BULLETIN OF THE
HELLENIC MATHEMATICAL SOCIETY
Volume 65, 2021 (9–16)

TOPOLOGICAL AND ALGEBRAIC GENERICITY IN CHAINS OF SEQUENCE SPACES AND FUNCTION SPACES

L. BERNAL-GONZÁLEZ AND V. NESTORIDIS

Abstract. We present the example of a chain of ℓ^p spaces, where we examine results of topological and algebraic genericity and spaceability. We also include other chains of spaces, mainly spaces of holomorphic functions, where a similar investigation is done.

0. Introduction

This paper is conceived as a research project where we consider chains of spaces $X_i \subset X_j$, $X_i \neq X_j$ for $i < j$ and we examine if X_i is an F_σ meager set in X_j (topological genericity); equivalently, if $X_j \setminus X_i$ is a G_δ-dense subset of X_j. The main tool towards this is Baire's Category theorem for complete metric spaces. Furthermore, we examine if $X_j \setminus X_i$ contains, except for 0, a vector space that is dense in X_j (algebraic genericity). Finally, we analyze if $X_j \setminus X_i$ contains, except for 0, a vector space which is infinite dimensional and closed in X_j (spaceability). One can also examine, if $\bigcup_{i<j} X_i$ is an F_σ meager in X_j (topological genericity) and if $X_j \setminus \left(\bigcup_{i<j} X_i\right)$ contains, except for 0, large vector spaces (algebraic genericity and spaceability).

In Sections 1, 2 and 3 we treat the example of ℓ^p spaces. Most, but not all, of the examples, concerning ℓ^p spaces, treated in Sections 1-2-3 can be found in [1] and [3], where the properties of algebraic genericity and spaceability for

$$\ell^p \setminus \bigcup_{q<\beta} \ell^q \quad (\beta \leq p)$$

are also included. From these sections, all the results concerning the space

$$\bigcap_{p>\alpha} \ell^p$$

seem to be new. However, since our proofs are close to known proofs they are omitted. Detailed proofs can be found in [8].

For algebraic genericity and spaceability we are also referring to the works mentioned in [1], [3], and especially those of F. Bayart and S. Charpentier, which have influenced us. For topological genericity we refer to [2] below and the references therein, especially the works of J.-P. Kahane and K.-G. Grosse-Erdmann.

The chain of ℓ^p spaces can be extended adding intersections of such spaces, as well as, the inclusions $\ell^p \subset c_0 \subset \ell^\infty \subset H(\mathbb{D}) \subset \mathbb{C}^{\mathbb{N}_0}$, where every sequence (a_n) in ℓ^∞ can be identified with the function $f(z) = \sum_{n\geq 0} a_n z^n$, which is holomorphic in the open unit disc $\mathbb{D}$ of the complex

Date: Received 28 January 2021. Accepted 14 September 2021.

2020 *Mathematics Subject Classification.* Primary: 15A03, 46E10, 46E15. Secondary, 30H10, 30H20, 30H35.

Key words and phrases. Topological genericity, algebraic genericity, spaceability, Baire theorem, ℓ^p spaces, spaces of holomorphic functions.

plane $\mathbb{C}$. Then we say that a function $f \in H(\mathbb{D})$ with MacLaurin coefficientes a_n belongs to ℓ^p whenever $(a_n) \in \ell^p$. The cartesian product $\mathbb{C}^{\mathbb{N}_0}$ can be identified with the set of formal power series $\sum_{n\geq 0} a_n z^n$, with $(a_n) \in \mathbb{C}^{\mathbb{N}_0}$.

In Section 4 we present a project with other chains of spaces, where similar questions may be investigated. They are mostly spaces of holomorphic functions in the open unit disc $\mathbb{D}$, but they also include some sequence spaces, via the identification $(a_n) \leftrightarrow \sum a_n z^n$. We can also add to the project other spaces of holomorphic functions such as Dirichlet spaces or Bloch spaces, as well as spaces of holomorphic functions on domains in $\mathbb{C}^d$, $d \geq 1$, and not only on $\mathbb{D}$. In all previous cases the spaces are complete metrizable topological vector spaces, in fact F-spaces, and the injections $X_i \subset X_j$ are continuous. What about spaces of harmonic functions on domains of $\mathbb{R}^{2n}$?

For background on concepts and results about Hardy spaces the reader is referred, for instance, to Duren's book [4]. In Sections 5 and 6 we shall provide two original theorems concerning Hardy spaces on $\mathbb{D}$: Theorem 8 about topological genericity and Theorem 9 about algebraic genericity. These results are in the frame of the project presented in Section 4.

1. Topological genericity for the ℓ^p spaces

We will deal with the following chain of spaces

$$\ell^\infty \supset c_0 \supset \bigcap_{p>b} \ell^p \supset \ell^b \supset \bigcap_{p>a} \ell^p \supset \ell^a \supset \bigcap_{p>0} \ell^p \qquad \text{(C)}$$

where $0 < a < b < +\infty$. Recall that c_0 is the space of all complex sequences tending to zero, while ℓ^p denotes the space of all complex sequences (a_n) such that $\sum_{n=1}^{p} |a_n|^p$ converges.

All inclusions are strict. Indeed, it is easy to verify that

$$(1, 1, \dots) \in \ell^\infty \setminus c_0,$$

$$\Big(\frac{1}{n^{\frac{1}{b+1}}}\Big)_{n=1}^\infty \in c_0 \setminus \bigcap_{p>b} \ell^p,$$

$$\Big(\frac{1}{n^{1/b}}\Big)_{n=1}^\infty \in \Big(\bigcap_{p>b} \ell^p\Big) \setminus \ell^b,$$

$$\Big(\frac{1}{n^{\frac{1}{\gamma}}}\Big)_{n=1}^\infty \in \ell^b \setminus \bigcap_{p<a} \ell^p, \text{ with } \gamma = \frac{a+b}{2},$$

$$\Big(\frac{1}{n^{\frac{1}{a}}}\Big)_{n=1}^\infty \in \bigcap_{p>a} \ell^p \setminus \ell^a,$$

$$\Big(\frac{1}{n^{\frac{1}{x}}}\Big)_{n=1}^\infty \in \ell^a \setminus \bigcap_{p>0} \ell^p, \text{ with } x = \frac{a}{2}.$$

In addition, all above spaces are metrizable complete topological vector spaces, when endowed with their natural topologies. The spaces ℓ^∞, c_0, ℓ^p with $1 \leq p < +\infty$ are Banach spaces. The space $\bigcap_{p>b} \ell^p$ with $1 \leq b < +\infty$ becomes a Fréchet space when we consider a strictly decreasing sequence $p_m \downarrow b$ and the distance in this space is defined by

$$d(f,g) = \sum_{m=1}^\infty \frac{1}{2^m} \frac{\|f-g\|_{p_m}}{1+\|f-g\|_{p_m}}, \quad \text{where } \|F\|_p = \Big(\sum_{n=1}^\infty |F(n)|^p\Big)^{1/p}.$$

For $0 < p < 1$ the space ℓ^p is not a Banach space. It is a metrizable complete topological vector space with metric $d_p(f,g) = \sum_{n=1}^{\infty} |f(n) - g(n)|^p$. For $0 \leq a < 1$ the space $\bigcap_{p>a} \ell^p$ is a metrizable complete topological space. Let $p_m \downarrow a$. Then the metric in this space is

$$d(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{d_{p_m}(f,g)}{1 + d_{p_m}(f,g)} \quad \text{where} \quad d_p(f,g) = \sum_{n=1}^{\infty} |f(n) - g(n)|^p.$$

Let Y and X be two of the previously mentioned spaces with $X \subset Y$ and $Y \setminus X \neq \emptyset$. Then, the injection map $I : X \to Y$, $I(f) = f$, is linear, continuous and it is not surjective. Then, according to a theorem of Banach (see, e.g. [11]), its image $I(X) = X$ is meager in Y; that is, X is contained in a denumerable union of closed subsets of Y with empty interior (in Y). In some cases we will show that X is equal to such a set; that is, X is an F_σ meager subset of Y; equivalently $Y \setminus X$ is a G_δ and dense subset in Y, while in the general case $Y \setminus X$ is residual in Y.

Let Y, X be two spaces in the chain (C) above, with $X \subset Y$ and $Y \setminus X \neq \emptyset$. If $X = c_0$, then $Y = \ell^\infty$. In this case X is a closed vector subspace of Y and hence it has empty interior in Y. Thus, $X = c_0$ is an F_σ meager subset of $Y = \ell^\infty$.

Next we consider the case $X = \ell^p$ with $0 < p < +\infty$.

As mentioned in the Introduction, all proofs in sections 1, 2 and 3 will be omitted and can be found in [8].

Proposition 1. *Let $X = \ell^p$ with $0 < p < +\infty$ and Y be one of the spaces in the chain (C) satisfying $Y \supset X$ and $Y \setminus X \neq \emptyset$. Then X is an F_σ meager subset in Y.*

The remaining case is $X = \bigcap_{p>a} \ell^p$, $0 \leq a < +\infty$. Then $Y \supset X$, $Y \setminus X \neq \emptyset$, is one of the previously mentioned spaces. Therefore, there exists $b \in (a, +\infty)$ such that $X \subset \ell^b \subset Y$ and $Y \setminus \ell^b \neq \emptyset$. It follows that ℓ^b is an F_σ meager subset of Y, according to Proposition 1. Thus, X is contained in ℓ^b which is a denumerable union of closed in Y sets with empty interiors. It follows that X is meager in Y and we arrived to this conclusion without using Banach's theorem. Since $X = \bigcap_m \ell^{p_m}$, where $p_m \in (a, +\infty)$ is a strictly decreasing sequence $p_m \downarrow a$ and each ℓ^{p_m} is an F_σ meager subset of Y, we cannot conclude that $X = \bigcap_{p>a} \ell^p$ is an F_σ subset of Y. In [5] V. Gregoriades, using descriptive set theory, has shown that $\bigcap_{p>a} \ell^p$ is not an F_σ subset of Y.

2. Algebraic genericity for the ℓ^p spaces

Let X and Y be two of the spaces in the chain (C) in the beginning of Section 1, such that $Y \supset X$ and $X \neq Y$. Thus, we say that *there is algebraic genericity for the couple* (Y, X) if there is a vector subspace F of Y dense in Y, such that $F \setminus \{0\} \subset Y \setminus X$.

If $Y = \ell^\infty$ we met the difficulty of the nonseparability of ℓ^∞. In this case D. Papathanasiou has proved algebraic genericity for the couple (ℓ^∞, c_0) in [10].

In all other cases we also have algebraic genericity. The essential lemma is the following.

Lemma 2. *Let* $0 < b < +\infty$, $Y = \bigcap_{p>b} \ell^p$ *and* $X = \ell^b$. *Then we have algebraic genericity for the couple* (Y, X).

Proposition 3. *Let* Y *and* X *be two of the spaces in the chain (C) such that* $Y \neq \ell^\infty$, $Y \supset X$, $Y \neq X$. *Then there is algebraic genericity for the couple* (Y, X).

3. Spaceability for the ℓ^p spaces

Let X and Y be two spaces as in the chain (C) with $X \subset Y$, $X \neq Y$. We say that *there is spaceability for the couple* (Y, X) if there exists a closed infinite dimensional subspace F of Y such that $F \setminus \{0\} \subset Y \setminus X$. We will show in this section that we always have spaceability.

Proposition 4. *Let* X *be one of the spaces in the chain (C) satisfying* $X \subset \ell^\infty$ *and* $X \neq \ell^\infty$. *Then we have spaceability for the couple* (ℓ^∞, X).

Proposition 5. *Let* $X = \ell^b$, $0 < b < +\infty$ *and* Y *be one of the spaces of the chain (C) satisfying* $Y \supset X$ *and* $Y \neq X$. *Then we have spaceability for the couple* (Y, ℓ^b).

Proposition 6. *Let* $X = \bigcap_{p>a} \ell^p$, $0 \leq a < +\infty$ *and* Y *be one of the spaces in the chain (C) satisfying* $Y \supset X$ *and* $Y \neq X$. *Then we have spaceability for the couple* (Y, X).

Combining Propositions 4, 5 and 6 we obtain the following assertion.

Theorem 7. *Let* Y, X *be among the spaces of the chain (C) satisfying* $Y \supset X$, $Y \neq X$. *Then we have spaceability of the couple* (Y, X).

4. Continuation of the project

Other chains of spaces, for which the same questions can be investigated, are the following:

(1) $A(\mathbb{D}) \subset H^\infty(\mathbb{D}) \subset BMOA(\mathbb{D}) \subset H^p(\mathbb{D}) \subset H(\mathbb{D})$ and intersections of those spaces.

(2) $A(\mathbb{D}) \subset H^\infty(\mathbb{D}) \subset OL^p(\mathbb{D}) \subset H(\mathbb{D})$, where $OL^p(\mathbb{D})$ denotes the Bergman space of order p on the unit disc.

(3) $0 \leq p \leq 1$, $1 \leq \gamma < +\infty$, $\ell^p \subset \ell^1 = \left\{ \sum_{n=0}^{\infty} a_n z^n : \sum |a_n| < +\infty \right\} \subset A(\mathbb{D}) \subset H^\infty(\mathbb{D}) \subset BMOA(\mathbb{D}) \subset H^\gamma(\mathbb{D}) \subset c_0 \subset \ell^\infty \subset H(\mathbb{D})$.

This chain can also be continued as follows: $H^\gamma(\mathbb{D}) \subset H^\beta(\mathbb{D}) \subset H(D)$ with $0 < \beta < \gamma < 1$.

(4) $0 \leq p \leq 2$, $2 \leq \delta \leq 1$, $\ell^p \subset \ell^2 = H^2(D) \subset H^\delta(D) \subset H^1(D) \subset c_0 \subset \ell^\infty \subset H(D)$.

(5) $0 \leq p \leq 2$, $0 < \varepsilon < 4$, $\ell^p \subset \ell^2 = H^2(D) \subset OL^4(\mathbb{D}) \subset OL^\varepsilon(\mathbb{D}) \subset H(\mathbb{D})$.

(6) $H^p(\mathbb{D}) \subset OL^{2p}(\mathbb{D}) \subset H(\mathbb{D})$.

(7) All previous ones with localized version of these spaces, as for example $H^1(\mathbb{D}) \subset H^1_{[\alpha,\beta]}(\mathbb{D})$, where the last space is defined in the next section.

5. Topological genericity in Hardy spaces

In this section, as well as in Section 6, we provide two original results in the frame of the project of Section 4 and especially for Hardy spaces on the unit disc.

Definition 5.1. *Let $a < b$ and $p \in (0, +\infty)$. A function $f \in H(\mathbb{D})$ is said to belong to the localised Hardy space $H^p_{[a,b]}(\mathbb{D})$ if* $\sup\limits_{0<r<1} \int_a^b |f(re^{i\vartheta})|^p d\vartheta < +\infty$.

The spaces $H^p_{[a,b]}(\mathbb{D})$ are F-spaces under their natural topologies. A sequence $(f_n) \subset H^p_{[a,b]}(\mathbb{D})$ converges in $H^p_{[a,b]}(\mathbb{D})$ to a function $f \in H^p_{[a,b]}(\mathbb{D})$ if and only if $f_n \to f$ uniformly on each compact subset of $\mathbb{D}$ and $\sup\limits_{0<r<1} \int_a^b |f_n(re^{i\vartheta}) - f(re^{i\vartheta})|^p d\vartheta \to 0$ as $n \to +\infty$.

The space $H^p_{[a,b]}(D)$ contains $H^p(D) = H^p$ and the inclusion map is continuous. Trivially, if $b - a \geq 2\pi$ then $H^p_{[a,b]}(D)$ coincides with the usual Hardy space $H^p = H^p(\mathbb{D})$.

Theorem 8. *Assume that $p \in (0, +\infty)$. Then the set $\mathcal{A}$ consisting of all $f \in H^p$ such that*

$$f \notin H^q_{[a,b]}(\mathbb{D}) \quad \text{for all} \quad q \in (p, +\infty) \quad \text{and all} \quad a < b$$

is a dense G_δ subset of H^p. In particular, $\mathcal{A}$ is non-void.

Proof. By using Baire's category theorem and the fact that H^p is a complete metric space, it is enough to show that the set $H^p \setminus \mathcal{A}$ can be written as a denumerable union of subsets of H^p that are closed and have empty interiors.

To this end, fix a strictly decreasing sequence $(q_j) \subset \mathbb{Q}$ converging to p and consider the countable collection $\{[a_k, b_k]\}_{k\geq 1}$ of closed real intervals with rational ends. For every $q > p$, every pair of reals a, b with $a < b$, every $M \in \mathbb{N}$ and every $r \in (0, 1)$, we define the sets

$$E(a, b, q, M) := \left\{ f \in H^p : \sup_{0<r<1} \int_a^b |f(re^{i\vartheta})|^q \, d\vartheta \leq M \right\}$$

and

$$E(a, b, q, M, r) := \left\{ f \in H^p : \int_a^b |f(re^{i\vartheta})|^q \, d\vartheta \leq M \right\}.$$

Observe that $E(a, b, q, M) = \bigcap\limits_{0<r<1} E(a, b, q, M, r)$ and that

$$H^p \setminus \mathcal{A} = \bigcup_{\substack{q>p \\ a<b}} H^q_{[a,b]}(\mathbb{D}) = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{M=1}^{\infty} E(a_k, b_k, q_j, M).$$

Consequently, it suffices to show that every set $E(a, b, q, M, r)$ is closed and that every set $E(a, b, q, M)$ has empty interior in H^p.

In order to reach the first goal, let $(f_n) \subset E(a, b, q, M, r)$ be a sequence converging to a function $f \in H^p$ in the topology of H^p. This implies that (f_n) converges to f uniformly on each compact subset of $\mathbb{D}$, in particular on each circle $|z| = r$ $(0 < r < 1)$. Then $|f_n|^q$ converges uniformly to $|f|^q$ on $|z| = r$. Thus

$$\int_a^b |f_n(re^{i\vartheta})|^q \, d\vartheta \longrightarrow \int_a^b |f(re^{i\vartheta})|^q \, d\theta \quad \text{as} \quad n \to +\infty.$$

As $\int_a^b |f_n(re^{i\vartheta})|^q \, d\vartheta \le M$ for all n, it follows that f satisfies the same inequality. In other words, $f \in E(a, b, q, M, r)$. We have proved the closedness of $E(a, b, q, M, r)$ in H^p.

It remains to show that $E(a, b, q, M)^\circ = \varnothing$. If not, pick $f \in E(a, b, q, M)^\circ$. In particular, $f \in H^q_{[a,b]}(\mathbb{D})$. Let $\omega := \frac{\alpha + \beta}{2}$. For an appropriate choice of $\gamma > 0$, the function $g(z) := \frac{1}{(z - e^{i\omega})^\gamma}$ belongs to $H^p \setminus H^q_{[a,b]}(\mathbb{D})$.

Since H^p is a topological vector space, the sequence $(f + \frac{1}{n} g)$ converges in the topology of H^p towards f, as $n \to +\infty$. Finally, since f belongs to $E(a, b, q, M)^\circ$, it follows that there exists $n \in \mathbb{N}$ so that

$$f + \frac{1}{n} g \in E(a, b, q, M)^\circ \subset E(a, b, q, M) \subset H^q_{[a,b]}(\mathbb{D}).$$

It follows that both functions f and $f + \frac{1}{n} g$ are in $H^q_{[a,b]}(\mathbb{D})$. But $H^q_{[a,b]}(\mathbb{D})$ is a vector space, from which we conclude that g belongs to $H^q_{[a,b]}(\mathbb{D})$. This is a contradiction and the proof is complete.

□

A result complementing Theorem 8 is the following. See also [6] and [9].

Theorem A. *Let $0 < p < +\infty$. Then there exists a holomorphic function f on the open unit disc $\mathbb{D}$, such that (1) and (2) below hold:*

(1) $\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^q d\theta < +\infty$ *for all* $0 < q < p$,

(2) $\sup_{0<r<1} \int_a^b |f(re^{i\theta})|^\delta d\theta = \infty$ *for all* $p \le \delta < +\infty$ *and all* $a < b$.

The set of such functions f is a G_δ and dense subset of the space $\bigcap_{q<p} \ell^q$ endowed with its natural topology.

For the proof we use the fact that for $\omega \in \mathbb{R}$ and $\gamma = \frac{1}{p}$ the function $g(z) = \frac{1}{(z - e^{i\omega})^\gamma}$ belongs to ℓ^q for all $q \in (0, p)$ but not to ℓ^p. The proof of the previous theorem is similar to that of Theorem 8 and is omitted.

6. Algebraic genericity in H^p spaces

In this final section we prove a second original result in the frame of the project in § 4.

Theorem 9. *Let $0 < p < q < +\infty$. Then the set $(H^p \setminus H^q) \cup \{0\}$ contains a vector space that is dense in H^p.*

Proof. There is $\gamma > 0$ so that the function $\dfrac{1}{(z-1)^\gamma}$ belongs to $H^p \setminus H^q$. Let $\omega_n = \dfrac{1}{n}$. We consider the functions

$$f_n(z) = \frac{c_n}{(z - e^{i\omega_n})^\gamma}, \quad n = 1, 2, \ldots,$$

where the real numbers c_n are chosen close enough to zero to obtain $d_p(f_n, 0) < \dfrac{1}{n}$, where d_p is the natural translation-invariant distance in H^p. Let $\{P_n : n = 1, 2, \ldots\}$ be an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. The set $\{P_n : n = 1, 2, \ldots\}$ is dense in H^p (see [11]). Since H^p does not contain isolated points, it follows that the sequence $\{P_n + f_n : n = 1, 2, \ldots\}$ is also dense in H^p (indeed, note that for any $g \in H^p$ and any $n \in \mathbb{N}$, we have $d_p(P_n + f_n, g) \le d_p(P_n, g) + d(f_n, g) < d_p(P_n, g) + \frac{1}{n}$). Thus, the linear space $F := \left\langle f_n + P_n \right\rangle_{n=1}^{\infty}$ is a dense vector subspace in H^p.

To finish, we must prove that $F \setminus \{0\} \subset H^p \setminus H^q$. With this aim, take a function $L \in F \setminus \{0\}$. Then there are $N \in \mathbb{N}$ and scalars $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ with $\lambda_N \neq 0$ such that

$$L = \lambda_1(f_1 + P_1) + \cdots + \lambda_N(f_N + P_N).$$

Of course, $L \in H^p$. We have to show that L does not belong to H^q.

Choose a pair of reals a, b with $a < \omega_N < b$ and $\omega_1, \ldots, \omega_{N-1} \notin [a, b]$. Then

$$\sup_{0<r<1} \int_a^b |\lambda_k(f_k + P_k)(re^{i\vartheta})|^q \, d\vartheta < +\infty \quad \text{for} \quad k = 1, \ldots, N-1$$

and

$$\sup_{0<r<1} \int_a^b |\lambda_N(f_N + P_N)(re^{i\vartheta})|^q \, d\theta = +\infty.$$

It follows from the triangle inequality that $\sup_{0<r<1} \int_a^b |L(re^{i\theta})|^q \, d\vartheta = +\infty$, and so

$$\sup_{0<r<1} \int_0^{2\pi} |L(re^{i\theta})|^q \, d\vartheta = +\infty.$$

Consequently, L does not belong to H^q and we are done. □

A result complementing Theorem 9 is the following statement, whose proof is similar to that of Theorem 9, and so it is omitted.

Theorem B. *Let $0 < p \le q \le +\infty$ and $\alpha < \beta$ be fixed. Then the set $\Big(\bigcap_{\beta<p} H^\beta \setminus H^q_{[a,b]}(\mathbb{D})\Big) \cup \{0\}$ contains a vector space dense in $\bigcap_{\beta<p} H^\beta$ endowed with it natural topology. In this result, the space $\bigcap_{\beta<p} H^\beta$ can be replaced by H^p provided that $p < q$.*

To finish this paper, we want to remark that *there is spaceability for the couple* (H^p, H^q) *if* $1 \le p < q \le +\infty$. Indeed, a result due to Drewnowski (see Proposition 2.4 in [7]) asserts that if Y and Z are Fréchet spaces and $T : Z \to Y$ is a continuous linear operator whose range $X = T(Z)$ is not closed, then the complement $Y \setminus X$ contains, except for 0, a closed infinite dimensional vector subspace of Y. In our case, just take $X = H^q = Z$, $Y = H^p$, and $T =$ the injection $f \in H^q \mapsto f \in H^p$. Observe that H^q is not closed in H^p because it is dense (it contains all polynomials) but $H^p \neq H^q$. We note that, in the previous assertion, the Hardy spaces may be replaced by intersections of such spaces, provided that these intersections are locally convex.

Acknowledgements. The authors are grateful to the referees for helpful comments and suggestions. We wish to thank R. Aron, I. Deliyanni and A. Siskakis for helpful communications. The research of the first author has been supported by the Plan Andaluz de Investigación de la Junta de Andalucía FQM-127 Grant P08-FQM-03543 and by MCIU Grant PGC2018-098474-B-C21.

References

[1] R.M. Aron, L. Bernal-González, D.M. Pellegrino and J.B. Seoane-Sepúlveda, *Lineability, The Search for Linearity in Mathematics*, Monographs and Research Notes in Mathematics, CRC Press, Taylor and Francis Group, A Chapman and Hall book.

[2] F. Bayart, K.-G. Grosse-Erdmann, V. Nestoridis and C. Papadimitropoulos, Abstract theory of universal series and applications, Proc. London Math. Soc. **96** (2009), No. 2, 417–468.

[3] L. Bernal - González, D. Pellegrino and J. B. Seone - Sepulveda, Linear subsets of non linear sets in topological vector spaces, Bull. Amer. Math. Soc. (New Series) **51** (2014), No. 1, 71–130.

[4] P.L. Duren, *Theory of H^p spaces*, Dover, New York, 2000.

[5] V. Gregoriades, Intersection of ℓ^p spaces in the Borel hierarchy, J. Math. Anal. Appl. **498** (2021), article 124922.

[6] K. Kioulafa, On Hardy type spaces in strictly pseudoconvex domains and the density in these spaces, of certain classes of singular functions, *J. Math. Anal. Appl.* **484** (2020), article 123697.

[7] D. Kitson and M. Timoney, Operator ranges and spaceability, J. Math. Anal. Appl. **378** (2011), 680–686.

[8] V. Nestoridis, A project about chains of spaces regarding topological and algebraic genericity and spaceability, available at ArXiv:2005.01023 (2020).

[9] V. Nestoridis, A.G. Siskakis, A. Stavrianidi and S. Vlachos, Generic non-extendability and total unboundedness in function spaces, *J. Math. Anal. Appl.* **475** (2019), No. 2, 1720–1731.

[10] D. Papathanasiou, Dense lineability and algebrability of $\ell^\infty \setminus c_0$, *Preprint* (2020), available at ArXiv:2102.03199.

[11] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill Book Co., New York, 1991.

Deparmento de Análisis Metemático, Facultad de Matemáticas, Instituto de Matematicas Antonio de Castro Brzezicki, Universidad de Sevilla, Avenida Reina Mercedes s/n, 41012 Sevilla, Spain.

Email address: lbernal@us.es

National and Kapodistrian University of Athens, Department of Mathematics, Panepistemiopolis, 157 84, Athens, Greece.

Email address: vnestor@math.uoa.gr