set of $n/2$ points $y_i \in \mathbb{R}^3$ with coordinates $(1+\cos i, \sin i, 0)$, where $i = 1, \ldots, n/2$. Let Q be the set of $n/2$ points $z_j \in \mathbb{R}^3$ with coordinates $(2, 0, j)$, where $j = 1, \ldots, n/2$. Now choose an arbitrary pair of $y_i \in P$ and $z_j \in Q$. We can verify that the origin, y_i, and z_j form a right angle by computing the dot product of the vector $v = y_i - (0,0,0)$ with the vector $w = y_i - z_j$.

$$\begin{aligned} v \cdot w &= (1+\cos i, \sin i, 0) \cdot (-1+\cos i, \sin i, j) \\ &= (-1+\cos i - \cos i + \cos^2 i) + (\sin^2 i) + 0 \\ &= -1 + (\sin^2 i + \cos^2 i) = 0. \end{aligned}$$

The idea is that the planes normal to each choice of y_i containing that particular point y_i all meet at the line containing Q. One could in principle choose any set of about n points on this circle containing P and any set of about n points on the line containing Q to get the same result.

□

We also have the following pinned versions of Theorems 5.2 and 5.3. These both follow by straightforward modifications of the proofs of their unpinned versions.

Theorem 6.2. *For any positive integer k, there is a set of n points in $\mathbb{R}^6$ forming at least $\left(\frac{n}{6}\right)^{k+1}$ right angle k-chains starting from the origin.*

For a set of n points, such a right angle k-chain comprises of some $k+1$ points of the set together with the origin. So there are $\lesssim n^{k+1}$ ways to choose these $k+1$ points, which this theorem shows is asymptotically tight.

Proof. For a set of n points, there are $k+1$ choices of n points not including the origin to form a k chain. So there are $\lesssim n^{k+1}$ right angle k-chains starting from the origin, which serves as x_1.

For the lower bound, consider the following set of points, in this order:

$$\begin{aligned} x_2 &= (0,0,2,0,1+\cos(a_1),\sin(a_1)) \\ x_3 &= (1+\cos(a_2),\sin(a_2),2,0,2,0) \\ x_4 &= (2,0,1+\cos(a_3),\sin(a_3),2,0) \\ x_5 &= (2,0,0,0,1+\cos(a_4),\sin(a_4)) \\ x_6 &= (1+\cos(a_5),\sin(a_5),0,0,0,0) \\ x_7 &= (0,0,1+\cos(a_6),\sin(a_6),0,0) \end{aligned}$$

where the $a_i's$ are positive integers between 1 and $n/6$.

For every ordered triple (x_i, x_{i+1}, x_{i+2}), where $i-1 \in \mathbb{Z}/6\mathbb{Z}$, these points form a right angle. Therefore, for any positive integer k, there are $n/6$ choices of the form x_1, and $n/6$ of x_2, and then of x_3, x_4, x_5, x_6, and then another $n/6$ going back to x_1, etc. This forms a total of $\left(\frac{n}{6}\right)^{k+1}$ k-chains starting from the origin.

□

Theorem 6.3. *For any choice of k positive acute angles $(\alpha_1, ..., \alpha_k)$, with $\cot \alpha_i < \tan \alpha_{i-1}$, there is a set of n points in $\mathbb{R}^6$ forming $\approx n^{k+1}$ instances of a k-chain with angles $(\alpha_1, ..., \alpha_k)$ starting from the origin.*

Proof. Consider the same $k+2$ points as in Theorem 5.3:

$$
\begin{aligned}
x_1 &= (c_1\cos(a_1), c_1\sin(a_1), 0, 0, 0, 0)\\
x_2 &= (0, 0, c_2\cos(a_2), c_2\sin(a_2), 0, 0)\\
x_3 &= (0, 0, 0, 0, c_3\cos(a_3), c_3\sin(a_3))\\
x_4 &= (c_4\cos(a_4), c_4\sin(a_4), 0, 0, 0, 0)\\
x_5 &= (0, 0, c_5\cos(a_5), c_5\sin(a_5), 0, 0)\\
x_6 &= (0, 0, 0, 0, c_6\cos(a_6), c_6\sin(a_6))\\
&\vdots\\
x_{k+2}&.
\end{aligned}
$$

Here x_i has $c_i\cos(a_i)$ in the $2i-1 \pmod 6$ entry, $c_i\sin(a_i)$ in the entry that is $2i \pmod 6$, and 0 in the other entries. Setting $c_1 = 0$ makes x_1 the origin. We then follow the proof of Theorem 5.3.

□

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